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COMPATIBILITY AND CENTRAL ELEMENTS IN EFFECT ALGEBRAS

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ABSTRACT. Relations between compatibility of elements (defined by Kôpka and Chovanec) and central elements (defined by Greechie, Foulis and Pulmannová) of effect algebras are studied.

1. Introduction

Recently, Foulis and Bennett [2] have introduced a new algebraic structure, called an effect algebra. Effects represent unsharp measurements or observations on a quantum mechanical system. For modelling unsharp measurements in Hilbert space, the set of all effects is the set of all selfadjoint operators T on a Hilbert space H with $0 \leq T \leq 1$. In a general algebraic form an effect algebra is defined as follows:

Definition 1.1. A partial algebra $(E; \oplus, 0, 1)$ is called an *effect-algebra* if $0, 1$ are two distinguished elements and $\oplus$ is a partially defined binary operation on E which satisfies the following conditions for any $a, b, c \in E$:

(Ei) $b \oplus a = a \oplus b$ if $a \oplus b$ is defined,

(Eii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if one side is defined,

(Eiii) for every $a \in E$ there exists the unique $b \in E$ such that $a \oplus b = 1$,

(Eiv) If $1 \oplus a$ is defined then $a = 0$.

Proposition 1.2. (Cancellation property) In an effect algebra $(E; \oplus, 0, 1)$ for all $a, b, c \in E$ with defined $a \oplus b$ and $a \oplus c$ it holds

$$a \oplus b = a \oplus c \qquad \text{implies} \qquad b = c\,.$$

Cancellation property guarantees that in every effect algebra E the partial binary operation $\ominus$ and the relation $\leq$ can be defined by

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(ED) $\quad a \le c$ and $c \ominus a = b$ iff $a \oplus b$ is defined and $a \oplus b = c$.

Under the partial order defined by (ED), 0 is the least and 1 the greatest element of E.

Unbounded versions (mutually equivalent) of effect algebras were studied by Foulis and Bennett (cones), Kalmbach and Riečanová (abelian RI-semigroups), Hedlíková and Pulmannová (cancellative positive partial abelian semigroups). Their common definition is the following:

Definition 1.3. A partial algebra $(E;\oplus,0)$ is called a *generalized effect algebra* if $0 \in E$ is a distinguished element and $\oplus$ is a partially defined binary operation on E which satisfies the following conditions for any $a,b,c \in E$:

(GEi) $a \oplus b = b \oplus a$, if one side is defined,

(GEii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$, if one side is defined,

(GEiii) $a \oplus 0 = a$ for all $a \in E$,

(GEiv) $a \oplus b = a \oplus c$ implies $b = c$ (cancellation law),

(GEv) $a \oplus b = 0$ implies $a = b = 0$.

In every generalized effect algebra E, the partial binary operation $\ominus$ and relation $\le$ can be defined by (ED). Then $\le$ is a partial order on E under which 0 is the least element of E.

Proposition 1.4. *If $(E;\oplus,0)$ is a generalized effect algebra and there is $1 \in E$ such that for all $a \in E$ there is $b \in E$ with $a \oplus b = 1$, then $(E;\oplus,0,1)$ is an effect algebra. Conversely, if $(E;\oplus,0,1)$ is an effect algebra, then partial operation $\oplus$ satisfies axioms* (GEi) – (GEv) *of a generalized effect algebra.*

The proof is left to the reader.

Remark 1.5. If for an effect algebra $(E;\oplus,0,1)$ we define on E a partial operation $\ominus$ and a partial order $\le$ by condition (ED), then $(E;\le,\ominus,0,1)$ is a D-poset introduced by Kôpka and Chovanec [11] and $(E;\ominus,0,1)$ is a D-algebra defined by Gudder [6]. Moreover, using condition (ED) for generalized effect algebra $(E;\oplus,0)$ we obtain generalized D-poset $(E;\ominus,0)$ studied by Hedlíková and Pulmannová [7]. Conversely, for mentioned partial algebras with operation $\ominus$ we can derive a partial operation $\oplus$ using the same condition (ED) and we obtain effect algebras $(E;\oplus,0,1)$ or generalized effect algebras $(E;\oplus,0)$.

The following theorem was proved for generalized D-poset by Hedlíková and Pulmannová [7].

Proposition 1.6. *Let $(P;\oplus_P,0_P)$ be a generalized effect algebra and let $\ominus_P$ and $\le_P$ on P be defined by* (ED). *Let $(P^*;\le^*)$ be the dual poset to $(P;\le_P)$.*

Suppose that $\mathbb{P} = P\dot{\cup}P^*$ is disjoint union and we define the partial order $\le$ and the partial binary operations $\oplus$ and $\ominus$ on $\mathbb{P}$ by the following conditions:

(1) *for $a, b \in P$, $a \le b$ iff $a \le_P b$ and then we put $b \ominus a = b \ominus_P a$,*

(2) *for $a^*, b^* \in P^*$, $a^* \le b^*$ iff $b \le_P a$ and then we put $b^* \ominus a^* = a \ominus_P b$,*

(3) *for $a \in P$, $b^* \in P^*$, $a \le b^*$ iff $b \le a^*$ iff $a \oplus_P b$ is defined and then we put $b^* \ominus a = a^* \ominus b = (a \oplus_P b)^*$,*

(4) *for all $a \in P$ and $b^* \in P^*$ $b^* \le a$ does not hold,*

(5) *for all $a, b \in \mathbb{P}$ with $a \le 0^* \ominus b$ we put $a \oplus b = 0_P^* \ominus \big((0_P^* \ominus a) \ominus b\big)$.*

Then $(\mathbb{P}; \oplus, 0, 1)$, where $0 = 0_P$ and $1 = 0_P^*$ is an effect algebra. Moreover, for $a, b \in \mathbb{P}$ with defined $a \oplus b$ it holds $a \oplus b \in P$ iff $a, b \in P$. In such case $a \oplus b = a \oplus_P b$.

2. Compatibility and central elements

In the remainder, for a partial operation $\oplus$ ($\ominus$) on a set X and for $a, b, c \in X$, if we write $a \oplus b = c$ ($c \ominus b = a$), we mean both that $a \oplus b$ ($c \ominus b$) is defined and $a \oplus b = c$ ($c \ominus b = a$).

The notion of a central element of an effect algebra $(E; \oplus, 0, 1)$ was introduced by Greechie–Foulis–Pulmannová [5]. A central element of a generalized effect algebra was defined by Riečanová [15].

Definition 2.1. (1) For an effect algebra $(E; \oplus, 0, 1)$ an element $z \in E$ is called *central* iff for every $x \in E$ there exist $x \wedge z$ and $x \wedge z'$ (where $z' = 1 \ominus z$) and $x = (x \wedge z) \vee (x \wedge z')$.

(2) For a generalized effect algebra $(E; \oplus, 0)$ an element $z \in E$ is called *central* iff for every $x \in E$ there exists $x \wedge z$ and the following conditions are satisfied:

(i) $x = (x \wedge z) \vee (x \ominus (x \wedge z))$,

(ii) $x \wedge z = 0$ implies that $x \oplus z$ is defined,

(iii) if $x \oplus y$ is defined and $x \wedge z = y \wedge z = 0$ then $(x \oplus y) \wedge z = 0$.

In [15] it was shown that for generalized effect algebra $(P; \oplus, 0)$ an element $z \in P$ is central iff z and z^* are central elements of the effect algebra $(\mathbb{P}; \oplus, 0, 1)$ constructed by Proposition 1.6. Moreover, it is iff $(P; \oplus, 0)$ is isomorphic to a direct product of the effect algebra $[0, z]$ and the generalized effect algebra $Q_z = \{x \in P \colon x \wedge z = 0\}$, both meaning with operations $\oplus$ inherited from P. In such case, $(\mathbb{P}; \oplus, 0, 1)$ is isomorphic to a direct product of effect algebras $[0, z]$ and $[0, z^*]$, both meaning with $\oplus$ inherited from $\mathbb{P}$.

We can also show that for elements of effect algebras conditions (1) and (2) of Definition 2.1 coincide.

THEOREM 2.2. *Let $(E;\oplus,0,1)$ be an effect algebra. Then $z\in E$ is central iff z is a central element of the generalized effect algebra $(E;\oplus,0)$.*

Proof. (a) By [5], if for $x\in E$ there exist $x\oplus y$ and $x\vee y$, then $x\wedge y$ exists and $x\oplus y=(x\vee y)\oplus(x\wedge y)$. Let $z\in E$ satisfy condition (1) of Definition 2.1. Then $1=z\vee z'=z\oplus z'$, which implies that $z\wedge z'=0$. Thus for every $x\in E$, we obtain $x=(x\wedge z)\vee(x\wedge z')=(x\wedge z)\oplus(x\wedge z')$ and $x\wedge z'=x\ominus(x\wedge z)$. Moreover, the condition $x\wedge z=0$ implies that $x=x\wedge z'\le 1\ominus z$. Hence $x\oplus z$ is defined and $x\oplus z=z\vee x$. Finally, if $x\wedge z=y\wedge z=0$ and $x\oplus y$ is defined, then $x\le 1\ominus z$ and $y\le 1\ominus x$ imply that $y\oplus z=y\vee z\le 1\ominus x$. It follows that $x\oplus(y\oplus z)$ is defined, hence $x\oplus y\le 1\ominus z$ and thus $(x\oplus y)\wedge z=0$.

(b) Let $z\in E$ satisfy condition (2) of Definition 2.1. Then for every $y\in E$ with $y\wedge z=0$ there exists $y\oplus z=\big((y\oplus z)\wedge z\big)\vee\Big((y\oplus z)\ominus\big((y\oplus z)\wedge z\big)\Big)=z\vee y$. Let $x\in E$, then $x=(x\wedge z)\vee\big(x\ominus(x\wedge z)\big)=(x\wedge z)\oplus\big(x\ominus(x\wedge z)\big)$, which implies that $(x\wedge z)\wedge\big(x\ominus(x\wedge z)\big)=0$ and hence $z\wedge\big(x\ominus(x\wedge z)\big)=0$. It follows that $x\ominus(x\wedge z)\le 1\ominus z=z'$. Suppose that $y\le x$ and $y\le z'$. Then $y\oplus z=y\vee z\le\big(x\ominus(x\wedge z)\big)\vee z=\big(x\ominus(x\wedge z)\big)\oplus z$. Thus $y\le x\ominus(x\wedge z)$, which implies that $x\ominus(x\wedge z)=x\wedge z'$. We conclude that $x=(x\wedge z)\vee\big(x\ominus(x\wedge z)\big)=(x\wedge z)\vee(x\wedge z')\le x$. □

In [10] the notion of compatibility of two elements in a D-poset was introduced. We note that the definition of compatibility makes sense also in effect algebras and generalized effect algebras.

DEFINITION 2.3. For an effect algebra $(E;\oplus,0,1)$ (or a generalized effect algebra $(E;\oplus,0)$) two elements $a,b\in E$ are *compatible* (written $a\leftrightarrow b$) if there exist $u,v\in E$ such that $v\le a\le u$, $v\le b\le u$ and $u\ominus a=b\ominus v$ (evidently then $u\ominus b=a\ominus v$).

LEMMA 2.4. *For elements of a generalized effect algebra $(E;\oplus,0)$ the following conditions are satisfied:*

(i) *if $a\oplus b$ and $a\vee b$ exist, then $a\wedge b$ exists and $a\oplus b=(a\vee b)\oplus(a\wedge b)$,*

(ii) *if $v\le a\le u$, then $a\ominus v=(u\ominus v)\ominus(u\ominus a)$,*

(iii) *if E is a lattice (under $\le$ defined by (ED)) and $a\oplus b$, $a\oplus c$ are defined, then $a\oplus(b\vee c)=(a\oplus b)\vee(a\oplus c)$,*

(iv) *if E is a lattice, then $a\leftrightarrow b$ iff $(a\vee b)\ominus b=a\ominus(a\wedge b)$,*

(v) *if $z\in E$ is central and $x\in E$ is such that $x\wedge z=0$, then there exists $x\vee z$ and $x\vee z=x\oplus z$,*

(vi) *if $z\in E$ is central, then $x\leftrightarrow z$ for all $x\in E$,*

(vii) *if $z\in E$ is central, then for every $x\in E$ there exists $x\vee z$.*

P r o o f . The condition (i) was proved in [15], (ii) in [11] and [6]. A proof of (iii) can be found in [13] and (iv) was proved in [11] for lattice D-poset. The same proof is possible for generalized effect algebra.

(v): Let $z \in E$ be a central element and $x \in E$ with $x \wedge z = 0$. Then by properties of central elements there exists $x \oplus z$ and $x \oplus z = \big((x \oplus z) \wedge z\big) \vee \Big((x \oplus z) \ominus \big((x \oplus z) \wedge z\big)\Big) = z \vee x$.

(vi): Let $z \in E$ be central and $x \in E$. Then $x = (x \wedge z) \vee \big(x \ominus (x \wedge z)\big) = (x \wedge z) \oplus \big(x \ominus (x \wedge z)\big)$ which implies that $0 = \big(x \ominus (x \wedge z)\big) \wedge (x \wedge z) = \big(x \ominus (x \wedge z)\big) \wedge z$. By (v) there exists $\big(x \ominus (x \wedge z)\big) \vee z = \big(x \ominus (x \wedge z)\big) \oplus z$ and hence there exists $x \vee z = \big(x \ominus (x \wedge z)\big) \vee z = \big(x \ominus (x \wedge z)\big) \oplus z$. It follows that $(x \vee z) \ominus z = x \ominus (x \wedge z)$ and hence $x \leftrightarrow z$.

(vii): By (v) there exists $\big(x \ominus (x \wedge z)\big) \oplus z = \big(x \ominus (x \wedge z)\big) \vee z = x \vee z$. □

If an effect algebra (a generalized effect algebra) is a lattice under the partial order defined by (ED), then it is called a *lattice effect algebra* (*a lattice generalized effect algebra*).

Theorem 2.5. (1) *For an element z of a lattice generalized effect algebra $(E; \oplus, 0)$ the following conditions are equivalent:*

(i) *z is central,*

(ii) *$z \leftrightarrow x$ and $\big(x \ominus (x \wedge z)\big) \wedge z = 0$ for every $x \in E$,*

(iii) *$x = (x \wedge z) \vee \big(x \ominus (x \wedge z)\big)$ for every $x \in E$.*

(2) *For a lattice effect algebra $(E; \oplus, 0, 1)$ and $z \in E$, any of the conditions* (i) – (iii) *is equivalent to any of the following conditions:*

(iv) *$z \leftrightarrow x$ and $z \wedge z' = 0$ for every $x \in E$,*

(v) *$x \ominus (x \wedge z) = x \wedge z'$ for every $x \in E$.*

P r o o f . (i) $\Longrightarrow$ (iii): This follows by Definition 2.1.

(iii) $\Longrightarrow$ (ii): By Lemma 2.4 and cancellation law we obtain that $x = (x \wedge z) \vee \big(x \ominus (x \wedge z)\big)$ implies $(x \wedge z) \wedge \big(x \ominus (x \wedge z)\big) = 0$ and hence $\big(x \ominus (x \wedge z)\big) \wedge z = 0$. Moreover, $(x \vee z) \ominus z = \Big((x \wedge z) \vee \big(x \ominus (x \wedge z)\big) \vee z\Big) \ominus z = \Big(\big(x \ominus (x \wedge z)\big) \vee z\Big) \ominus z = \Big(\big(x \ominus (x \wedge z)\big) \oplus z\Big) \ominus z = x \ominus (x \wedge z)$. It follows that $x \leftrightarrow z$ by Lemma 2.4, (iv).

(ii) $\Longrightarrow$ (i): If $\big(x \ominus (x \wedge z)\big) \wedge z = 0$ then $\big(x \ominus (x \wedge z)\big) \wedge (x \wedge z) = 0$ and hence $x = (x \wedge z) \oplus \big(x \ominus (x \wedge z)\big) = (x \wedge z) \vee \big(x \ominus (x \wedge z)\big)$. Moreover, $z \leftrightarrow x$ for every $x \in E$ implies that $(x \vee z) \ominus z = x \ominus (x \wedge z)$ for every $x \in E$. It follows that if $x \wedge z = 0$ then $(x \vee z) \ominus z = x$, hence $x \vee z = x \oplus z$. Finally, if $x \oplus y$ is defined and $x \wedge z = y \wedge z = 0$ then by Lemma 2.4, (iii) $(x \oplus y) \vee (x \oplus z) = x \oplus (y \vee z) = x \oplus (y \oplus z)$. Thus $(x \oplus y) \oplus z = (x \oplus y) \vee (x \vee z) = (x \oplus y) \vee z$. It follows that $(x \oplus y) \wedge z = 0$ by Lemma 2.4, (i) and cancellation property. We conclude that z is central.

(2) If $(E;\oplus,0,1)$ is a lattice effect algebra and $z\in E$, then conditions (i)–(iii) are equivalent by Theorem 2.2. Moreover, $1=(1\wedge z)\vee\big(1\ominus(1\wedge z)\big)=z\vee z'$ by (iii), hence $z\wedge z'=0$. By Theorem 2.5, (1) we have $x\ominus(x\wedge z)\le 1\ominus z=z'$. Hence $x=(x\wedge z)\vee\big(x\ominus(x\wedge z)\big)\le(x\wedge z)\vee(x\wedge z')\le x$. We obtain $x=(x\wedge z)\vee(x\wedge z')=(x\wedge z)\oplus(x\wedge z')$, which implies that $x\ominus(x\wedge z)=x\wedge z'$.

Conversely, if $x\ominus(x\wedge z)=x\wedge z'$ for every $x\in E$ then $z\wedge z'=z\ominus z=0$ and hence $x=(x\wedge z)\oplus\big(x\ominus(x\wedge z)\big)=(x\wedge z)\oplus(x\wedge z')=(x\wedge z)\vee(x\wedge z')$, which implies (i). If $z\leftrightarrow x$ for every $x\in E$ and $z\wedge z'=0$, then $x\ominus(x\wedge z)=(x\vee z)\ominus z\le 1\ominus z=z'$, hence $x=(x\wedge z)\oplus\big(x\ominus(x\wedge z)\big)=(x\wedge z)\vee\big(x\ominus(x\wedge z)\big)$. □

In [5] an element x of an effect algebra is called *isotropic* if $x\oplus x$ is defined. This definition makes sense also for a generalized effect algebra.

Theorem 2.6. *For elements x,y of a lattice effect algebra $(E;\oplus,0,1)$ without nonzero isotropic elements the following conditions are equivalent:*

(i) $x=(x\wedge y)\vee(x\wedge y')$,

(ii) $y=(x\wedge y)\vee(x'\wedge y)$,

(iii) $x\leftrightarrow y$.

Proof. If $x\in E$ then $x\wedge x'=0$ since $u\le x\wedge x'$ implies $u\oplus u$ exists and thus $u=0$.

(iii)$\Rightarrow$(i): If $x\leftrightarrow y$ then by Lemma 2.4 $(x\vee y)\ominus y=x\ominus(x\wedge y)$, which implies that $x\ominus(x\wedge y)\le 1\ominus y=y'$. It follows that $x=(x\wedge y)\oplus\big(x\ominus(x\wedge y)\big)\le(x\wedge y)\oplus(x\wedge y')=(x\wedge y)\vee(x\wedge y')\le x$. Hence $x=(x\wedge y)\vee(x\wedge y')$.

(i)$\Rightarrow$(iii): If $x=(x\wedge y)\vee(x\wedge y')$ then $x\wedge y'=x\ominus(x\wedge y)$. It follows that $x\vee y=(x\wedge y)\vee\big(x\ominus(x\wedge y)\big)\vee y=\big(x\ominus(x\wedge y)\big)\vee y=\big(x\ominus(x\wedge y)\big)\oplus y$. Thus $(x\vee y)\ominus y=x\ominus(x\wedge y)$, which implies that $x\leftrightarrow y$.

(i)$\Leftrightarrow$(ii): This follows from the fact that (i)$\Leftrightarrow$(iii)$\Leftrightarrow$(ii). □

Corollary 2.7. *For an element z of a lattice effect algebra $(E;\oplus,0,1)$ without nonzero isotropic elements the following conditions are equivalent:*

(i) *z is central,*

(ii) *$z\leftrightarrow x$ for every $x\in E$,*

(iii) *$x=(x\wedge z)\vee(x\wedge z')$ for every $x\in E$.*

Corollary 2.8. *For a lattice effect algebra $(E;\oplus,0,1)$ without nonzero isotropic elements the following conditions are equivalent:*

(i) *every element $x\in E$ is central,*

(ii) *$x\leftrightarrow y$ for all $x,y\in E$,*

(iii) $x = (x \wedge y) \vee (x \wedge y')$ *for all* $x, y \in E$,

(iv) $(E, \vee, \wedge, ', 0, 1)$ *under the partial order on* E *defined by* (ED) *and* $x' = 1 \ominus x$ *for every* $x \in E$, *is a Boolean algebra.*

COROLLARY 2.9. *For a lattice effect algebra* $(E; \oplus, 0, 1)$ *the following conditions are equivalent:*

(i) *every element* $x \in E$ *is central,*

(ii) $x = (x \wedge y) \vee (x \wedge y')$ *for all* $x, y \in E$,

(iii) $(E, \vee, \wedge,', 0, 1)$ *is a Boolean algebra.*

Proof. Each of the conditions (i), (ii) and (iii) implies that nonzero elements of E are not isotropic. Conditions (i)–(iii) are equivalent by Corollary 2.8. □

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