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NEARLY FRANK T-NORMS

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ABSTRACT. Recently, M. Navara has introduced a new family of t-norms playing an important role in the theory of measures of fuzzy sets — a family of nearly Frank t-norms. We characterize this family by means of functional equations to be fulfilled by the corresponding additive (multiplicative) generators. Some positive and some negative examples are given.

1. Introduction

Klement [4] in 1982 introduced a $\boldsymbol{T}$-measure $\mathbf{m}$ as a measure of fuzzy sets (from some $\boldsymbol{T}$-tribe, i.e., a system of fuzzy subsets generalizing the classical notion of a σ-algebra). Here $\boldsymbol{T}$ is some given triangular norm which models the fuzzy intersection. The fuzzy union is modelled by the corresponding dual t-conorm $\boldsymbol{S}$. For more details about t-norms and t-conorms we refer to [2, 5, 12]. Several interesting results for $\boldsymbol{T}$-tribes and $\boldsymbol{T}$-measures can be found in [1, 2, 6, 7]. However, the characterization and representation theorems in these papers concern the t-norms from the Frank family of t-norms [3], see also [2] with parameter $s \in \,]0,\infty[$, i.e., the strict Frank t-norms. The main reason lies in the fact that Frank t-norms (for $s > 0$) are the only Archimedean solutions of the functional equation

$$\boldsymbol{T}(x,y) + \boldsymbol{S}(x,y) = x + y\,, \quad x,y \in [0,1]\,. \tag{1}$$

Functional equation (1) when rewritten for the fuzzy unions and fuzzy intersections modeled by $\boldsymbol{T}$ and $\boldsymbol{S}$, respectively, leads to the extension of the classical equality $\mathbf{1}_{A\cap B} + \mathbf{1}_{A\cup B} = \mathbf{1}_A + \mathbf{1}_B$ for the fuzzy subsets, i.e., $\mu_{A\cap B} + \mu_{A\cup B} = \mu_A + \mu_B$.

In [7], we have shown that each $\boldsymbol{T}$-measure based on a strict Frank t-norm is a sum of a support measure and a fuzzy probability measure introduced by Zadeh [16]. The characterization of $\boldsymbol{T}$-measures for a general strict t-norm was

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solved only recently by N a v a r a [8, 9, 10], where a new family of t-norms related to Frank t-norms was introduced. This family is called the family of nearly Frank t-norms and it will be discussed in Section 2. By [8], if a strict t-norm $\boldsymbol{T}$ is not nearly Frank, then the only $\boldsymbol{T}$-measures are just the support measures (hence eliminating any fuzziness hidden in fuzzy sets taken into account). Further, if a strict t-norm $\boldsymbol{T}$ is nearly Frank t-norm, then each $\boldsymbol{T}$-measure is a sum of a support measure and a distorted fuzzy probability measure,

$$\boldsymbol{m}(A) = \boldsymbol{M}_1(\operatorname{supp}\mu_A) + \int h \circ \mu_A \, d\boldsymbol{M}_2 \,,$$

where $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are some classical σ-additive measures.

Nearly Frank t-norms form an important family in the theory of measures of fuzzy sets. We can expect their applications in other domains where Frank t-norms are applied (e.g., the game theory, see [2]). The aim of this paper is to solve the problem given by N a v a r a in [8], i.e., to investigate this family and to find some hints how to recognize whether a given strict t-norm $\boldsymbol{T}$ is nearly Frank or not.

2. Nearly Frank t-norms

Frank t-norms are such t-norms which (together with some not necessarily dual t-conorm $\boldsymbol{S}$) are solutions of the functional equation (1). All Frank t-norms are continuous, they include the strongest t-norm $\boldsymbol{T_M}$ (minimum), the Łukasiewicz t-norm $\boldsymbol{T}_L$, the product t-norm $\boldsymbol{T}_P$. Further, Frank t-norms are closed under the ordinal sum compositions. For more details, we recommend [2, 3].

Recall first the strict Frank t-norms described in [3].

Definition 1. *Strict Frank t-norms* are defined for parameters $s \in \,]0,\infty[$. For $s = 1$, $\boldsymbol{T}_1^F = \boldsymbol{T}_P$ is just the product t-norm. For remaining parameters $s \in \,]0,1[\,\cup\,]1,\infty[$, $\boldsymbol{T}_s^F$ is defined by

$$\boldsymbol{T}_s^F(x,y) = \log_s\bigl(1 + (s^x - 1)(s^y - 1)/(s-1)\bigr)\,.$$

The corresponding additive generators are given by $f_1(x) = -\log x$ and for $s \in \,]0,1[\,\cup\,]1,\infty[$, $f_s(x) = -\log(s^x - 1)/(s-1)$.

The corresponding multiplicative generators are given by $\phi_1(x) = x$ and for $s \in \,]0,1[\,\cup\,]1,\infty[$, $\phi_s = (s^x - 1)/(s-1)$.

Recall that the pointwise limit $\boldsymbol{T}_\infty^F$ is also a t-norm. However, it is nilpotent, and $\boldsymbol{T}_\infty^F(x,y) = \max(0, x+y-1)$, i.e., $\boldsymbol{T}_\infty^F$ is just the Łukasiewicz t-norm. On the other side, $\boldsymbol{T}_0^F = \boldsymbol{T}_M$. Any other Frank t-norm different from $\boldsymbol{T_M}$ is an ordinal sum with summands chosen from the system $\bigl(\boldsymbol{T}_s^F\bigr)$, $s \in \,]0,\infty]$.

The complement in fuzzy set theory is modelled by means of some strong negation $\boldsymbol{n}$, i.e., a non-increasing involutive mapping $\boldsymbol{n}\colon [0,1] \to [0,1]$. Each strong negation $\boldsymbol{n}$ can be derived from a generator $g\colon [0,1] \to [0,1]$, g is an increasing automorphism, in the following way [13]:

$$\boldsymbol{n}(x) = g^{-1}\bigl(1 - g(x)\bigr) . \tag{2}$$

Vice-versa, any generator g defines via (2) a strong negation $\boldsymbol{n}$. However, the system of all generators leading to the same strong negation $\boldsymbol{n}$ is rather rich (it is, in fact, uncountable). A principal role in the investigations of Navara [8, 9, 10] of $\boldsymbol{T}$-measures was played by those generators g which generate the standard Zadeh [15] negation $\boldsymbol{n}(x) = 1 - x$. Throughout this paper, such generators will be called *standard generators*. We can easily show the following characterization of standard generators.

PROPOSITION 1. *A mapping $g\colon [0,1] \to [0,1]$ is a standard generator if and only if it is an increasing automorphism such that for all $x \in [0,1]$,*

$$g(x) + g(1-x) = 1 .$$

It is immediate that each standard generator should fulfill $g(0.5) = 0.5$ and if g is twice differentiable in point 0.5 then $g''(0.5) = 0$ (these conditions are necessary!).

Now, we are ready to introduce nearly Frank t-norms following Navara [8, 9, 10].

DEFINITION 2. A t-norm $\boldsymbol{T}$ is called a *nearly Frank t-norm* if and only if there is a Frank t-norm $\boldsymbol{T}^*$ and a standard generator g such that $\boldsymbol{T} = \boldsymbol{T}^*_g$, i.e.,

$$\boldsymbol{T}(x,y) = g^{-1}\Bigl(\boldsymbol{T}^*\bigl(g(x),\, g(y)\bigr)\Bigr) .$$

Continuous Archimedean t-norms are characterized by the corresponding additive (multiplicative) generators [5, 11, 12]. Note that a multiplicative generator of a continuous Archimedean t-norm $\boldsymbol{T}$ is unique up to a positive power constant, while the additive generators are unique up to a positive multiplicative constant. For generated t-norms, we have the following lemma which can be proved straightforwardly.

LEMMA 1. *Let $\boldsymbol{T}$ be a t-norm generated by the additive generator f (by the multiplicative generator ϕ) and let g be an increasing automorphism of the unit interval. Then the t-norm $\boldsymbol{T}_g$ defined by*

$$\boldsymbol{T}_g(x,y) = g^{-1}\Bigl(\mathbf{T}\bigl(g(x),\, g(y)\bigr)\Bigr)$$

is generated by the additive generator $f \circ g$ (by the multiplicative generator $\phi \circ g$).

Now, it is easy to characterize any nilpotent nearly Frank t-norm $\boldsymbol{T}$.

THEOREM 1. *Let $\boldsymbol{T}$ be a nilpotent t-norm with an additive generator f (multiplicative generator ϕ). $\boldsymbol{T}$ is a nearly Frank t-norm if and only if $g = 1 - f/f(0)$ ($g = 1 - \log_{\phi(0)} \phi$) is a standard generator.*

P r o o f. The only nilpotent Frank t-norms is the Łukasiewicz t-norm $\boldsymbol{T}_L$ with the normed additive generator $f_L(x) = 1 - x$. Let f be an additive generator of $\boldsymbol{T}$. Then $f/f(0)$ is the only normed additive generator of $\boldsymbol{T}$ and by Lemma 1, $\boldsymbol{T}$ is a nearly Frank t-norm if and only if $f/f(0) = f_L \circ g$ for some standard generator g, i.e., $f/f(0) = 1 - g$. Equivalently, $g = 1 - f/f(0)$. The result for multiplicative generators follows from the fact that if ϕ is a multiplicative generator of $\boldsymbol{T}$ then $f = -\log \phi$ is an additive generator of $\boldsymbol{T}$. □

Situation with strict t-norms with respect to be nearly Frank is more complicated. First, there is no such distinguished additive generator of a strict t-norm as the normed generator in the case of nilpotent t-norms. Secondly, there is a class of strict Frank t-norms parametrized by parameter $s \in\,]0, \infty[$. All these t-norms are different from the nearness point of view [9], i.e., if for two Frank t-norms $\boldsymbol{T}$ and $\boldsymbol{T}^*$ there is some standard generator g so that if $\boldsymbol{T} = \boldsymbol{T}^*_g$ then $g = \mathrm{id}$ and $\boldsymbol{T} = \boldsymbol{T}^*$. Consequently, if an Archimedean t-norm $\boldsymbol{T}$ is a nearly Frank t-norm then there are a unique Frank t-norm $\boldsymbol{T}^*$ and a unique standard generator g such that $\boldsymbol{T} = \boldsymbol{T}^*_g$.

THEOREM 2. *Let $\boldsymbol{T}$ be a strict t-norm with the multiplicative generator ϕ such that $\phi(0.5) = 0.5$. Then $\boldsymbol{T}$ is a nearly Frank t-norm corresponding to the Frank t-norm $\boldsymbol{T}^F_s$ for some $s \in\,]0, \infty[$ if and only if the strong negation $\boldsymbol{n}_\phi$ generated by the generator ϕ^{-1}, $\boldsymbol{n}_\phi(x) = \phi\bigl(1 - \phi^{-1}(x)\bigr)$, is just the strong negation $\boldsymbol{n}_s$ given by $\boldsymbol{n}_1(x) = 1 - x$ if $s = 1$ (i.e., ϕ is a standard generator) and by*

$$\boldsymbol{n}_s(x) = \left(\frac{1 - x^k}{1 + (s-1)x^k}\right)^{1/k}, \qquad \textit{where} \quad k = \log_2\bigl(1 + s^{0.5}\bigr),$$

in all remaining cases.

P r o o f. Recall that a multiplicative generator of $\boldsymbol{T}^F_1$ $(= \boldsymbol{T}_P)$ is $\phi_1(x) = x$ and that $\phi_1(0.5) = 0.5$. Then $\boldsymbol{T}$ is a nearly Frank t-norm which is a transformation of $\boldsymbol{T}^F_1$ if and only if $\phi = \phi_1 \circ g$ for some standard generator g (see Lemma 1).

Now, suppose that $s \in\,]0,1[\, \cup\,]1, \infty[$ and that $\boldsymbol{T}$ is a nearly Frank t-norm which is a transformation of the strict Frank t-norm $\boldsymbol{T}^F_s$. Due to Lemma 1, this is equivalent to the fact that $\phi = \phi^p_s \circ g$ for some positive power $p \in\,]0, \infty[$. Recall that $\phi = (s^x - 1)/(s - 1)$. In point 0.5, we have $\phi(0.5) = 0.5 = \phi^p_s\bigl(g(0.5)\bigr) = \phi^p_s(0.5) = \bigl((s^{0.5} - 1)/(s - 1)\bigr)^p$. Consequently,

$$p = 1/\log_2\bigl(1 + s^{0.5}\bigr).$$

So, $\boldsymbol{T}$ is a nearly Frank t-norm transformed from a strict Frank t-norm $\boldsymbol{T}_s^F$ different from the product if and only if the composite function

$$g = \left(\phi_s^p\right)^{-1} \circ \phi$$

is a standard generator, i.e., for all $x \in [0,1]$ it is

$$\left(\phi_s^p\right)^{-1}\left(\phi(x)\right) + \left(\phi_s^p\right)^{-1}\left(\phi(1-x)\right) = 1 . \tag{3}$$

Denoting the standard Zadeh strong negation by $\boldsymbol{n}$, $\boldsymbol{n}(x) = 1-x$, the functional equation (3) can be rewritten in the form

$$\boldsymbol{n} \circ \left(\phi_s^p\right)^{-1} \circ \phi = \left(\phi_s^p\right)^{-1} \circ \phi \circ \boldsymbol{n} . \tag{4}$$

Consequently,

$$\phi = \phi_s^p \circ \boldsymbol{n} \circ \left(\phi_s^p\right)^{-1} \circ \phi \circ \boldsymbol{n}$$

and

$$\phi \circ \boldsymbol{n} \circ \phi^{-1} = \phi_s^p \circ \boldsymbol{n} \circ \left(\phi_s^p\right)^{-1} . \tag{5}$$

It is evident that both sides in exp. (5) define a strong negation. The strong negation $\phi_s^p \circ \boldsymbol{n} \circ \left(\phi_s^p\right)^{-1}$ is generated by the generator $\left(\phi_s^p\right)^{-1}$ while the strong negation $\phi \circ \boldsymbol{n} \circ \phi^{-1}$ is generated by the generator ϕ^{-1}. The equality (5) means that both ϕ^{-1} and $\left(\phi_s^p\right)^{-1}$ generate the same strong negation. Denote by $\boldsymbol{n}_s$ the strong negation defined by $\boldsymbol{n}_s = \phi_s^p \circ \boldsymbol{n} \circ \left(\phi_s^p\right)^{-1}$. Then for all $x \in [0,1]$ we have

$$\boldsymbol{n}_s(x) = \left(\frac{1-x^k}{1+(s-1)x^k}\right)^{1/k}, \qquad \text{where} \quad k = \log_2\left(1+s^{0.5}\right) .$$

□

As a by-product, we have the following equivalent characterization of nearly Frank strict t-norms.

Corollary 1. *Let $\boldsymbol{T}$ be a strict t-norm with multiplicative generator ϕ such that $\phi(0.5) = 0.5$. Then $\boldsymbol{T}$ is a nearly Frank t-norm corresponding to the Frank t-norm $\boldsymbol{T}_s^F$ for some $s \in \,]0,\infty[$ if and only if either ϕ is a standard generator (and then $\boldsymbol{T}$ is nearly product) or $\left(\phi_s^p\right)^{-1} \circ \phi$ (with $s \in \,]0,1[\,\cup\,]1,\infty[$ and $p = 1/\log_2(1+s^{0.5})$) is a standard generator (and then $\boldsymbol{T}$ is nearly $\boldsymbol{T}_s^F$).*

The properties of a standard generator give the next necessary condition for strict t-norms to be nearly Frank.

Corollary 2. *Let $\boldsymbol{T}$ be a strict t-norm with multiplicative generator ϕ such that $\phi(0.5) = 0.5$. Let ϕ be twice differentiable at point 0.5. Then $\boldsymbol{T}$ is a nearly Frank t-norm only if either $\phi''(0.5) = 0$ (in the case when $\boldsymbol{T}$ is nearly* *product) or the following equality holds (in the case when $\boldsymbol{T}$ is nearly $\boldsymbol{T}_s^F$ with $s \in]0,1[\cup]1,\infty[$):*

$$s^{0.5}\left(\phi'(0.5)^2 - \phi''(0.5)/2\right) = \phi'(0.5)^2 \log_2\left(1 + s^{0.5}\right). \tag{6}$$

Note that the equality (6) follows from the fact that $\left(\phi_s^p\right)^{-1} \circ \phi$ should be a standard generator and hence its second derivative in point 0.5 should be zero.

For a given strict t-norm $\boldsymbol{T}$ we check whether it is nearly Frank t-norm as follows: we find its multiplicative generator ϕ crossing the point $(0.5, 0.5)$. We derive the corresponding strong negation $\boldsymbol{n}_\phi$. If it is the standard negation, $\boldsymbol{T}$ is a nearly Frank t-norm transformed from the product t-norm. If $\boldsymbol{n}_\phi$ is not the standard negation, we have to compare it with negations $\boldsymbol{n}_s$ for $s \in]0,1[\cup]1,\infty[$. We already know that if $\boldsymbol{T}$ is nearly Frank t-norm then the corresponding s is unique. To check this constant s, it is enough to compute the value $\boldsymbol{n}_\phi(0.25)$ and to find the solution of the equation

$$\boldsymbol{n}_\phi(0.25) = \left(\frac{2 + s^{0.5}}{2 + 2s^{0.5}}\right)^{1/\log_2(1+s^{0.5})}.$$

Note that the above equation has solution only if $\boldsymbol{n}_\phi(0.25) \in \left]2^{-0.5}, 1\right[$ and then this solution is unique. If for that unique s there holds $\boldsymbol{n}_\phi = \boldsymbol{n}_s$ then $\boldsymbol{T}$ is a nearly Frank t-norm (transformed from $\boldsymbol{T}_s^F$), covering also the already discussed case when $s = 1$. In the opposite case (and also if there is no solution of the above equation), $\boldsymbol{T}$ is not nearly Frank. Corollary 2 gives another hint how to exclude the nearness to Frank's t-norms.

For a general continuous t-norm $\boldsymbol{T}$ which is not Archimedean (and neither the strongest t-norm $\boldsymbol{T}_M$), we have to investigate the nearness of ordinal sums. Note that if $\boldsymbol{T}$ is an ordinal sum $\boldsymbol{T} \sim \left(\langle a_k, b_k, \boldsymbol{T}_k\rangle : k \in \mathcal{K}\right)$ with continuous Archimedean summands $\boldsymbol{T}_k$ and if g is some standard generator, then $\boldsymbol{T}_g$ is also an ordinal sum, $\boldsymbol{T}_g \sim \left(\langle g^{-1}(a_k),\, g^{-1}(b_k),\, \boldsymbol{T}_k^*\rangle : k \in \mathcal{K}\right)$, where $\boldsymbol{T}_k^* = \left(\boldsymbol{T}_k\right)_{h_k}$, with

$$h_k(x) = \left(g\left(g^{-1}(a_k) + \left(g^{-1}(b_k) - g^{-1}(a_k)\right)g^{-1}(x)\right) - a_k\right) \Big/ \left(b_k - a_k\right).$$

Then $\boldsymbol{T}$ is nearly Frank if and only if all $\boldsymbol{T}_k^*$, $k \in \mathcal{K}$, are Archimedean Frank t-norms. This means that all summands $\boldsymbol{T}_k$, $k \in \mathcal{K}$, should be nearly Frank t-norms with special standard generators h_k^{-1} depending on k and on some common standard generator g^{-1}.

3. Examples

EXAMPLE 1.

i) Each normed additive generator symmetric with respect to the point $(0.5, 0.5)$ generates a nearly Frank nilpotent t-norm. Take, e.g., $f(x) = \cos^2(\prod x/2)$. The corresponding nearly Frank t-norm $\boldsymbol{T}$ is given by

$$\boldsymbol{T}(x,y) = (2/\prod)\arccos\bigl(\min\bigl(1, \cos^2(\prod x/2) + \cos^2(\prod y/2)\bigr)\bigr)^{0.5}.$$

ii) No Yager t-norm [14] is nearly Frank up to the Łukasiewicz t-norm $\boldsymbol{T}_L$ (which is the Yager t-norm with parameter $p = 1$).

EXAMPLE 2.

i) Let t-norm $\boldsymbol{T}$ be generated by a multiplicative generator $\phi(x) = \sin^2(\prod x/2)$, i.e., $\boldsymbol{T}$ is a strict t-norm given by

$$\boldsymbol{T}(x,y) = (2/\prod)\arcsin\bigl(\sin(\prod x/2)\sin(\prod y/2)\bigr).$$

As far as ϕ is a standard generator, by Theorem 1, $\boldsymbol{T}$ is nearly product. Note that $\phi''(x) = (\prod^2/2)\cos(\prod x)$ and hence $\phi''(0.5) = 0$.

ii) Let a strict t-norm $\boldsymbol{T}$ be generated by a multiplicative generator $\phi(x) = \bigl((9^x - 1)/8\bigr)^{0.5}$.

Then $\phi'(0.5) = (3/8)\ln 9$ and $\phi''(0.5) = (3/32)\ln^2 9 \neq 0$ (i.e., $\boldsymbol{T}$ cannot be nearly product). If $\boldsymbol{T}$ is nearly Frank then the equality (6) should be fulfilled for some parameter s,

$$\bigl(s^{0.5}/64\bigr)\bigl(9\ln^2 9 - 3\ln^2 9\bigr) = (9/64)\ln^2 9\,\log_2\bigl(1 + s^{0.5}\bigr).$$

The only solution of the above equation (for s) is $s = 9$. Now, it is easy to recognize that ϕ is a multiplicative generator of Frank t-norm with parameter $s = 9$, i.e., $\boldsymbol{T} = \boldsymbol{T}_9^F$.

iii) Let $\boldsymbol{T}$ be the Hamacher product, i.e., a strict t-norm with multiplicative generator $\phi(x) = 2^{(x-1)/x}$ (see [5]), which is given by $\boldsymbol{T}(x,y) = xy/(x + y - xy)$ whenever $(x,y) \neq (0,0)$.

Then $\phi'(0.5) = 2\ln 2$ and $\phi''(0.5) = 8(\ln 2 - 1)\ln 2 \neq 0$. Consequently, $\boldsymbol{T}$ cannot be a nearly product. Concerning equation (6), it is to be fulfilled for parameter s such that $s^{0.5} = \ln(1 + s^{0.5})$, i.e., there is no such s so that (3) is fulfilled for given ϕ. Then $\boldsymbol{T}$ cannot be nearly Frank (for $s \in \,]0,1[\, \cup \,]1,\infty[$). Hence, the Hamacher product is not a nearly Frank t-norm (this result was observed also by Navara [8] by entirely different methods.)

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