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REACHABILITY OF FUZZY MATRIX PERIODS

MARTIN GAVALEC — GÜNTER ROTE

ABSTRACT. The computational complexity of the matrix period reachability problem (MPR) in a fuzzy algebra $\mathscr{B}$ is studied. Given an $n \times n$ matrix A with elements in $\mathscr{B}$, the problem is to decide whether there is an n-vector x such that the sequence of matrix powers $A, A^2, A^3, \ldots$ has the same period length as the sequence $Ax, A^2x, A^3x, \ldots$ of iterates of x. In general, the MPR problem is NP-complete. Two conditions are described, both of which together imply that MPR can be solved in $O(n^3)$ time. If only one of the conditions is satisfied, the problem remains NP-complete.

1. Introduction

Power sequences of matrices, and their convergence and periodicity for special classes of matrices in fuzzy algebra were studied by M. G. Thomason [10], and subsequently by many other authors. Li Jian-Xin [7, 8] considered the periodicity of fuzzy matrices in the general case and gave an upper estimate for the period of a matrix. The convergence of the power sequence of a square matrix in a fuzzy algebra was studied using digraphs by K. Cechlárová [2], and a necessary and sufficient condition was given for a matrix A to be stationary. Similar methods of working with digraphs to describe of powers of nonnegative matrices are summarized by R. A. Brualdi and H. J. Ryser in [1].

The computational complexity of several problems connected with finding the exact value of the matrix and orbit period in a fuzzy algebra $\mathscr{B}$ was considered in [4]. In [9], a polynomial algorithm for computing the matrix period in a max-plus algebra, for matrices with zero-weight cycles, is described. It was shown in [4] that the period of a matrix A is the least common multiple of the periods of at most n non-trivial strongly connected components in the threshold

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digraphs $\mathscr{G}(A,h)$ for some threshold levels h, and an algorithm was suggested for computing the matrix period in $O(n^3)$ time.

On the other hand, the matrix period $\mathrm{per}(A)$ is the least common multiple of the orbit periods $\mathrm{per}(A,x)$ for all vectors $x \in \mathscr{B}(n)$. In [3], the question of reaching the matrix period by some orbit period is discussed. The matrix period reachability (MPR) problem is shown to be NP-complete in [5].

The aim of this paper is to discuss some conditions under which the MPR problem can be solved in polynomial time. In it, two conditions are described, both of which together imply that the MPR problem is polynomially solvable (Theorem 5.1). We show that if only one of the conditions holds, then MPR is NP-complete (Theorems 5.2 and 5.3).

2. Matrix and orbit periods

In this section we define the notions mentioned informally in the introduction. To simplify the notation of index sets, we shall use the convention by which any natural number n is identified with the set of all smaller natural numbers, i.e., $n = \{0, 1, \dots, n-1\}$. We denote by $\mathbb{N}$ the set of all natural numbers (nonnegative integers). The greatest common divisor and the least common multiple of a set $S \subseteq \mathbb{N}$ are denoted by the abbreviations $\gcd S$ and $\mathrm{lcm}\, S$, respectively.

If $\mathscr{G} = (V, E)$ is a digraph (directed graph), then a walk in $\mathscr{G}$ (from a vertex $v_0 \in V$ to a vertex $v_k \in V$) is defined to be a sequence $w = (v_0, \dots, v_k)$ with $k \ge 0$ vertices in V, such that $(v_{i-1}, v_i) \in E$ for all $i = 1, \dots, k$. The number k is called the length of w. If $v_0 = v_k$ then w is called a cycle, and if all vertices in w are distinct then w is called an elementary path.

A strongly connected component of $\mathscr{G}$ is defined to be a subdigraph $\mathscr{K} = \big(K, E \cap K^2\big)$ generated by a non-empty subset $K \subseteq V$ such that there is a walk in $\mathscr{G}$ from every vertex $x \in K$ to every other vertex $y \in K$, and K is a maximal subset with this property. The vertex sets K of the strongly connected components form a partition of V. For $i \in V$, we denote by $\mathscr{K}[i]$ the unique strongly connected component to which i belongs. For strongly connected components $\mathscr{K} = \big(K, E \cap K^2\big)$ and $\mathscr{L} = \big(L, E \cap L^2\big)$ we define $\mathscr{K} \le \mathscr{L}$ iff there are vertices $x \in K$, $y \in L$ and a walk w from x to y. Clearly, $\le$ is a partial ordering on the set of all strongly connected components of $\mathscr{G}$.

A strongly connected component $\mathscr{K}$ is called *non-trivial* if it contains at least one arc, otherwise it is called *trivial*. By $\mathrm{SCC}^*(\mathscr{G})$ we denote the set of all non-trivial strongly connected components of $\mathscr{G}$. For arbitrary $\mathscr{K} \in \mathrm{SCC}^*(\mathscr{G})$, the period $\mathrm{per}(\mathscr{K})$ is defined as the gcd of the lengths of all cycles in $\mathscr{K}$. If $\mathscr{K}$ is trivial, then $\mathrm{per}(\mathscr{K}) = 0$.

Definition 2.1. A *fuzzy algebra* $\mathscr{B}$ is a triple $(\mathscr{B}, \oplus, \otimes)$, where $\mathscr{B}$ is a linearly ordered set and $\oplus, \otimes$ are the binary operations of maximum and minimum, respectively, on $\mathscr{B}$. For any natural number $n > 0$, $\mathscr{B}(n)$ denotes the set of all n-dimensional column vectors over $\mathscr{B}$, and $\mathscr{B}(n, n)$ denotes the set of all square matrices of order n over $\mathscr{B}$. For $x \in \mathscr{B}(n)$, $A = (a_{ij}) \in \mathscr{B}(n, n)$, we denote by $\bar{x}$ and $\bar{A}$ the input sets $\{x_i \colon i \in n\}$ and $\{a_{ij} \colon i, j \in n\}$, respectively. The matrix operations over $\mathscr{B}$ are defined formally in the same manner (with respect to $\oplus, \otimes$) as the matrix operations over a ring. For arbitrary $A \in \mathscr{B}(n, n)$, $x \in \mathscr{B}(n)$, the *orbit* of x generated by A is the sequence $x^{(0)}, x^{(1)}, x^{(2)}, \dots$, where $x^{(0)} := x$ and $x^{(r)} := A^r x$ for $r > 0$.

Definition 2.2. Let $A \in \mathscr{B}(n, n)$, $h \in \mathscr{B}$. Then the *threshold digraph* $\mathscr{G}(A, h)$ is the digraph $\mathscr{G} = (n, E)$, with the vertex set $n = \{0, 1, \dots, n-1\}$ and with the arc set $E = \big\{(i, j) \colon i, j \in n,\ a_{ij} \ge h\big\}$. For any natural number r and for any two vertices i, j of a graph $\mathscr{G}$, we write $i \xrightarrow[\mathscr{G}]{r} j$ if there is a walk of length r in $\mathscr{G}$, beginning in vertex i and ending in j. If the graph $\mathscr{G}$ is understood from the context, we shall simply write $i \xrightarrow{r} j$.

The following lemma, which is due to Cechlárová [2], gives a connection between the existence of walks in threshold graphs and values of matrix powers A^r and orbit vectors $x^{(r)}$.

Lemma 2.1. *Let $A \in \mathscr{B}(n, n)$, $x \in \mathscr{B}(n)$, $h \in \mathscr{B}$, $r \in \mathbb{N}$, $r \ge 1$, $i, j \in n$. Then*

(i) $(A^r)_{ij} \ge h \iff i \xrightarrow[\mathscr{G}(A,h)]{r} j$,

(ii) $(A^r x)_i \ge h \iff (\exists\, j \in n)\Big[x_j \ge h \wedge i \xrightarrow[\mathscr{G}(A,h)]{r} j\Big]$.

Proof. By induction on r. □

Definition 2.3. Let $A \in \mathscr{B}(n, n)$, $x \in \mathscr{B}(n)$. The *period* of an infinite sequence $a_1, a_2, \dots$ is the smallest positive number p for which there is an R such that for all $r > R$ we have $a_r = a_{r+p}$, if such a number exists. The *matrix period*, the *orbit period* and the *ith coordinate-orbit period* of x with respect to A, respectively denoted by: $\operatorname{per}(A)$, $\operatorname{per}(A, x)$ and $\operatorname{per}(A, x, i)$, are defined as the periods of the sequences A^r, $x^{(r)}$, or $x_i^{(r)}$ $(r = 1, 2, \dots)$, respectively.

The period of a set $S \subseteq \mathbb{N}$ is the period of its characteristic sequence

$$a_r := \begin{cases} 1 & \text{if } r \in S, \\ 0 & \text{otherwise}. \end{cases}$$

Remark 2.1. Since the ordering of $\mathscr{B}$ is linear, every element of every power of the matrix A is equal to an element of A. Hence, the sequence of powers of A contains only finitely many different matrices. Consequently, the periods $\mathrm{per}(A)$, $\mathrm{per}(A,x)$, $\mathrm{per}(A,x,i)$ are always well-defined.

The connection between matrix periods, orbit periods and coordinate-orbit periods is described by the following theorem.

THEOREM 2.2. *Let $A \in \mathscr{B}(n,n)$, $x \in \mathscr{B}(n)$. Then*

(i) $\mathrm{per}(A) = \operatorname*{lcm}_{x\in\mathscr{B}(n)} \mathrm{per}(A,x)$,

(ii) $\mathrm{per}(A,x) = \operatorname*{lcm}_{i\in n} \mathrm{per}(A,x,i)$.

P r o o f . (i) It is easy to see that $\mathrm{per}(A)$ is a multiple of $\mathrm{per}(A,x)$ for any $x \in \mathscr{B}(n)$ and therefore, $\mathrm{per}(A)$ is a multiple of $\operatorname*{lcm}_{x\in\mathscr{B}(n)} \mathrm{per}(A,x)$.

Conversely, let us assume that p is a common multiple of $\mathrm{per}(A,x)$ for all $x \in \mathscr{B}(n)$, i.e., for arbitrary $x \in \mathscr{B}(n)$ there is a number R_x such that $(\forall r > R_x)\ x^{(r)} = x^{(r+p)}$. In general, there are infinitely many vectors $x \in \mathscr{B}(n)$, but we may restrict our attention to only finitely many R_x, because the orbit period of a given vector $x \in \mathscr{B}(n)$ depends only on the way in which the elements of $\bar{x}$ are comparable with all elements of $\bar{A}$ and with each other, and there are only finitely many different possibilities for placing the elements of $\bar{x}$ between the elements of $\bar{A}$ or for permuting them. Therefore, there are only finitely many equivalence classes in $\mathscr{B}(n)$ with respect to comparability with elements of $\bar{A}$, in the sense that $x, x' \in \mathscr{B}(n)$ are equivalent if and only if $(\forall\, i,j,k \in n)[a_{ij} \le x_k \iff a_{ij} \le x'_k]$. Thus, we conclude that

$$(\exists R \in \mathbb{N})(\forall r > R)\big(\forall x \in \mathscr{B}(n)\big)\ x^{(r)} = x^{(r+p)}\,.$$

By the simple fact that $A = B \iff \big(\forall x \in \mathscr{B}(n)\big)\ Ax = Bx$, we get

$$(\exists R \in \mathbb{N})(\forall r > R)\ A^{(r)} = A^{(r+p)}\,,$$

i.e., p is a multiple of $\mathrm{per}(A)$.

(ii) follows from the definitions. □

DEFINITION 2.4. Let $A \in \mathscr{B}(n,n)$, $h \in \mathscr{B}$, let $\mathscr{G}(A,h)$ be the threshold digraph of A. Recall that $\mathrm{SCC}^*\,\mathscr{G}(A,h)$ denotes the set of all non-trivial strongly connected components of $\mathscr{G}(A,h)$.

(i) $\mathrm{SCC}^*(A) := \bigcup \big\{\mathrm{SCC}^*\colon h \in \bar{A}\big\}$,

(ii) $\mathrm{SCC}^{\min}(A) := \big\{\mathscr{K} \in \mathrm{SCC}^*(A)\colon \mathscr{K} \text{ is minimal in } \big(\mathrm{SCC}^*(A), \subseteq\big)\big\}$.

The period of a matrix in a fuzzy algebra is characterized by the periods of the non-trivial strongly connected components in the threshold graphs of the matrix:

THEOREM 2.3. [4] *Let $A \in \mathscr{B}(n,n)$. Then*

$$\mathrm{per}(A) = \mathrm{lcm}\,\big\{\mathrm{per}(\mathscr{K})\colon \mathscr{K} \in \mathrm{SCC}^*(A)\big\} = \mathrm{lcm}\,\big\{\mathrm{per}(\mathscr{K})\colon \mathscr{K} \in \mathrm{SCC}^{\min}(A)\big\}\,.$$

3. Three examples

In view of Theorem 2.2(i), a natural question arises: under what conditions is the value of the matrix period achieved by some orbit period [5]?

Definition 3.1. Matrix Period Reachability Problem (MPR)

Given a matrix $A \in \mathscr{B}(n,n)$, does there exist $x \in \mathscr{B}(n)$ such that $\mathrm{per}(A) = \mathrm{per}(A,x)$?

To illustrate the situations that can occur in this problem, we begin with three examples.

Example 1. Let $\mathscr{B} = \{0,1\}$ and let $A \in \mathscr{B}(n,n)$ be the adjacency matrix of the digraph $\mathscr{G}$ in Figure 1. The digraph consists of $n = 13$ vertices $0, 1, \dots, 12$, in 3 disjoint cycles, C_0, C_1, C_2, of lengths 3, 4 and 6.

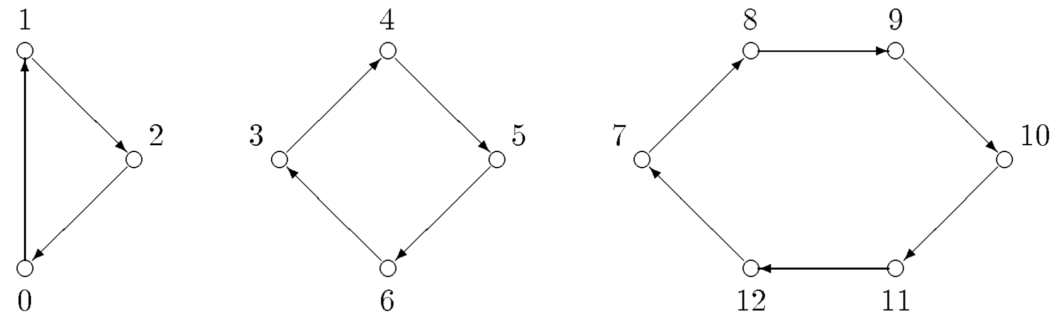

Figure 1

The digraph $\mathscr{G}$ is the threshold digraph of matrix A for the threshold level $h = 1$, and the cycles C_0, C_1, C_2 are its only (non-trivial) strongly connected components. The component periods are $3, 4, 6$. Therefore, the matrix period $\mathrm{per}(A)$ is equal to $\mathrm{lcm}(3,4,6) = 12$. The components are non-comparable, in the sense of the ordering of components of the digraph $\mathscr{G}$, i.e., they are not connected by any walk in $\mathscr{G}$. As a consequence of the non-comparability, the vector $x \in \mathscr{B}(n)$ defined on the vertex set by putting $x(i) = 1$ for $i = 0, 3$ and $x(i) = 0$ otherwise, has orbit period $\mathrm{per}(A,x) = \mathrm{lcm}(3,4) = 12$. That means that this example of the MPR problem has a positive answer.

We may obtain the same result by putting, e.g., $x'(i) = 1$ for $i = 4, 7$ and $x'(i) = 0$ otherwise. In this case, the orbit period is $\mathrm{per}(A,x') = \mathrm{lcm}(4,6) = 12$. In general, we may choose two vertices, $i_0 \in C_0$ (or $i_0 \in C_2$) and $i_1 \in C_1$ and put $x(i) = 1$ for $i = i_0, i_1$ and $x(i) = 0$ otherwise.

Example 2. A is the adjacency matrix of the digraph $\mathscr{G}$ in Figure 2. The digraph consists of $n = 7$ vertices $0, 1, \dots, 6$ in 2 disjoint cycles C_0, C_1, of lengths 3 and 4. The cycles are connected by an arc leading from the vertex 2 to 3, i.e., $C_0 < C_1$, in the ordering of components.

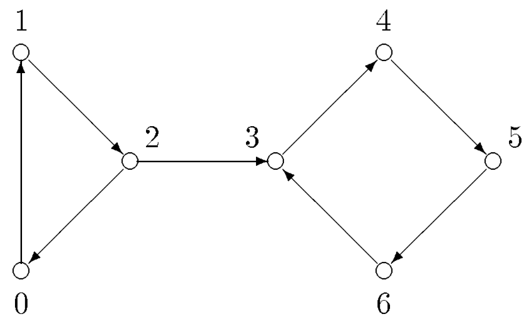

FIGURE 2

The cycles C_0, C_1 are the only non-trivial strongly connected components of $\mathscr{G}$. The component periods are 3 and 4 and the matrix period is $\mathrm{per}(A) = \mathrm{lcm}(3,4) = 12$. However, in this case there is no vector $x \in \mathscr{B}(n)$ with the property $\mathrm{per}(A,x) = \mathrm{per}(A)$. The reason is that the periods of both cycles have the greatest common divisor equal to 1, which implies that every sufficiently large integer can be expressed as a linear combination with nonnegative coefficients, of the component periods 3, 4. Consequently, for arbitrary vertices $i \in C_0$ and $j \in C_1$, there are walks beginning in i and ending in j, of arbitrary given length greater than some value. If $x(j) = 1$ for some j, then the ith coordinate in the orbit of the vector x eventually has constant value 1 (see Lemma 2.1 below). This means that the number 3 will not occur as a coordinate-orbit period of x. Thus, the orbit period $\mathrm{per}(A,x)$ will not be a multiple of 3. On the other hand, if $x(j) = 0$ for every vertex $j \in C_1$, then 4 cannot occur as the coordinate period and the orbit period $\mathrm{per}(A,x)$ will not be a multiple of 4.

EXAMPLE 3. In this example, the digraph $\mathscr{G}$ in Figure 3 consists of $n = 10$ vertices $0, 1, \dots, 9$ in 2 disjoint cycles C_0, C_1, of lengths 4 and 6. The cycles are connected by an arc leading from the vertex 2 to 4, i.e., $C_0 < C_1$. The situation differs from the previous example by the fact that the lengths of the cycles have a non-trivial common divisor 2.

Again, the cycles C_0, C_1 are the only non-trivial strongly connected components $\mathscr{G}$. The component periods are 4 and 6 and the matrix period is $\mathrm{per}(A) = \mathrm{lcm}(4,6) = 12$. In this case, we may find a vector $x \in \mathscr{B}(n)$ with the property $\mathrm{per}(A,x) = \mathrm{per}(A)$. We choose two vertices, $i \in C_0$ and $j \in C_1$ and put $x(i) = x(j) = 1$ and $x(k) = 0$ otherwise. If there is a walk of odd length from i to j, then all walks from i to j have odd lengths and, by Lemma 2.1(ii), the ith coordinate in the orbit sequence of x in odd positions eventually has

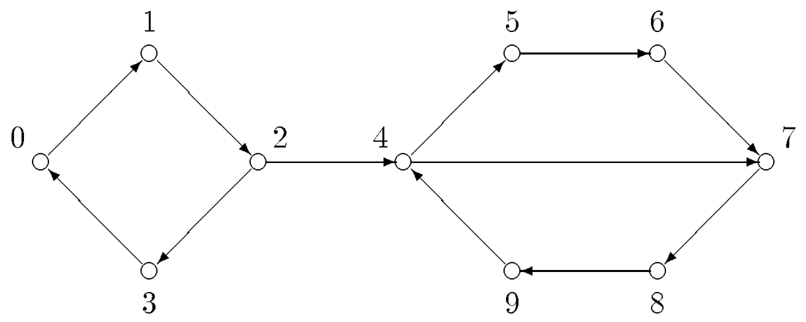

FIGURE 3

constant value 1, while in even positions it oscillates between the values 0 and 1. Thus, the ith coordinate-orbit period is $\mathrm{per}(A, x, i) = 4$. On the other hand, $\mathrm{per}(A, x, j) = 6$ and, by Theorem 2.2(ii), the orbit period is $\mathrm{per}(A, x) = 12$.

We note that, if we choose the vertices $i \in C_0$ and $j \in C_1$ in such a way that the walks from i to j have even lengths, then by a similar argument to the above, the ith coordinate in the orbit of vector x eventually has the constant value 1 on even positions and 0 on odd positions, therefore $\mathrm{per}(A, x, i) = 2$. Thus, the number 4 does not occur as a coordinate-orbit period of x and the orbit period is $\mathrm{per}(A, x) = 6$.

Our three examples indicate that in searching for a vector solution of the equation $\mathrm{per}(A, x) = \mathrm{per}(A)$, we have to find a system of strongly connected components with suitable periods and to choose a suitable system of vertices, one in each of these components. The precise statement of this idea can be found in Theorem 4.5.

It can be easily seen that if the components can be found in such a way that they are pairwise non-comparable, then any choice of vertices (with one vertex in each component) induces a solution of the MPR problem. If there exist comparable components in the threshold graph $\mathscr{G}(A, h)$, then the solvability of the problem depends on the greatest common divisors of the periods for comparable pairs of components.

4. Reachability of the matrix period

In this section we describe a necessary and sufficient condition for the MPR problem to have the positive answer “yes”. The condition is described in Theorem 4.5. This result is in accordance with our observations made about the examples in the previous section.

The following definition and lemma are crucial for investigating the periodicity of the orbit sequence $(x^{(r)} \colon r = 1, 2, \ldots)$, since they establish a connection with the periods of paths and components in a digraph.

Definition 4.1. For vertices i and j in a digraph $\mathscr{G}$, we denote by $W(i,j)$ the set of all *elementary* paths from i to j (i.e., walks with distinct vertices) in a digraph. For any walk w we define

$$\operatorname{per}(w) = \gcd\left\{\operatorname{per}(\mathscr{K})\colon \mathscr{K} \in \mathrm{SCC}^*(\mathscr{G}) \wedge \mathscr{K} \cap w \neq \emptyset\right\}$$

and set

$$R(i,j,w) := \left\{r \in \mathbb{N}\colon \text{there is a walk } w' \text{ from } i \text{ to } j \text{ with } r \text{ arcs and } w \subseteq w'\right\}.$$

Remark 4.1. If the walk w meets no non-trivial component $\mathscr{K} \in \mathrm{SCC}^*(\mathscr{G})$, then by Definition 4.1, $\operatorname{per}(w) = 0$ and $R(i,j,w)$ contains only the element $|w|$.

The notation $\operatorname{per}(w)$ for the period of the walk w is justified by part (ii) of the following lemma.

Lemma 4.1. *For an arbitrary digraph $\mathscr{G}$, there are numbers R_1 and R_2 for which the following holds.*

(i) *If a vertex i is contained in a component $\mathscr{K}$, then*

$$(\forall r > R_1)\left[r \equiv 0 \bmod \operatorname{per}(\mathscr{K}) \Leftrightarrow i \xrightarrow{r} j\right]. \tag{4.1}$$

(ii) *For any two vertices i and j and arbitrary $w \in W(i,j)$, we have*

$$(\forall r > R_2)\left[r \equiv |w| \bmod \operatorname{per}(w) \Leftrightarrow r \in R(i,j,w)\right].$$

(iii) *For any two vertices i and j and for an arbitrary walk w from i to j, we have*

$$(\forall r > R_2)\left[r \equiv |w| \bmod \operatorname{per}(w) \Rightarrow i \xrightarrow{r} j\right].$$

Proof. First we recall that for natural numbers a, b, the congruence $a \equiv b$, mod 0 holds iff $a = b$, i.e., congruence modulo 0 is the same as the usual equality. Therefore, for $\operatorname{per}(\mathscr{K}) = 0$ or $\operatorname{per}(w) = 0$, the statements of the lemma are trivially satisfied, in view of Remark 4.1. In the remaining part of the proof, we assume $\operatorname{per}(\mathscr{K}) \neq 0$ and $\operatorname{per}(w) \neq 0$.

In part (i), the existence of R_1 for any specific vertex i follows from the definition of $\operatorname{per}(\mathscr{K})$. As the number of vertices is finite, by taking R_1 sufficiently large we may ensure that (4.1) holds for *all* vertices i in $\mathscr{G}$.

For part (ii), we first note that if w does not meet any $\mathscr{K} \in \mathrm{SCC}^*(\mathscr{G})$, then, by Remark 4.1, $\operatorname{per}(w) = 0$. Now, we show that whenever $\mathscr{K} \in \mathrm{SCC}^*(\mathscr{G})$ is a non-trivial component with $\mathscr{K} \cap w \neq \emptyset$, then

$$r \in R(i,j,w) \wedge c \in \mathbb{N} \wedge c \cdot \operatorname{per}(\mathscr{K}) \geq R_1 \Longrightarrow r + c \cdot \operatorname{per}(\mathscr{K}) \in R(i,j,w).$$

If $k \in \mathscr{K} \cap w$ for some $\mathscr{K} \in \mathrm{SCC}^*(\mathscr{G})$, then, by part (i) of the lemma, there is a cycle C through k with $c \cdot \mathrm{per}(\mathscr{K})$ arcs whenever $c \cdot \mathrm{per}(\mathscr{K}) \geq R_1$. This cycle may be added to w to obtain a walk of length $r + c \cdot \mathrm{per}(\mathscr{K})$ from i to j. Since $\mathrm{per}(w)$ is the greatest common divisor of $\mathrm{per}(\mathscr{K})$ for all non-trivial components $\mathscr{K} \in \mathrm{SCC}^*(\mathscr{G})$ which meet w, then, by a well-known fact from number theory, an arbitrary sufficiently large multiple of $\mathrm{per}(w)$ can be expressed as a linear combination, with nonnegative coefficients, of the corresponding component periods $\mathrm{per}(\mathscr{K})$. Therefore, if the set $R(i,j,w)$ contains an element r in some congruence class modulo $\mathrm{per}(w)$, then all sufficiently large elements in that congruence class are also contained in $R(i,j,w)$. Thus, $R(i,j,w)$ either contains all elements of any given congruence class except a finite number, or none at all. Since there are only finitely many congruence classes, finitely many vertices i and j, and finitely many elementary paths w, R_2 can be selected to by sufficiently large so that the statement of the lemma holds.

For part (iii), note first that we may assume w.l.o.g. that w is an elementary path. If w contains a vertex k twice, the cycle C that w describes between the first and last visit to k has a length which is a multiple of $\mathrm{per}\big(\mathscr{K}[k]\big)$. Thus, if we cut C out of w, the length $r \bmod \mathrm{per}(w)$ remains unchanged. Furthermore, C does not intersect any additional components, by the definition of strongly connected components. Thus, the set of components $\mathscr{K}$ in the definition of $\mathrm{per}(w)$ is also unchanged. By repeatedly cutting cycles out of w, we eventually arrive at an elementary path with fewer than n arcs that has the same period $\mathrm{per}(w)$ and the same length modulo $\mathrm{per}(w)$ as the original walk. The statement of the lemma is now a simple consequence of part (ii). $\square$

The following sequence of lemmas will allow us to convert a given vector x into another vector having the same orbit period $\mathrm{per}(A,x)$ and satisfying additional constraints. For a given vector $x \in \mathscr{B}(n)$ and for given $h \in \mathscr{B}$, we set

$$S(x,h) := \big\{ i \in n \colon x(i) \geq h \big\}.$$

For $i \in n$, we denote the strongly connected component in $\mathscr{G}(A,h)$ containing the vertex i by $\mathscr{K}[i,h]$.

Lemma 4.2. *Let $A \in \mathscr{B}(n,n)$, $x \in \mathscr{B}(n)$. Then there is a vector $x' \in \mathscr{B}(n)$ with $\mathrm{per}(A,x') = \mathrm{per}(A,x)$, such that, for arbitrary $h \in \mathscr{B}$, every vertex $i \in S(x',h)$ is contained in a non-trivial component of the threshold digraph $\mathscr{G}(A,h)$.*

Proof. Let $H = \{h_1, h_2, \dots\} = \bar{A} \cup \bar{x}$ be the elements occurring in A and x, in decreasing order. We take $h = h_k \in H$ and we assume by induction that for arbitrary $h_l > h$, $i \in S(x,h_l)$, the component $\mathscr{K}[i,h_l]$ is non-trivial. Then we put $x' := x$ and modify x' by the following two rules. (All walks mentioned in the proof are in the digraph $\mathscr{G}(A,h)$.)

1. For arbitrary $j \in S(x,h)$ with $\mathscr{K}[j,h]$ trivial, we define $x'(j) = h_{k+1}$ (i.e., j is removed from $S(x',h)$, but it remains in $S(x',h_l)$ for arbitrary $h_l \in H$, $h_l < h$).

2. For arbitrary $j \in S(x,h)$ with $\mathscr{K}[j,h]$ trivial and for arbitrary $w\colon i \xrightarrow{s} j$ such that i is the only vertex in w with non-trivial component $\mathscr{K}[i,h]$, we choose the vertex $j_w \in \mathscr{K} = \mathscr{K}[i,h]$ such that $i \xrightarrow{s} j_w$. As $\mathscr{K}$ is non-trivial, such a vertex j_w can always be found in $\mathscr{K}$. Then we set $x'(j_w) = \max\bigl(h, x(j_w)\bigr)$. (This means that j_w is added to $S(x',h)$ if it is not there already, but not to $S(x',h_l)$ for $h_l > h$. The vertex j_w is intended to replace the vertex j which was removed by rule 1.)

All remaining values $x'(j)$ are unchanged at this stage of the recursion, i.e., we have $x'(j) = x(j)$. We note that the rules 1, 2 apply only to vertices j with $x(j) = h$. Indeed, if $x(j) < h$, then $j \notin S(x,h)$ and if $x(j) > h$, then, as a consequence of the inductive assumption, $\mathscr{K}[j,h]$ is non-trivial. In view of Lemma 2.1(ii) and Theorem 2.2, it is sufficient to verify that for arbitrary $i \in n$ and for arbitrary sufficiently large r, the equality $(A^r x)_i = (A^r x')_i$ holds, i.e., the following two statements are equivalent:

$$(\exists j \in n)\bigl[x(j) \geq h \wedge i \xrightarrow{r} j\bigr] \iff (\exists j' \in n)\bigl[x'(j') \geq h \wedge i \xrightarrow{r} j'\bigr].$$

Let the first condition be satisfied; in other words, let there exist $j \in n$ such that $x(j) \geq h$ and $w\colon i \xrightarrow{r} j$. If r is sufficiently large, then the walk w cannot be an elementary path, i.e., w must meet some non-trivial component in $\mathscr{G}(A,h)$. If the component $\mathscr{K}[j,h]$ is non-trivial, then we have $x'(j) = x(j)$ and the second statement holds for $j' = j$. If the component $\mathscr{K}[j,h]$ is trivial, then we denote by i_1 the last vertex in w with non-trivial $\mathscr{K}[i_1,h] =: \mathscr{K}$ and we denote $w_1 := w(i_1,j)$ (the subwalk of w from i_1 to j). By rule 2 of the definition of x', there is a vertex $j' = j_{w_1} \in \mathscr{K}$ and a walk $w_1'\colon i_1 \xrightarrow{s} j'$ such that $s = |w_1|$ and $x'(j') \geq h$. If we denote $w_0 := w(i,i_1)$, then the walks $w = w_0 w_1$ and $w' = w_0 w_1'$ are of the same length r and, therefore, the second statement holds.

Conversely, let the second condition be satisfies, i.e., let $j' \in n$ be such that $x'(j') \geq h$ and $w\colon i \xrightarrow{r} j'$. By the definition of x', the component $\mathscr{K} := \mathscr{K}[j',h]$ is non-trivial. If $x(j') \geq h$, then the first statement holds for $j = j'$. If $x(j') < h$, then $x(j') < x'(j')$ and $x(j')$ was modified by rule 2. Therefore there exist vertices $i_1 \in \mathscr{K}$, $j \notin \mathscr{K}$ and walks $w_1\colon i_1 \xrightarrow{s} j$, $w_1'\colon i_1 \xrightarrow{s} j'$ such that $x(j) \geq h$ and i_1 is the only vertex in w_1 with non-trivial component $\mathscr{K}[i,h] =: \mathscr{K}$. We choose a walk $u\colon j' \xrightarrow{t} i_1$, $u \subseteq \mathscr{K}$. The concatenation $c = uw_1'$ is a cycle in $\mathscr{K}$ and, therefore, its length $s+t$ is a multiple of $\mathrm{per}(\mathscr{K})$ and $r - s \equiv r + t \bmod \mathrm{per}(\mathscr{K})$. For sufficiently large r, we may assume that $r - s > R_2$, and as $\mathrm{per}(\mathscr{K})$ is a multiple of $\mathrm{per}(w) = \mathrm{per}(wu)$, we have $r - s \equiv r + t \bmod \mathrm{per}(wu)$. Applying Lemma 4.1(iii) to the concatenation wu of length

$r+t$ we get a walk $w_0 \colon i \xrightarrow{r-s} i_1$. Then $w_0 w_1 \colon i \xrightarrow{r} j$ and the first statement holds true.

We have shown that $\mathrm{per}(A, x') = \mathrm{per}(A, x)$. By the inductive assumption and by rules 1, 2, for arbitrary $h_l > h_{k+1}$, $i \in S(x', h_l)$, the component $\mathscr{K}[i, h_l]$ is non-trivial. Thus, we may replace x by x' and repeat the inductive step with $h = h_{k+1}$. □

The key notions in Theorem 4.5 are the notions of *dominance of vertices* in a digraph $\mathscr{G}$ and that of *deciding components* in $\mathrm{SCC}^*(\mathscr{G})$.

Definition 4.2. Let $i, j \in n$, $h \in \mathscr{B}$. We say that j *dominates* i *at level* h (in notation: $i \preceq_h j$), if

$$(\exists R \in \mathbb{N})(\forall r > R)(\forall k \in n)\left[k \xrightarrow[\mathscr{G}(A,h)]{r} i \Rightarrow k \xrightarrow[\mathscr{G}(A,h)]{r} j \right].$$

Remark 4.2. For fixed h, the relation $\preceq_h$ is reflexive and transitive (i.e., $\preceq_h$ is a quasi-order on n). Note that the threshold $R := R_{ijh}$ in Definition 4.2 may depend on i, j, and h. By taking the maximum R of the finitely many numbers R_{ijh}, (for $i, j \in n$, $h \in \bar{A}$) we have a global constant R with the property:

$$(\forall i, j \in n)(\forall h \in \bar{A})\left[i \preceq_h j \Rightarrow (\forall r > R)(\forall k \in n)\left[k \xrightarrow[\mathscr{G}(A,h)]{r} i \Rightarrow k \xrightarrow[\mathscr{G}(A,h)]{r} j \right]\right].$$

Remark 4.3. If the component $\mathscr{K} = \mathscr{K}[i, h]$ is non-trivial, then $i \preceq_h j$ holds if and only if

$$(\forall r > R)\left[i \xrightarrow[\mathscr{G}(A,h)]{r} i \Rightarrow i \xrightarrow[\mathscr{G}(A,h)]{r} j \right]. \tag{4.2}$$

Indeed, we may assume w.l.o.g. that R is greater than the constants R_1, R_2 in Lemma 4.1. Suppose that there is a walk $w \colon k \xrightarrow{r} i$ for some $r > R$. As $\mathscr{K}$ is non-trivial, there is an $r' > R$ such that $i \xrightarrow{r'} i$ and, by Lemma 4.1(i), $r' \equiv 0 \mod \mathrm{per}(\mathscr{K})$. Then, by assumption (4.2), there is walk $w \colon i \xrightarrow{r'} j$. The period $\mathrm{per}(ww')$ divides $\mathrm{per}(w)$ and $\mathrm{per}(\mathscr{K})$, therefore we get $r = |w| \equiv |ww'| \mod \mathrm{per}(ww')$. Then, by Lemma 4.1(iii), we have $k \xrightarrow{r} j$.

(4.2) can be expressed equivalently as

$$(\forall r > R)\left[r \equiv 0 \mod \mathrm{per}(\mathscr{K}[i, h]) \Rightarrow i \xrightarrow[\mathscr{G}(A,h)]{r} j \right],$$

or even, in view of Lemma 4.1(i), as

$$(\exists r > R)\left[r \equiv 0 \mod \mathrm{per}(\mathscr{K}[i, h]) \wedge i \xrightarrow[\mathscr{G}(A,h)]{r} j \right]. \tag{4.3}$$

Lemma 4.3. *Let $A \in \mathscr{B}(n,n)$, $x \in \mathscr{B}(n)$. Then there is a vector $x' \in \mathscr{B}(n)$ such that $\operatorname{per}(A,x') = \operatorname{per}(A,x)$ and arbitrary vertex $i \in n$ with $x'(i) = h > \min(\bar{A})$ is not dominated at level h by any vertex $j \neq i$ with $x'(j) \geq x'(i)$.*

P r o o f. In view of Lemma 4.2, we may assume that for arbitrary $h > \min(\bar{A})$, $x(i) \geq h$, the component $\mathscr{K}[i,h]$ is non-trivial. If there are vertices $i \neq j$ such that $\min(\bar{A}) < h = x(i) \leq x(j)$ and $i \preceq_h j$, then we denote by h' the greatest element in $\bar{A} \cup \bar{x}$ with $h' < h$ and define

$$x'(l) := \begin{cases} h' & \text{if } l = i\,, \\ x(l) & \text{otherwise}. \end{cases}$$

By the assumption that $i \preceq_h j$, for arbitrary sufficiently large r and for arbitrary $k \in n$, $k \xrightarrow[\mathscr{G}(A,h)]{r} i$ implies $k \xrightarrow[\mathscr{G}(A,h)]{r} j$. Thus, $k \xrightarrow[\mathscr{G}(A,h)]{r} l$ holds for some $l \in n$, $x(l) \geq h$, if and only if there is $l' \in n$, $x'(l') \geq h$, such that $k \xrightarrow[\mathscr{G}(A,h)]{r} l'$. Clearly, the same equivalence is valid for level h' and for all remaining threshold levels.

Therefore, in view of Lemma 2.1(ii), the orbits of the vectors x and x' are eventually equal, and therefore, $\operatorname{per}(A,x) = \operatorname{per}(A,x')$. This procedure is repeated finitely many times until x' has the desired property. □

Remark 4.4. As $\mathscr{K}[i,h] \subseteq \mathscr{K}[i,h']$ holds for $h \geq h'$, the procedure in the above proof preserves the property of the vector x' formulated in Lemma 4.2.

The following lemma provides a lower bound for the period of a sequence, if the period of some subsequence is known.

Lemma 4.4. *If a sequence $a_1, a_2, \dots$ has period p, then the period of the subsequence $a_d, a_{2d}, a_{3d}, \dots$ formed by taking every dth element divides $p/\gcd(p,d)$.*

P r o o f. Since p divides $\operatorname{lcm}(p,d) = pd/\gcd(p,d)$, we have

$$a_{id} = a_{id+pd/\gcd(p,d)} = a_{(i+p/\gcd(p,d))d}\,,$$

for sufficiently large i. Clearly, the period of a sequence (or, of a subsequence) does not depend on the first finitely many elements, hence $p/\gcd(p,d)$ is a multiple of the period of the subsequence. □

Definition 4.3. Let $\operatorname{per}(A) = p_0^{\alpha_0} p_1^{\alpha_1} \dots p_{k-1}^{\alpha_{k-1}}$ be a decomposition of the matrix period $\operatorname{per}(A)$ into powers of distinct primes. Then a subset $D \subseteq \mathrm{SCC}^*(A)$ which contains a component $\mathscr{K}$ with $p_t^{\alpha_t} \,|\, \operatorname{per}(\mathscr{K})$, for each $t = 0, \dots, k-1$, is called a *deciding* set of components. If $\mathscr{K} \in \mathrm{SCC}^*\big(\mathscr{G}(A,h)\big)$ for some $\mathscr{K} \in D$, we use the notation $h_{\mathscr{K}} := h$ for the *level of* $\mathscr{K}$.

The level of a component may not be unique; in the following theorem we can make an arbitrary choice.

Theorem 4.5. *For arbitrary $A \in \mathscr{B}(n,n)$ with decomposition of the matrix period into powers of distinct primes $\mathrm{per}(A) = p_0^{\alpha_0} p_1^{\alpha_1} \dots p_{k-1}^{\alpha_{k-1}}$, the following statements are equivalent.*

(i) *There exist $x \in \mathscr{B}(n)$ such that $\mathrm{per}(A,x) = \mathrm{per}(A)$.*

(ii) *There exist a deciding set D of pairwise disjoint components and a set $I = \{i_{\mathscr{K}} \in \mathscr{K} : \mathscr{K} \in D\}$ of vertices, with one vertex chosen from each component in D, such that, if $h_{\mathscr{K}} \le h_{\mathscr{L}}$ and $\mathscr{K} \ne \mathscr{L}$, then the vertex $i_{\mathscr{L}}$ does not dominate $i_{\mathscr{K}}$ at level $h_{\mathscr{K}}$.*

Proof. (ii) $\Rightarrow$ (i): We define a vector $x \in \mathscr{B}(n)$ as follows. We put

$$x(i) := \begin{cases} h_{\mathscr{K}} & \text{if } i = i_{\mathscr{K}} \text{ for some } \mathscr{K} \in D\,, \\ \min(\bar{A}) & \text{otherwise}. \end{cases}$$

By Theorem 2.2(ii), it is sufficient to prove that

$$(\forall t \in k)(\exists\, i \in n)\ p_t^{\alpha_t} \,|\, \mathrm{per}(A,x,i)\,.$$

Let $t \in k$ be fixed and let $\mathscr{K} \in D$ be a maximal component with the property $p_t^{\alpha_t} \,|\, \mathrm{per}(\mathscr{K})$ (maximal in the sense of the ordering induced by $\mathscr{G}(A, h_{\mathscr{K}})$; there is no walk leading from $\mathscr{K}$ to another component with this property). We denote by $i := i_{\mathscr{K}}$ the vertex chosen by the system I. All the walks in this part of the proof are understood to be in the threshold digraph $\mathscr{G} := \mathscr{G}(A, h_{\mathscr{K}})$. We will first show that

$$(\exists R \in \mathbb{N})(\forall r > R)\Big[\big(r \equiv 0 \bmod \mathrm{per}(\mathscr{K})/p_t\big) \Longrightarrow \big(x^{(r)}(i) = h_{\mathscr{K}} \Leftrightarrow r \equiv 0 \bmod \mathrm{per}(\mathscr{K})\big)\Big]\,.$$

By Lemma 2.1 and by Lemma 4.1(i) we may conclude that $\mathrm{per}(\mathscr{K})|r$ implies $x^{(r)}(i) \ge h_{\mathscr{K}}$, for sufficiently large r. By the non-dominance assumption, (4.3) holds for none of the vertices $i_{\mathscr{L}}$ with $h_{\mathscr{L}} \ge h_{\mathscr{K}}$, and we get $x^{(r)}(i) = h_{\mathscr{K}}$. Now, if we had $x^{(r)}(i) = h_{\mathscr{K}}$ for some r with $(\mathrm{per}(\mathscr{K})/p_t)|r$ and $\mathrm{per}(\mathscr{K}) \not| \, r$, this can only happen if $w\colon i \xrightarrow{r} j$ for some $j \in I$ which was selected from another component $\mathscr{K}'$. By the maximality of $\mathscr{K}$ we know that $p_t^{\alpha_t} \not| \, \mathrm{per}(\mathscr{K}')$, and hence, applying Lemma 4.1(iii), we conclude that $i \xrightarrow{r'} j$ for all sufficiently large r' with $r' \equiv r \bmod \mathrm{per}(w)$. Since $p_t^{\alpha_t} \not| \, \mathrm{per}(w)$, we have $\gcd\big(p_t^{\alpha_t}, \mathrm{per}(w)\big) | p_t^{\alpha_t - 1} | (\mathrm{per}(\mathscr{K})/p_t)$, and the two equations $r' \equiv r \bmod \mathrm{per}(w)$, $r' \equiv 0 \bmod \mathrm{per}(\mathscr{K})$ have infinitely many solutions. So we have $i \xrightarrow{r'} j$ for some $r' \equiv 0 \bmod \mathrm{per}(\mathscr{K})$. This means that $i \preceq_{h_{\mathscr{K}}} j$, contradicting the assumption of non-dominance of vertices in I.

We have seen that the subsequence $\big(x^{(r)}(i)\big)$ of all elements with $r \equiv 0 \bmod \mathrm{per}(\mathscr{K})/p_t$ has period p_t. Applying Lemma 4.4 with increment $d = \mathrm{per}(\mathscr{K})/p_t$, we obtain that $p_t \cdot \gcd\bigl(\mathrm{per}(\mathscr{K})/p_t, \mathrm{per}(A,x,i)\bigr) = \gcd\bigl(\mathrm{per}(\mathscr{K}), p_t \cdot \mathrm{per}(A,x,i)\bigr)$ divides $\mathrm{per}(A,x,i)$, and hence $\gcd\bigl(p_t^{\alpha_t}, p_t \cdot \mathrm{per}(A,x,i)\bigr) \,\big|\, \mathrm{per}(A,x,i)$, from which it follows that $p_t^{\alpha_t} \mid \mathrm{per}(A,x,i)$.

(i) $\Rightarrow$ (ii): Let $x \in \mathscr{B}(n)$, $\mathrm{per}(A) = \mathrm{per}(A,x)$. By Lemma 4.3, we may assume that x fulfills the non-dominance requirement: an arbitrary vertex $i \in n$ with $x(i) = h > \min(\bar{A})$ is not dominated at level h by any vertex $j \neq i$ with $x(j) \geq x(i)$. By Lemma 4.2 and Remark 4.4, we may assume that an arbitrary vertex $i \in S(x,h)$ is contained in a non-trivial component of $\mathscr{G}(A,h)$.

We only have to ensure that, for every $t \in k$, there is a level $h \in \bar{A} \cup \bar{x}$ such that $S(x,h)$ contains a vertex j in a component $\mathscr{K} \in \mathrm{SCC}^*\bigl(\mathscr{G}(A,h)\bigr)$ with $p_t^{\alpha_t} \mid \mathrm{per}(\mathscr{K})$. We prove this by contradiction. Suppose that there is a t such that, for all $h \in \bar{A} \cup \bar{x}$ and for all $j \in S(x,h)$, we have $p_t^{\alpha_t} \not| \, \mathrm{per}\,\mathscr{K}[j,h]$. In what follows, we shall use the notations $W(i,j,h)$, $R(i,j,w,h)$, and $\mathrm{per}(w,h)$ for the corresponding notions $W(i,j)$, $R(i,j,w)$, and $\mathrm{per}(w)$ from Definition 4.1, defined with respect to the threshold digraph $\mathscr{G} = \mathscr{G}(A,h)$. By Lemma 2.1 and Lemma 4.1, we have for any $h \in \bar{A} \cup \bar{x}$ and $i \in n$:

$$\begin{aligned} x^{(r)}(i) \geq h &\iff \bigl(\exists j \in S(x,h)\bigr)\ i \xrightarrow{r} j \\ &\iff \bigl(\exists j \in S(x,h)\bigr)\bigl(\exists w \in W(i,j,h)\bigr)\ r \in R(i,j,w,h) \\ &\iff r \in R(i,h) := \bigcup_{j \in S(x,h)} \ \bigcup_{w \in W(i,j,h)} R(i,j,w,h)\,. \end{aligned}$$

Under the above assumption, we have $p_t^{\alpha_t} \not| \, \mathrm{per}(w,h)$, for all possible paths w in this statement, since $\mathrm{per}(w,h)$ is defined as the gcd of the periods of certain components which include the component $\mathscr{K}[j,h]$, whose period is not a multiple of $p_t^{\alpha_t}$. By Lemma 4.1(ii), the set $R(i,j,w,h)$ is periodic with period $\mathrm{per}(w,h)$. It follows that the set $R(i,h)$, being a finite union of sets $R(i,j,w,h)$, is also periodic, and its period divides

$$\mathrm{lcm}\,\bigl\{\mathrm{per}(w,h)\colon j \in S(x,h),\ w \in W(i,j,h)\bigr\}\,,$$

which is not a multiple of $p_t^{\alpha_t}$. It is easy to see that $\mathrm{per}(A,x,i)$ divides the lcm of the periods of sets $R(i,h)$ for $h \in \bar{A} \cup \bar{x}$ and, therefore, it is not a multiple of $p_t^{\alpha_t}$, for arbitrary i. Then, by Theorem 2.2(ii), the orbit period $\mathrm{per}(A,x) = \mathrm{lcm}_{i \in n}\, \mathrm{per}(A,x,i)$ is not a multiple of $p_t^{\alpha_t}$. Hence, $\mathrm{per}(A,x)$ is different from $\mathrm{per}(A)$, which contradicts our assumption (i).

Now, in view of the fact that two components $\mathscr{K}, \mathscr{L} \in \mathrm{SCC}^*(A)$ are either disjoint or comparable, and if $\mathscr{K} \subseteq \mathscr{L}$, then $\mathrm{per}(\mathscr{L}) \mid \mathrm{per}(K)$, we can easily choose a deciding system D of pairwise disjoint components and a vertex set I satisfying the statement (ii). $\square$

5. The computational complexity of the MPR problem

It was shown in [5] that, in general, the MPR problem is NP-complete. We describe here two conditions under which the problem is polynomially solvable.

By Theorem 4.5, solving of the MPR problem amounts to performing two choices: we have to choose a suitable deciding set of components in such a way that it is possible to choose a set of pairwise non-dominating vertices, one from each component. In spite of the fact that the problem of performing these two choices is NP-complete, we may hope that under some restrictive conditions it can be solved in polynomial time.

A similar situation occurs with the classical satisfiability problem (SAT) for disjunctive boolean formulas. If the number of variables in each disjunctive clause is restricted to 2, the problem 2-SAT is polynomially solvable. However, it is well-known that the restriction to 3-disjunctive formulas is not sufficient: the problem 3-SAT remains NP-complete.

First we extend the definition of the partial ordering of strongly connected components to components belonging to threshold digraphs with different levels.

Definition 5.1. If $\mathscr{K}$ and $\mathscr{L}$ are strongly connected components in the threshold digraphs $\mathscr{G}(A,h)$ and $\mathscr{G}(A,h')$, respectively, with $h \le h'$, then we define $\mathscr{K} \le_h \mathscr{L}$ in $\mathscr{G}(A,h)$ iff there are vertices $x \in \mathscr{K}$, $y \in \mathscr{L}$ and a walk w in $\mathscr{G}(A,h)$ from x to y. We write $\mathscr{K} <_h \mathscr{L}$ iff $\mathscr{K} \le_h \mathscr{L}$ and $\mathscr{K} \ne \mathscr{L}$. Then we say that $\mathscr{K}$ and $\mathscr{L}$ are comparable.

Theorem 5.1. MPR *problem with the two restrictive conditions*

(i) *the matrix A has a unique minimal deciding set of components D,*

(ii) *for any two components $\mathscr{K} \in D$ at level h and $\mathscr{L} \in D$ at level h' with $h \le h'$, $\mathscr{K} <_h \mathscr{L}$ in $\mathscr{G}(A,h)$, we have $\gcd\big(\mathrm{per}(\mathscr{K}), \mathrm{per}(\mathscr{L})\big) \le 2$,*

is solvable in $O(n^2)$ time for a given matrix $A \in \mathscr{B}(n,n)$ if a deciding set D of components is known, and in $O(n^3)$ time otherwise.

Proof. It is easy to show, using Theorem 2.3, that there is always at least one minimal deciding set D of components, and D is necessarily pairwise disjoint. If D is not known, it can be found in $O(n^3)$ time [4]. The rest of the computation is performed in $O(n^2)$ time. Condition (i) reduces the problem to the choice of pairwise non-dominating vertices in D. Condition (ii) implies that, for any comparable components $\mathscr{K}, \mathscr{L} \in D$, we have $\gcd\big(\mathrm{per}(\mathscr{K}), \mathrm{per}(\mathscr{L})\big) = 1$, or $\gcd\big(\mathrm{per}(\mathscr{K}), \mathrm{per}(\mathscr{L})\big) = 2$.

If the case $\gcd\big(\mathrm{per}(\mathscr{K}), \mathrm{per}(\mathscr{L})\big) = 1$ occurs for some $h \le h'$ and for $\mathscr{K}, \mathscr{L} \in D$, $\mathscr{K} \in \mathrm{SCC}^*\big(\mathscr{G}(A,h)\big)$, $\mathscr{L} \in \mathrm{SCC}^*\big(\mathscr{G}(A,h')\big)$, with $\mathscr{K} < \mathscr{L}$ in $\mathscr{G}(A,h)$, then for any $i \in \mathscr{K}$, $j \in \mathscr{L}$, the vertex i is dominated by j at level h and in this case the MPR has the negative answer "no".

If the case $\gcd(\mathrm{per}(\mathscr{K}), \mathrm{per}(\mathscr{L})) = 2$ occurs for all comparable pairs $\mathscr{K}, \mathscr{L} \in D$, then the dominance $i \preceq_h j$ for $i \in \mathscr{K} \in \mathrm{SCC}^*(\mathscr{G}(A,h))$, $j \in \mathscr{L} \in \mathrm{SCC}^*(\mathscr{G}(A,h'))$, $h \le h'$ depends only on the parity of the positions of the vertices i, j and on the existence or non-existence of a walk w in the digraph $\mathscr{G}(A,h)$, of even length and connecting points of the same parity in components $\mathscr{K}, \mathscr{L}$, or of odd length and connecting points of different parities in $\mathscr{K}$ and $\mathscr{L}$. Therefore, the choice of pairwise non-dominating vertices in D is equivalent to the choice of pairwise compatible representatives from a system of n two-element sets. It is shown in [6] that such a choice can be made in time $O(n^2)$. □

The conditions (i), (ii) in Theorem 5.1 may seem rather restrictive. However, the next two theorems show that each of the conditions alone is too weak to make the MPR problem polynomially solvable.

THEOREM 5.2. *The* MPR *problem with the restrictive condition* (i) *from Theorem 5.1 is NP-complete.*

P r o o f . As a special case of MPR, the problem belongs to NP. We show that the well-known NP-complete problem of 3-coloring for graphs (3-COL) polynomially transforms to MPR(i) problem (i.e., MPR with the restrictive condition (i)). We describe a polynomial algorithm which assigns, to an arbitrary instance of 3-COL, an instance of MPR(i) such that both instances are equivalent, i.e., they have the same answer *yes* or *no*.

Let $\mathscr{G}' = (V', E')$ be an instance of 3-COL, let us denote $m' = |V'|$, $n' = |E'|$. We choose $m' + n'$ distinct primes

$$(p_v : v \in V'),\ (q_e : e \in E')$$

and we define a digraph $\mathscr{G} = (V, E)$ consisting of $m' + n'$ disjoint cycles

$$(C_v : v \in V'),\ (D_e : e \in E')$$

of lengths

$$|C_v| = 3p_v,\ |D_e| = 3q_e, \qquad \text{for any } v \in V', e \in E'.$$

The vertices in each cycle are numbered beginning with 0, in the sense of the orientation. The vertices will be referred to as $C_v(i)$, $D_e(j)$. Besides the arcs contained in the cycles, there are additional arcs between cycles in $\mathscr{G}$, defined as follows: we fix a linear ordering $<$ of vertices in V' and for an arbitrary edge $e = (u,v) \in E'$, $u < v$ we add the arcs $(D_e(0), C_u(0))$, $(D_e(0), C_u(2))$, $(D_e(0), C_v(1))$ to E.

The additional arcs are schematically drawn in Figure 4. For the sake of simplicity, the cycles C_u, C_v, D_e, are depicted with 3 vertices only. This simplification is based on the fact that the greatest common divisor of the periods of the cycles is equal to 3.

The matrix A is defined as the adjacency matrix of the digraph $\mathscr{G}$. Denoting $\mathscr{B} = \{0,1\}$ and $n = 3(m' + n')$, we have $A \in \mathscr{B}(n,n)$. Clearly, A has exactly one deciding set of components, namely the set of all cycles $\{C_v : v \in V'\}$ and $\{D_e : e \in E'\}$. Therefore, A is an instance in MPR (i). By a theorem in number theory, there exist a constant a such that the kth prime number is less than $ak \log k$. Thus, the primes used in the construction of A can be found in polynomial time, with respect to the size of G', and the whole construction of A is polynomial. In the following we show that the instances A and G' are equivalent.

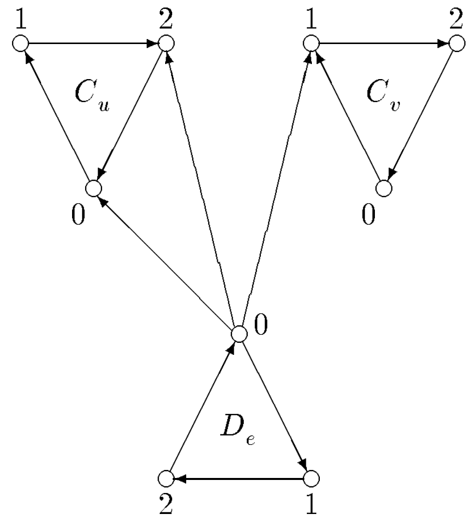

FIGURE 4

Let A be a *yes* instance of MPR(i). By Theorem 4.5, there is a set

$$I = \{i_u \in C_u : u \in V'\} \cup \{j_e \in D_e : e \in E'\}$$

of pairwise non-dominating vertices in $\mathscr{G}$ (at level 1).

We shall show that the graph G' is a *yes*-instance of 3-COL, i.e., there is a coloring $F \colon V' \to \{0,1,2\}$ with $F(u) \neq F(v)$ for every $e = (u,v) \in E'$. The coloring F is defined for each $u \in V'$ by the formula

$$F(u) := \begin{cases} 0 & \text{if } \ i_u = C_u(3k) \text{ for some } k \in \mathbb{N}\,, \\ 1 & \text{if } \ i_u = C_u(3k+1) \text{ for some } k \in \mathbb{N}\,, \\ 2 & \text{if } \ i_u = C_u(3k+2) \text{ for some } k \in \mathbb{N}\,. \end{cases}$$

Let us suppose that $F(u) = F(v)$ for some $e = (u,v) \in E'$, $u < v$. If $F(u) = F(v) = 0$, then the vertex $i_u = C_u(3k)$ dominates all the vertices of the form $D_e(3l+1)$, $D_e(3l+2)$ and the vertex $i_v = C_v(3k)$ dominates all the vertices

$D_e(3l)$. Consequently, the vertex j_e is dominated either by i_u or by i_v. The assumption that $F(u) = F(v) = 1$, or $F(u) = F(v) = 2$ leads similarly to a contradiction. Therefore, $F(u) \neq F(v)$ holds for any adjacent vertices of the graph $\mathscr{G}'$.

Conversely, let $\mathscr{G}'$ be a *yes*-instance of 3-COL with the coloring $F\colon V' \to \{0,1,2\}$. Then we define $i_u = C_u\bigl(F(u)\bigr)$ for any $u \in V'$ and $j_e = D_e\bigl(F(u)\bigr)$ for any $e = (u,v) \in E'$, $u < v$. By the above reasoning, the vertex i_u does not dominate j_e. The adjacency property of F implies that the vertex i_v does not dominate j_e, as well. Therefore by Theorem 4.5, the matrix A is a *yes*-instance of MPR(i). ☐

Remark 5.1. To underline the analogy with 2-SAT and 3-SAT, we note that Theorem 5.2 holds even if the modified condition (ii), with 3 instead of 2, is added.

THEOREM 5.3. *The* MPR *problem in which condition* (ii) *from Theorem 5.1 holds for every deciding set of components D, is NP-complete.*

P r o o f . It is shown in [5] that the NP-complete problem of satisfiability for 3-disjunctive boolean formulas (3-SAT) polynomially transforms to MPR problem. It is easy to verify that the construction described in [5] satisfies the condition of the theorem. ☐

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Department of Mathematics
Faculty of Electrical Engineering and Informatics
Technical University
Hlavná 6
SK–04213 Košice
SLOVAKIA

Institut für Mathematik
Technische Universität Graz
Steyrergasse 30
A–8010 Graz
AUSTRIA

E-mail: gavalec@tuke.sk
rote@opt.math.tu-graz.ac.at