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TRANSLATING BOOLEAN SETS TO FUZZY SETS (REALS AND THEIR TOPOLOGY)

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ABSTRACT. A Boolean universe (possibly over a set of urelements) can be translated to the fuzzy universe using a measure on the Boolean algebra. Conjecturing that Boolean reals can be represented as distribution functions we prove it for binary rationals over the algebra of half-open intervals (which is a quite efficient implementation of Boolean sourced fuzzy rationals). The induced topology is defined.

In this paper we continue the research we started in [V]. Before we recall notation, definitions and results of [V], we give some motivation. Dealing with multivalued mathematics and/or logic we are forced to choose appropriate methods to describe such phenomena. Besides applicability there are several methods of back-coupling, mainly concerned with relations to the classical-crisp-two valued case (which should, unchanged, be covered too). The first possibility just states we are generalizing two valued notions in the language of characteristic functions to many valued notions with values in a structure having smallest and largest elements (unit interval, lattice). This we show is not always fully satisfactory alone. The second possibility asks for the same categorical witness for the notion (construction) as for the two valued one. This we show, at least in the case of reals, gives similar result as the third approach does, see [H]. The third being the use of Boolean model as a model of ZFC and hence all notions being definable in the same way as crisp ones. Using a measure over the algebra we can translate it to the fuzzy sets (loosely speaking: instead of Boolean value assign the measure of this Boolean element — and do it hereditarily).

There is yet another aspect of our approach — urelements. As in applications fuzziness appears from a certain level on, we find it more convenient to construct our universes starting from urelements on, those being crisp.

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Definition 1. ([J]) For a Boolean algebra B and a set of urelements U we define the *Boolean valued universe* $\mathscr{V}^B(U)$ by transfinite induction through ordinal numbers:

(1) $\mathscr{V}_0^B(U) = U$,

(2) $\mathscr{V}_{\alpha+1}^B(U) = \{f\colon \operatorname{dom}(f) \subseteq \mathscr{V}_\alpha^B(U) \text{ and } \operatorname{rng}(f) \subseteq B\}$,

(3) $\mathscr{V}_\lambda^B(U) = \bigcup\limits_{\alpha<\lambda} \mathscr{V}_\alpha^B(U)$ for λ limit,

(4) $\mathscr{V}^B(U) = \bigcup\limits_{\alpha\in\mathrm{On}} \mathscr{V}_\alpha^B(U)$.

Definition 2. ([V]) For a set of urelements U we define the *fuzzy universe* $\mathscr{F}(U)$ by transfinite induction through ordinal numbers:

(1) $\mathscr{F}_0(U) = U$,

(2) $\mathscr{F}_{\alpha+1}(U) = \{f\colon \operatorname{dom}(f) \subseteq \mathscr{F}_\alpha \text{ and } \operatorname{rng}(f) \subseteq [0,1]\}$,

(3) $\mathscr{F}_\lambda(U) = \bigcup\limits_{\alpha<\lambda}$ for λ limit,

(4) $\mathscr{F}(U) = \bigcup\limits_{\alpha\in\mathrm{On}} \mathscr{F}_\alpha(U)$.

To distinguish between $\mathscr{V}^B(U)$ and $\mathscr{F}(U)$, elements of $\mathscr{V}^B(U)$ will be denoted by $f^B, g^B, h^B, \dots$ (possibly indexed) and elements of $\mathscr{F}(U)$ will be denoted by $f^F, g^F, h^F, \dots$ (also possibly indexed). If there is no danger of confusion we omit upper indexes $(\cdot)^B$ and $(\cdot)^F$.

Definition 3. ([V], [TT]) Assume that $\mu\colon B \to [0,1]$ is a (positive) probabilistic measure on a Boolean algebra and U is a set of urelements. We define

$$i_\mu\colon \mathscr{V}^B(U) \to \mathscr{F}(U)$$

by transfinite induction as follows:

(1) for $x \in U = \mathscr{V}_0^B$ put $i_\mu(x) = x$,

(2) having $i_\mu \restriction \mathscr{V}_\alpha^B\colon \mathscr{V}_\alpha^B \to \mathscr{F}_\alpha(U)$ already defined, for $f \in \mathscr{V}_{\alpha+1}^B(U)$ we define:

$$\operatorname{dom}\bigl(i_\mu(f)\bigr) = \bigl\{i_\mu(h)\colon h \in \operatorname{dom}(f) \subseteq \mathscr{V}_\alpha^B(U)\bigr\} \subseteq \mathscr{F}_\alpha(U)$$

and

$$i_\mu(f)\bigl(i_\mu(h)\bigr) = \mu\bigl(f(h)\bigr) \text{ (for some fixed choice of } h\text{)}.$$

To look for reals (for simplicity in the unit interval, with operations mod 1) in this setting we can start (to generalize the classical-crisp-two valued case) from either $U = [0,1]$ or $U = \mathbb{N}$ — the set of natural numbers (think of a real in dyadic expansion).

I. $U = [0,1]$. A real $x \in [0,1]$ can be viewed as a one-element set $\{x\} \subseteq [0,1]$ and its characteristic function $\chi_x \colon [0,1] \to \{0,1\}$. If we generalize it this way, we could be tempted to say that a fuzzy real is an arbitrary function $f \colon [0,1] \to [0,1]$ (possibly requesting that it takes value 1). We do not deal with this case now, but using Boolean models over $\mathbb{N}$ we get more information about it.

II. $U = \mathbb{N}$. A real $x \in [0,1]$ can be viewed also as $f \colon \mathbb{N} \to \{0,1\}$, where $0.f(0)f(1)\cdots f(n)\cdots$ is its dyadic expansion. In this sense, a fuzzy real is arbitrary $f : \mathbb{N} \to [0,1]$ i.e., $f^F \in \mathscr{F}_1(\mathbb{N})$. We will deal with this case to get out more about arithmetical operations and representation of reals in $\mathscr{F}\big([0,1]\big)$. In order to do this we look for $\mathscr{V}^B(\mathbb{N})$ and/or $\mathscr{V}^B\big([0,1]\big)$, because these are models of ZFC and there is no doubt what reals are (and arithmetic and topology).

We start with an example on which we illustrate our approach.

Example 4. Consider $f^F, g^F \in \mathscr{F}_1(\mathbb{N})$ defined as follows:

$$f^F(1) = \frac{1}{2}\,,\ f^F(2) = \frac{1}{4}\,,\ f^F(3) = \frac{3}{4} \qquad \text{and} \quad f^F(i) = 0 \quad \text{for} \quad i > 3\,;$$
$$g^F(1) = 0\,,\ g^F(2) = \frac{2}{3}\,,\ g^F(3) = \frac{1}{3} \qquad \text{and} \quad g^F(i) = 0 \quad \text{for} \quad i > 3\,.$$

Moreover, consider the Boolean algebra B of subsets of the unit interval consisting of finite disjoint unions of intervals (left closed right open) with rational endpoints equipped with the Lebesgue measure (carrying the translations from $\mathscr{V}^B(\mathbb{N})$ to $\mathscr{F}(\mathbb{N})$).

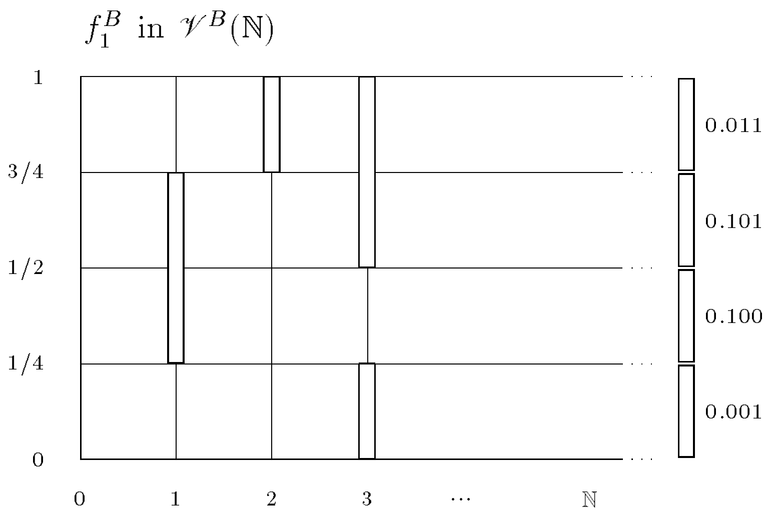

Figure 1.1

Note that there are many possible Boolean sources for f^F and g^F. In our example we consider just f_1^B and g_1^B satisfying $i_\mu(f_1^B) = f^F$ and $i_\mu(g_1^B) = g^F$. The further explanation is only for f_1^B and g_1^B (for other possible Boolean sources it holds similarly). Namely take

$$f_1^B(1) = \left[\tfrac{1}{4}, \tfrac{3}{4}\right),\ f_1^B(2) = \left[\tfrac{3}{4}, 1\right),\ f_1^B(3) = \left[0, \tfrac{1}{4}\right) \cup \left[\tfrac{1}{2}, 1\right),\ f_1^B(i) = \emptyset \quad \text{for } i > 3\,,$$
$$g_1^B(1) = \emptyset\,,\ g_1^B(2) = \left[0, \tfrac{1}{3}\right) \cup \left[\tfrac{2}{3}, 1\right)\,,\ g_1^B(3) = \left[0, \tfrac{1}{3}\right)\,,\ g_1^B(i) = \emptyset \quad \text{for } i > 3\,.$$

(See pictures Fig. 1.1 – 1.2, intervals are denoted by boxes.)

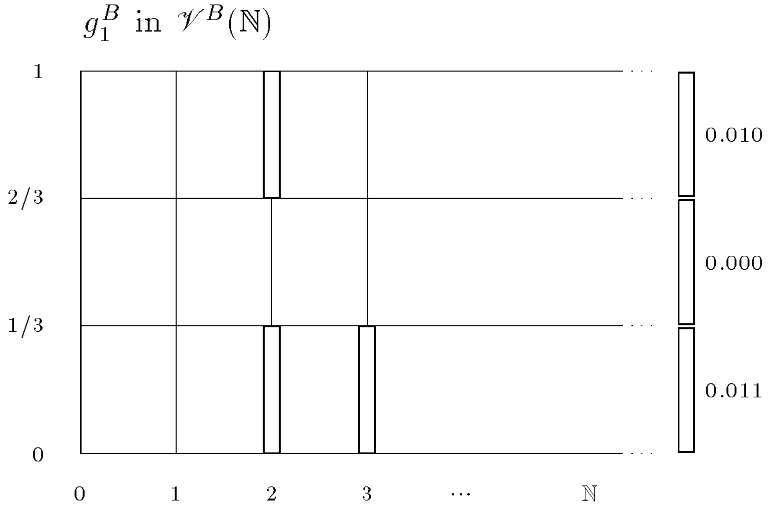

FIGURE 1.2

Note, moreover, that if we refine partitions $\{f_1^B(1), -f_1^B(1)\}$, $\{f_1^B(2), -f_1^B(2)\}$, $\{f_1^B(3), -f_1^B(3)\}$ we get the four element partition on the right, showing us that Boolean values (as defined in [J], see also [V]) fulfill: $\|f_1^B = 0.001\| = \left[0, \tfrac{1}{4}\right)$, $\|f_1^B = 0.100\| = \left[\tfrac{1}{4}, \tfrac{1}{2}\right)$, $\|f_1^B = 0.101\| = \left[\tfrac{1}{2}, \tfrac{3}{4}\right)$, and $\|f_1^B = 0.011\| = \left[\tfrac{3}{4}, 1\right)$, and similarly we get $\|g_1^B = 0.011\| = \left[0, \tfrac{1}{3}\right)$, $\|g_1^B = 0.000\| = \left[\tfrac{1}{3}, \tfrac{2}{3}\right)$, and $\|g_1^B = 0.010\| = \left[\tfrac{2}{3}, 1\right)$.

The Boolean sum fulfills the same rules as convolution, e.g., the value 0.100 can be obtained by both $0.001 + 0.011$ witnessed by $\left[0, \tfrac{1}{4}\right) \cap \left[0, \tfrac{1}{3}\right)$ and by $0.100 + 0.000$ witnessed by $\left[\tfrac{1}{4}, \tfrac{1}{2}\right) \cap \left[\tfrac{1}{3}, \tfrac{2}{3}\right)$, i.e., we have $\|f_1^B + g_1^B = 0.100\| = \left[0, \tfrac{1}{4}\right) \cup \left[\tfrac{1}{3}, \tfrac{1}{2}\right)$. Similarly we get $\|f_1^B + g_1^B = 0.101\| = \left[\tfrac{1}{2}, \tfrac{2}{3}\right) \cup \left[\tfrac{3}{4}, 1\right)$ and $\|f_1^B + g_1^B = 0.111\| = \left[\tfrac{1}{4}, \tfrac{1}{3}\right) \cup \left[\tfrac{2}{3}, \tfrac{3}{4}\right)$, Fig. 1.3 (for this see more in [V]).

There is yet another feature of this Boolean approach, namely, statements like $\|f_1^B = 0.001\| = \left[0, \tfrac{1}{4}\right)$ can be viewed as a representation of f_1^B from $\mathscr{V}_1^B(\mathbb{N})$ by an element of $\mathscr{V}_1^B([0,1])$, namely the function (denoted again by f_1^B) $f_1^B(0.001) = \left[0, \tfrac{1}{4}\right)$, $f_1^B(0.100) = \left[\tfrac{1}{4}, \tfrac{1}{2}\right)$, $f_1^B(0.101) = \left[\tfrac{1}{2}, \tfrac{3}{4}\right)$ and $f_1^B(0.011) =$

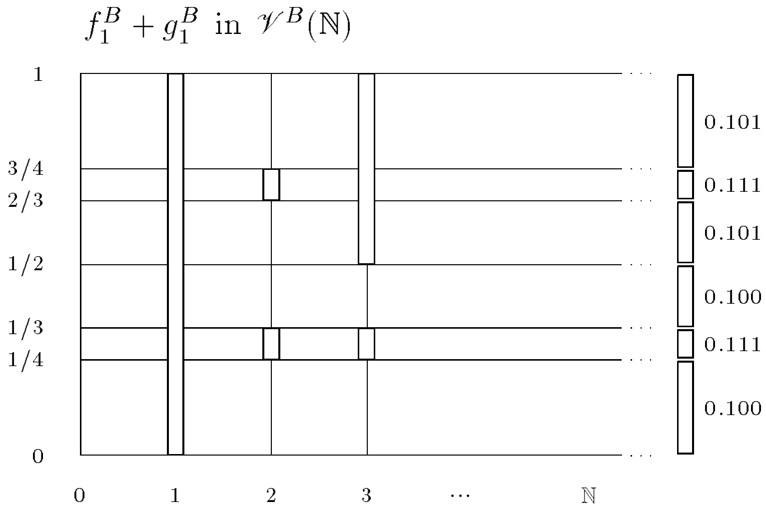

FIGURE 1.3

$\left[\frac{3}{4}, 1\right)$, for all other $x \in [0,1]$ is $f_1^B(x) = \emptyset$. The point is that $i_\mu(f_1^B)$ defined from $\mathscr{V}_1^B([0,1])$ into $\mathscr{F}_1([0,1])$ gives $f_1^F = i_\mu(f_1^B)\colon [0,1] \to [0,1]$, namely $f_1^F\left(\frac{1}{8}\right) = f_1^F\left(\frac{1}{2}\right) = f_1^F\left(\frac{5}{8}\right) = f_1^F\left(\frac{3}{8}\right) = \frac{1}{4}$ and for all other $x \in [0,1]$ is $f_1^F(x) = 0$. (Similarly for g_1^B as seen in $\mathscr{V}_1^B([0,1])$ and its image g_1^F). The sum $f_1^B + g_1^B$ seen in $\mathscr{V}_1^B([0,1])$ is the function $h_1^B\left(\frac{1}{2}\right) = \left[0, \frac{1}{4}\right) \cup \left[\frac{1}{3}, \frac{1}{2}\right)$, $h_1^B\left(\frac{5}{8}\right) = \left[\frac{1}{2}, \frac{2}{3}\right) \cup \left[\frac{3}{4}, 1\right)$ and $h_1^B\left(\frac{7}{8}\right) = \left[\frac{1}{4}, \frac{1}{3}\right) \cup \left[\frac{2}{3}, \frac{3}{4}\right)$, everywhere else $= \emptyset$.

However, this representation is satisfactory only for rationals. Though we deal here only with rationals, we have to notice that for a Boolean real f^B with infinite expansion it can happen that there is no partition refining all $\{f^B(i), -f^B(i)\}$ (because no measure algebra is distributive) and that for all crisp reals $x \in [0,1]$ is $\|f^B = x\| = 0_B$, (see [V]). This problem can be solved by switching to distribution functions, both Boolean and those known from the probability theory (see [L]).

Namely observe that:

$\left\| f_1^B \le x \right\| = \emptyset$ for $x \in \left[0, \frac{1}{8}\right)$ (in both $\mathscr{V}_1^B(\mathbb{N})$ and $\mathscr{V}_1^B([0,1])$),

$\left\| f_1^B \le x \right\| = \left[0, \frac{1}{4}\right)$ for $x \in \left[\frac{1}{8}, \frac{3}{8}\right)$,

$\left\| f_1^B \le x \right\| = \left[0, \frac{1}{4}\right) \cup \left[\frac{3}{4}, 1\right)$ for $x \in \left[\frac{3}{8}, \frac{1}{2}\right)$,

$\left\| f_1^B \le x \right\| = \left[0, \frac{1}{2}\right) \cup \left[\frac{3}{4}, 1\right)$ for $x \in \left[\frac{1}{2}, \frac{5}{8}\right)$,

$\left\| f_1^B \le x \right\| = [0,1] = 1_B$ for $x \in \left[\frac{5}{8}, 1\right)$.

This is emphasized on pictures Fig. 1.4 – 1.6 by the area in the unit square, where domain are urelements, range is the underlying set of our algebra and the vertical section by a real x of the area is the Boolean value $\left\| f_1^B \le x \right\|$.

The measure translation of this is the classical distribution function (right continuous, nondecreasing, $f(1) = 1$) on the unit interval. It represents the

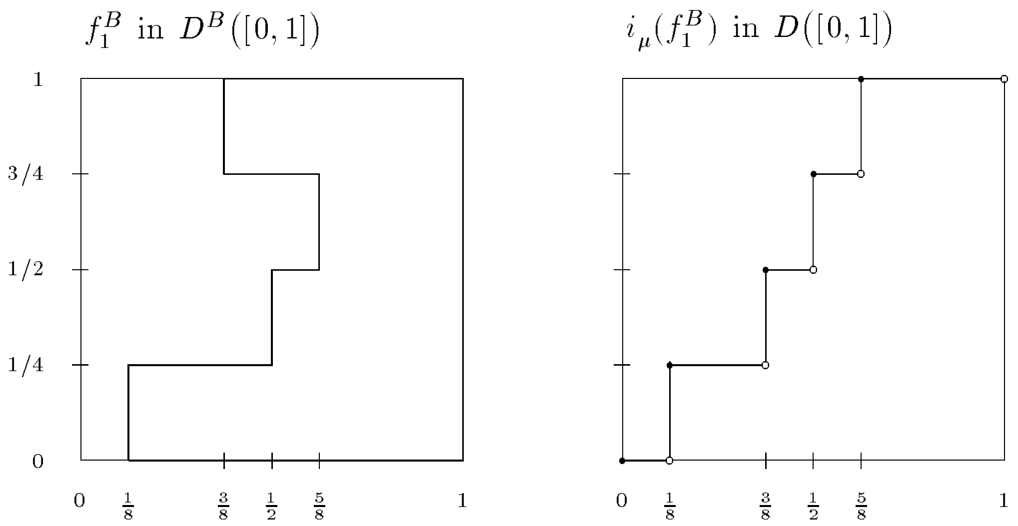

FIG. 1.4

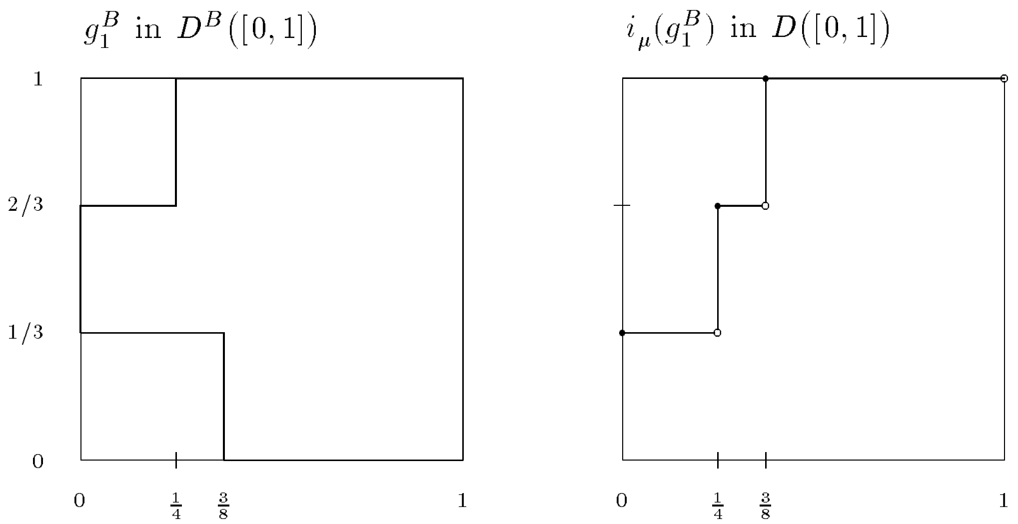

FIG. 1.5

event, where all four values $\{\frac{1}{8}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}\}$ have probability $\frac{1}{4}$. Moreover, note that the sum $f_1^B + g_1^B$ is represented by a Boolean distribution which is the convolution (in [L] called composition), of the two former, but in fuzzy case the sum is not the convolution, but the image of the Boolean convolution and as for $a, b \in B$ it is

$$\max\bigl(0, \mu(a) + \mu(B) - 1\bigr) \le \mu(a \wedge b) \le \min\bigl(\mu(a), \mu(b)\bigr)$$

and dually for $a \vee b$, the result is somewhere between the values for extremal t-norms T_0 and T_∞, see [M] (applied in the discrete formula for convolution — more about this see in later theorems). For other possible Boolean sources for f^F, g^F it is worth to note that though fuzzy target in $\mathscr{F}(\mathbb{N})$ is the same, our

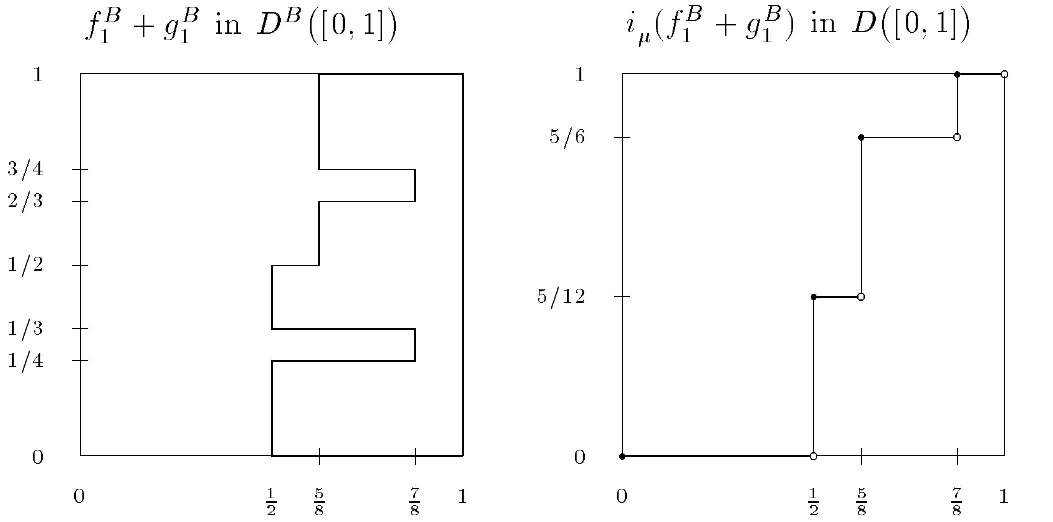

FIG. 1.6

translation to $\mathscr{F}([0,1])$ through Boolean reals gives possibly different distributions (this is caused by the lack of information the fuzzy reals are suffering from).

DEFINITION 5. We define

(i) the set of Boolean distributions:
$D^B([0,1]) = \{F\colon [0,1] \to B\colon F(1) = 1_B,\ x \leq y \Rightarrow F(x) \leq F(y),$ $\forall x\ \forall \varepsilon > 0\ \exists \delta\ \forall y > x\ (|x-y| < \delta \Rightarrow \mu(F(y) - F(x)) < \varepsilon\}$ (note that the last condition expresses that F is right continuous).
$D_2^B([0,1])$ are Boolean distributions of discrete events taking only dyadic rational values.

(ii) the Boolean convolution (also called composition, i.e., the additive one): $(F *_B G)(x) = \bigvee_{u+v=x} F(u) \wedge G(v) = \bigvee_y F(x-y) \wedge G(y)$ which we can denote, thinking about some sort Boolean-Rieman-like integral in our discrete case as $\int^B F(x-y)\, d^B G(y)$.

(iii) the multiplicative counterpart
$(F \,\square_B\, G)(x) = \bigvee_{u\cdot v=x} F(u) \wedge G(v) = \bigvee_y F\left(\frac{x}{y}\right) \wedge G(y) = \int^B F\left(\frac{x}{y}\right) d^B G(y)$
(to be sure that $y \neq 0$, define it over $(0,1)$).

(iv) the ordering; to define it we look for $F \in D^B([0,1])$, in our case B is the interval algebra over $[0,1]$, as of $F \subseteq [0,1] \times [0,1]$ (identifying the function with its graph). Horizontal sections by some y will be denoted by F^y. We define $F <_{BD} G$ iff $\mu\{y\colon F^y \subseteq G^y\} = 0$ (i.e., roughly speaking if $F \supseteq G$ in $[0,1] \times [0,1]$).

THEOREM 6. *Assume that B is the above interval Boolean algebra over the unit interval. Then there is an isomorphism of* $\Big(\bigcup_{n\in\mathbb{N}} \mathscr{V}_1^B(n), +_B, \cdot_B, <_B\Big)$ *onto* $\big(D_2^B([0,1]), *_B, \square_B, <_{BD}\big)$.

Proof. A typical element J of B looks like $J = [a_0, b_0) \cup \cdots \cup [a_n, b_n)$ where $a_0 < b_0 < a_1 < \cdots < b_n$. For a system $J_0, \dots, J_n \in B$ there is a unique shortest sequence $p_0, \dots, p_k$ (also called a refinement of $J_0, \dots, J_n$) such that

(1) $p_0 = 0 < p_1 < \dots p_{n-1} < p_n = 1$,

(2) $(\forall i < k)(\forall n)\big([p_i, p_{i+1}) \subseteq J_i$ or $[p_i, p_{i+1}) \subseteq [0,1] \setminus J_i\big)$.

Take $f \in \mathscr{V}_1^B(n)$, i.e., $f\colon n \to B$, where $f(i) = J_i$ (to emphasize that f has only finite binary Boolean expansion). Take $P_f = \{p_0, \dots, p_{k_1}\}$, the shortest refinement of $J_0 = f(0), \dots, J_n = f(n)$. For $i < k_1$ and $j < n$ define

$$\varepsilon_j^i = \begin{cases} 1 & \text{if } [p_i, p_{i+1}) \subseteq J_j\,, \\ 0 & \text{if } [p_i, p_{i+1}) \subseteq [0,1] \setminus J_j\,. \end{cases}$$

Recall the definitions of $\|f = g\|$, $\|f \le x\|$ and $\|f = x\|$ from [J], see also [V]. □

LEMMA 7. $\|f = 0.\varepsilon_0^i \cdots \varepsilon_n^i\| = [p_i, p_{i+1})$.

Proof. $x = 0.\varepsilon_0^i \cdots \varepsilon_n^i$ is represented in $\mathscr{V}_1^B(n)$ (as a crisp real) by $h_i\colon n \to \{0_B, 1_B\}$ defined $h_i(j) = 0_B$ if $\varepsilon_j^i = 0$ and $h_i(j) = 1_B$ if $\varepsilon_j^i = 1$. For two reals $f, h_i \in \mathscr{V}_1^B(\mathbb{N})$ is the Boolean value

$$\begin{aligned} \|f = h_i\| &= \bigwedge_{e\in\mathrm{dom}(f)} \big(f(e) \Rightarrow \|e \in h_i\|\big) \wedge \bigwedge_{e\in\mathrm{dom}(h_i)} \big(h_i(e) \Rightarrow \|e \in f\|\big) \\ &= \bigwedge_{j\in n} \big(-f(j) \vee h_i(j)\big) \wedge \bigwedge_{j\in n} \big(-h_i(j) \vee f(j)\big) \\ &= \bigwedge_{h_i(j)=0_B} -f(j) \wedge \bigwedge_{h_i(j)=1_B} f(j) \\ &= \bigwedge_{\varepsilon_j^i=0} -f(j) \wedge \bigwedge_{\varepsilon_j^i=1} f(j) \\ &= [p_i, p_{i+1})\,. \end{aligned}$$

In the last equality, $\supseteq$ is easy by definition of ε's and $\subseteq$ is true by the minimality of the sequence of p_i's.

We construct the isomorphism: having f^B, $p_0, \dots, p_{k_1}$, $x_i = 0.\varepsilon_0^i \cdots \varepsilon_n^i$ we assign to f^B an $F_{f^B}^B \in D_2^B([0,1])$ defined as follows

$$F_{f^B}^B(x) = \|f^B \le x\| = \bigcup \big\{[p_i, p_{i+1})\colon x_i \le x\big\}\,.$$

We denote sometimes $F^B_{f^B}$ simply by F_f. Note that the mapping is one-to-one. Namely, if $\|f \neq g\| \neq 0_B$, then there is (in our case of interval algebra) a minimal i such that (wlog) some $\emptyset \neq [p,q) \subset f(i) \setminus g(i)$. But then there is a k with $F_f\big(\frac{k}{2^n}\big) \supseteq [p,q)$ and $F_g\big(\frac{k}{2^n}\big) \cap [p,q) = \emptyset$.

For f, P_f and x_i's as above assume we have $g \in \mathscr{V}_1^B(n)$, $P_g = \{q_0, \dots, q_{k_2}\}$ and $y_j = 0.\delta^j_0 \cdots \delta^j_n$. We have to show that $F_{f+g} = F_f *_B F_g$ and $F_{f.g} = F_f \,\square_B\, F_g$, with operations (mod 1). To this end just note that in our discrete case we have

$$\|f+g \le x\| = \bigvee_{x_i+y_j \le x} \|f \le x_i\| \wedge \|g \le y_j\| = \int^B F_f(x-y)\, d^B F_g(y)$$

and

$$\|f \cdot g \le x\| = \bigvee_{x_i \cdot y_j \le x} \|f \le x_i\| \wedge \|g \le y_j\| = \int^B F_f\left(\frac{x}{y}\right) d^B F_g(y)\,.$$

To decide whether ordering is preserved we have to go back to $\mathscr{V}^B(\mathbb{N})$ and look for what it means. In our discrete case (considering Boolean rationals)

$$\|f < g\| = \bigvee_{x_i < y_j} \|f = x_i\| \wedge \|g = y_j\| = 1\,.$$

Let $P = \{r_0, \dots, r_k\}$ be a minimal refining of P_f and P_g. Then for all $l < k$ $\exists! x_i\, \exists! y_j\ x_i < y_j$ and $\|f = x_i\| \wedge \|g = y_j\| \supseteq [r_l, r_{l+1})$, that is (looking to F_f, F_g in $[0,1] \times [0,1]$) $F_f \cap \big([0,1] \times [r_l, r_{l+1})\big) \supseteq F_g \cap \big([0,1] \times [r_l, r_{l+1})\big)$, i.e., $F_f <_{BD} F_g$. ☐

The general case for all Boolean reals will be studied in a forthcoming paper.

Example 8. In our former example note that $\|f_1^B < g_1^B\| = \big[0, \frac{1}{4}\big)$ and $\|f_1^B > g_1^B\| = \big[\frac{1}{4}, 1\big)$. Moreover, note that for the classical distributions G_1 induced by $f_1^F = i_\mu(f_1^B)$ and G_2 induced by $g_1^F = i_\mu(f_1^B)$ we have that the classical convolution $(G_1 * G_2)\big(\frac{1}{2}\big) = \sum\limits_y G_1\big(\frac{1}{2} - y\big) \cdot G_2(y) = G_1\big(\frac{1}{8}\big) \cdot G_2\big(\frac{3}{8}\big) + G_1\big(\frac{1}{2}\big) \cdot G_2(0) = \frac{1}{4} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{2}{12} + \frac{2}{12} = \frac{1}{3}$ but the i_μ-image of $F_{f_1^B} *_B F_{g_1^B} = F$ gives $F\big(\frac{1}{2}\big) = \frac{5}{12}$. So i_μ-images of $D^B\big([0,1]\big)$ besides they glue together lot of Boolean distributions, they preserve neither the additive nor the multiplicative convolution (composition) of distribution functions.

Analogously we can define $D_2\big([0,1]\big)$ being classical-crisp-distribution function on $[0,1]$ of events taking dyadic rational values (though representing dyadic fuzzy reals). The mapping $i_\mu\colon D_2^B\big([0,1]\big) \to D_2\big([0,1]\big)$, as we saw, is not one-to-one and does not preserve $*$ and $\Box$. Nevertheless, for a family of Boolean rationals where the Boolean value is only interval of the form $[0,x]$, the image of $*_B$ is again a convolution but evaluated in the min-max algebra, i.e., for $F, G \in D_2\big([0,1]\big)$ define

$$(F *_m G)(x) = \max_{x_i+y_j=x} \Big(\min\big(F(x_i), G(y_j)\big)\Big) = \max_y \Big(\min\big(F(x-y), G(y)\big)\Big)\,,$$

that is, in a sense, the Rieman integral $\overset{m}{\int} F(x-y)\,d^m G(y)$ over the min-max algebra. Analogously for

$$\begin{aligned}(F \Box_m G)(x) &= \max_{x_i \cdot y_j = x} \big(\min\big(F(x_i), G(y_j)\big)\big) \\ &= \max_y \Big(\min\big(F\big(\tfrac{x}{y}\big), G(y)\big)\Big) = \overset{m}{\int} F\big(\tfrac{x}{y}\big)\,d^m G(y)\,.\end{aligned}$$

Note that with a quite good intuition we can define the ordering on distributions:

Definition 9. For F, $G \in D\big([0,1]\big)$ define $F <_D G$ iff $F(x) > G(x)$ for a.e. x.

Theorem 10. $i_\mu\colon D_2^B\big([0,1]\big) \to D_2\big([0,1]\big)$ *is an order preserving mapping.*

P r o o f. In our discrete case $F_1^B <_{BD} F_2^B$ means $F_1^B \supseteq F_2^B$ and then for all x $i_\mu(F_1^B)(x) > i_\mu(F_2^B)(x)$. $\Box$

This gives us the possibility (and a good reason, though we proved our theorem only for rationals) to consider $D\big([0,1]\big)$ — the set of all classical distribution functions as a model of fuzzy reals and $<_D$ the canonical (natural) ordering on fuzzy reals. Note, that $\big(D\big([0,1]\big), <_D\big)$ is not a linearly ordered space. This gives us the possibility to define the interval topology on Boolean based fuzzy reals.

Definition 11. τ_D is a (crisp) topology on $D\big([0,1]\big)$ generated by intervals $(F,G) = \big\{H \in D\big([0,1]\big) \colon F <_D H <_D G\big\}$.

It is behind the scope of this paper to study this topology here. Just note that this is a crisp topological space over fuzzy reals. Other question could be to study the fuzzy topology over fuzzy reals induced by the Boolean topology in $\mathscr{V}^B(\mathbb{N})$.

REFERENCES

[H] HÖHLE, U.: *Representation theorems for fuzzy quantities*, Fuzzy Sets and Systems **5** (1981), 83–109.

[J] JECH, T.: *Set Theory*, Academic Press, New York, 1978.

[L] LOÈVE, M.: *Probability Theory*, Van Nostrand, Toronto, 1955.

[M] MESIAR, R.: *Fuzzy sets and probability theory*, Tatra Mt. Math. Publ. **1** (1992), 105–123.

[TT] TAKEUTI, G.—TITANI, S.: *Fuzzy logic and fuzzy set theory*, Arch. Math. Logic **32** (1992), 1–32.

[V] VOJTÁŠ, P.: *Boolean universe versus fuzzy sets*, Tatra Mt. Math. Publ. **6** (1995), 179–186.

Appendix 12. Extending our example we display here second possible Boolean source f_2^B, g_2^B with all pictures as in previous case.

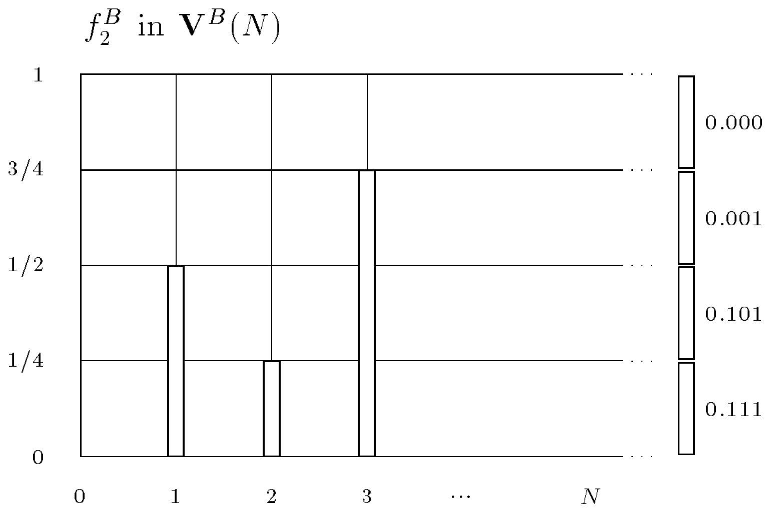

FIGURE 2.1

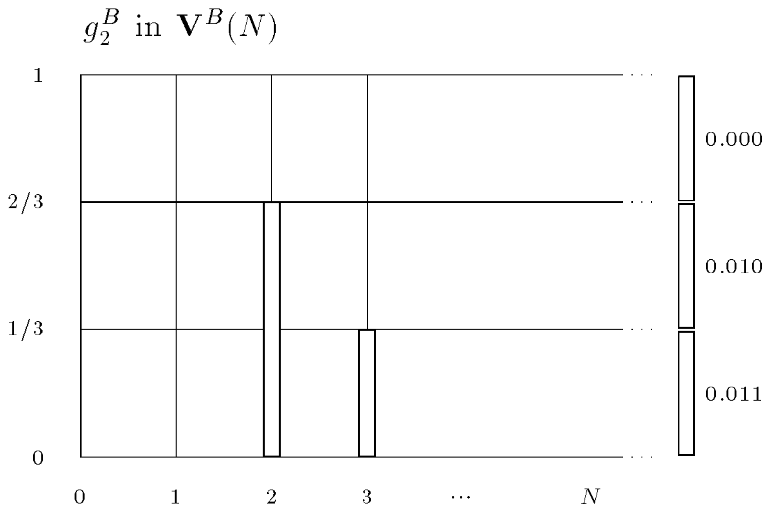

FIGURE 2.2

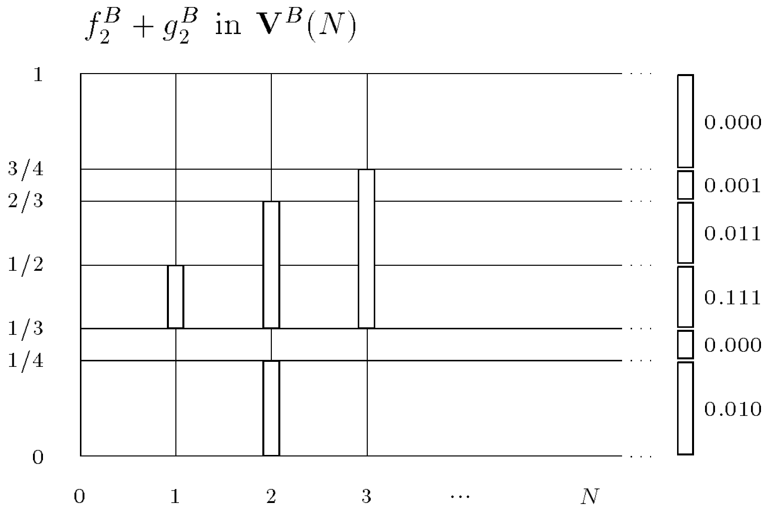

FIGURE 2.3

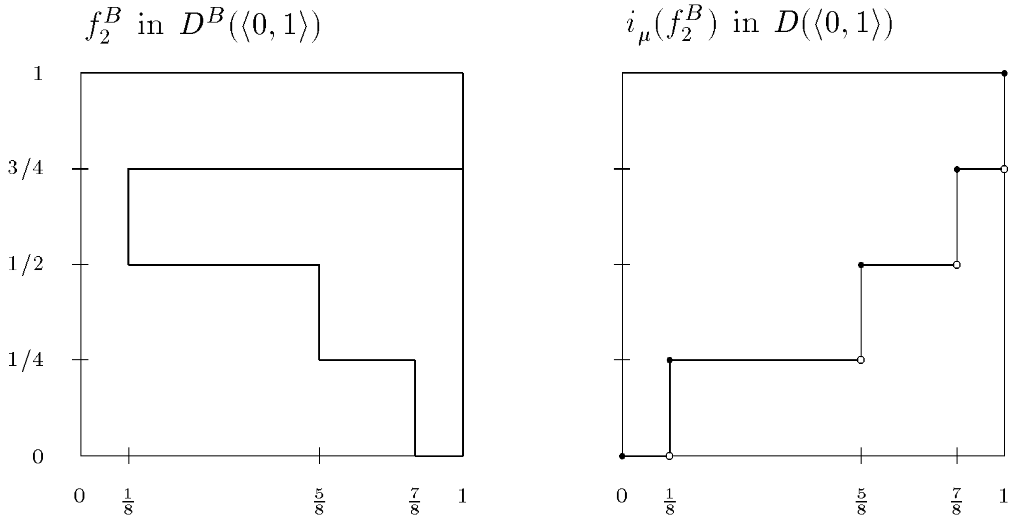

FIGURE 2.4

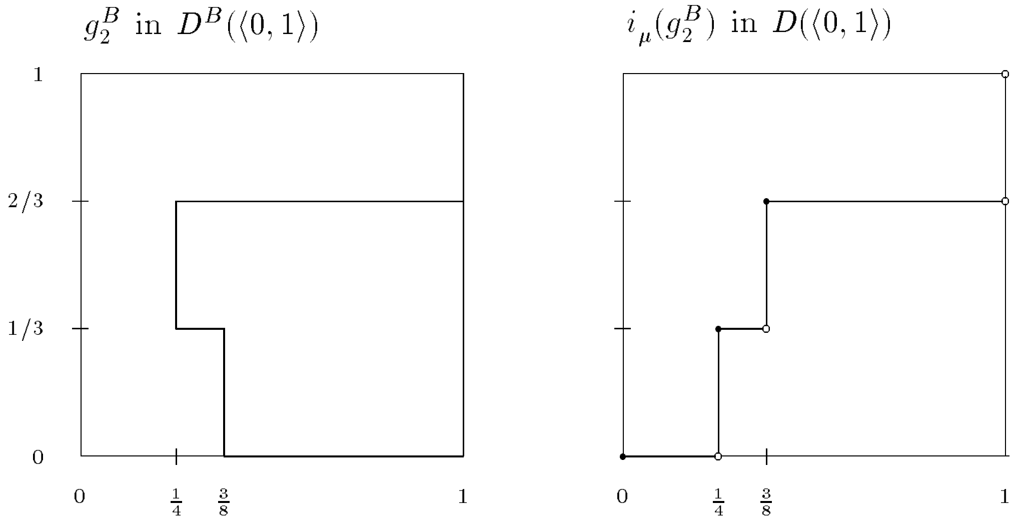

FIGURE 2.5

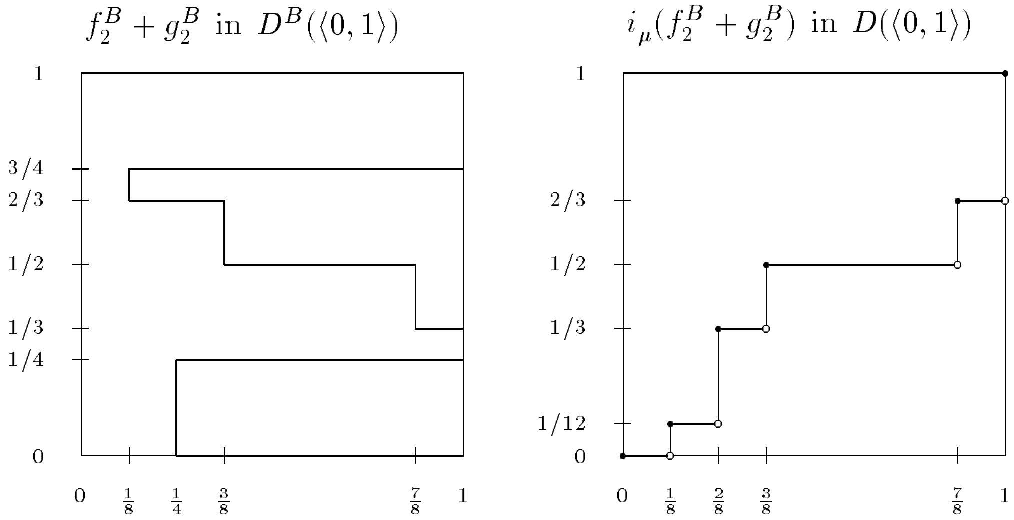

FIGURE 2.6

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