Tatra Mt. Math. Publ. **16** (1999), 95–108

TATRA MOUNTAINS
Mathematical Publications

FUZZY PROJECTIVE GEOMETRIES FROM FUZZY GROUPS

Leen Kuijken — Hendrik Van Maldeghem — Etienne Kerre

ABSTRACT. First we construct a fuzzy group from a fuzzy projective geometry, and then we construct a fuzzy projective geometry from a fuzzy group. This provides a natural link between fuzzy vector spaces and fuzzy groups, which generalizes the classical case.

1. Introduction

Fuzzy groups have been introduced by Rosenfeld in 1971, in [7]. Fuzzy vector spaces have been introduced by Katsaras and Liu in 1977, in [4]. In [5], we introduced fuzzy projective geometries.

These fuzzy projective geometries were derived from fuzzy vector spaces. In this paper, we derive a fuzzy group corresponding to such a fuzzy projective geometry (Section 4), thus obtaining a relationship between fuzzy vector spaces and fuzzy groups by means of these fuzzy projective geometries.

Moreover, we will give a construction of fuzzy projective geometries from fuzzy groups that yields *the same* fuzzy projective geometries as defined in [5].

The fuzzy projective geometries we constructed are thus an important link in the connection between the theories of fuzzy vector spaces and fuzzy groups.

2. Preliminaries

In this paper, $(G, \cdot)$ or briefly G will always denote a group, and its neutral element will be denoted by e.

2000 Mathematics Subject Classification: 20N25.
Keywords: fuzzy group, fuzzy projective geometry, fuzzy vector space.

The first author is a Research Fellow supported by the Flemish Institute for the Promotion of Scientific and Technological Research in Industry (IWT), grant No. 961336.
The second author is a Research Director of the Fund for Scientific Research FWO — Flanders (Belgium).

Definition 2.1. ([7]) A fuzzy set μ on a group G is a *fuzzy subgroup* of G if the following holds $\forall x, y \in G$:

(1) $\mu(x \cdot y) \geq \mu(x) \wedge \mu(y)$, and
(2) $\mu(x^{-1}) = \mu(x)$.

From (1) and (2) we immediately see that $\mu(e) \geq \mu(a)$, for all $a \in G$.

Note that the conditions (1) and (2) are equivalent to:

(3) $\mu(x \cdot y^{-1}) \geq \mu(x) \wedge \mu(y)$

which looks like the classical condition for a subset H of a group G to be a subgroup of G: if $a, b \in H$, then $a \cdot b^{-1} \in H$.

Definition 2.2. ([10]) Let μ be a fuzzy set of some set X. Then for $t \in [0,1]$, the set $\mu_t = \{x \in X : \mu(x) \geq t\}$ is called a *level subset* of the fuzzy set μ.

Proposition 2.1. (Propositions 2.1 and 2.2 in [1]) *A fuzzy set μ on the group G is a fuzzy subgroup of G if and only if μ_t is a subgroup of G for every $t \in [0, \mu(e)]$.*

Proposition 2.2. (Proposition 3.3 in [6]) *If G is a group having a chain*

$$\{e\} = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_n = G$$

of subgroups, with n maximal, then a fuzzy subgroup μ of G is a step function from $G \to [0,1]$ having at most $n+1$ steps.

If the subgroups G_i are as in the proposition above, the membership degree of e must be higher than the membership degrees of the elements of $G_1 \backslash \{e\}$, which in turn have to be higher than the membership degrees of the elements in the set $G_2 \backslash G_1$ and so on, leaving the elements of $G_n \backslash G_{n-1}$ with the lowest membership degree (this follows from the definition of fuzzy subgroup). This means that the level subsets of μ are given by $G_i \backslash G_{i-1}$. Note that the chain of subgroups of G doesn't need to be chosen to be maximal.

In fact, in [6] Proposition 2.2 is stated for normal fuzzy subgroups, but it holds for fuzzy subgroups as well.

Definition 2.3. The *projective space* $PG(V) = (D(V), I)$ corresponding to a vector space V, is defined as the collection $D(V)$ of subspaces of V together with the following incidence relation I on the class of subspaces: U and U' are incident if $U \subseteq U'$ or $U' \subseteq U$, which we denote by $U \; I \; U'$ (for more details, see, e.g., [3]). The *dimension* $\mathrm{d}(U)$ of a subspace U is equal to the number of its base vectors. The *projective dimension* $\mathrm{pd}(U)$ of U is defined by $\mathrm{pd}(U) = \mathrm{d}(U) - 1$. The definition of projective points, lines, planes, ... is based on this projective dimension: subspaces having projective dimension 0 are called projective points, projective lines have projective dimension 1 and so on. Throughout this paper, we shall denote a projective space by $\mathscr{P}$. The *order of a projective space* is the number of points on a line minus one. If we work with projective spaces over a finite field of order q, the number of points on a line equals $q+1$, so the order of the projective space will be q. The projective space of dimension n over a field of order q is denoted by $PG(n,q)$.

Definition 2.4. Suppose $\mathscr{P}$ is an n-dimensional projective space. A *flag* in $\mathscr{P}$ is a sequence of distinct, non-trivial (i.e., different from $\emptyset$ and $\mathscr{P}$) subspaces $(U_0, U_1, \dots, U_m)$ such that $U_j \subset U_i$ for all $j < i \leq m \leq n-1$. The *rank* of a flag is the number of subspaces it contains. A *maximal flag* in $\mathscr{P}$ is a flag of length n.

Suppose from now on that the dimension of U_i equals i. A *flag of type* $\{i, i+1, \dots, m\}$, where $m > i$, is a sequence of distinct, non-trivial subspaces $(U_i, U_{i+1}, \dots, U_m)$ such that $U_j \subset U_k$ for all $i \leq j < k \leq m \leq n-1$. We denote this briefly as an $[i,m]$-*flag*.

Suppose $\mathscr{P}$ is an n-dimensional projective space.

Definition 2.5. The fuzzy set λ on $\mathscr{P}$ is a *fuzzy n-dimensional projective space* on $\mathscr{P}$ if $\lambda(p) \geq \lambda(q) \wedge \lambda(r)$, for all collinear points p, q, r of $\mathscr{P}$. We denote $[\lambda, \mathscr{P}]$. The projective space $\mathscr{P}$ is called the *base projective space* of $[\lambda, \mathscr{P}]$. If $\mathscr{P}$ is a fuzzy point, line, plane, …, we use base point, base line, base plane, …, respectively.

Property 2.3. *A fuzzy n-dimensional space $[\lambda, \mathscr{P}]$ is of the following form (see [5]):*

$$\begin{aligned}
&\lambda\colon \mathscr{P} \longrightarrow [0,1], \\
&p \mapsto a_0 \qquad \textit{if} \quad p = q, \\
&p \mapsto a_1 \qquad \textit{for} \quad p \in U_1 \backslash \{q\}, \\
&p \mapsto a_2 \qquad \textit{for} \quad p \in U_2 \backslash U_1, \\
&\quad \dots \\
&p \mapsto a_{n-1} \qquad \textit{for} \quad p \in U_{n-1} \backslash U_{n-2}, \\
&p \mapsto a_n \qquad \textit{for} \quad p \in \mathscr{P} \backslash U_{n-1},
\end{aligned}$$

for a certain maximal flag $(q, U_1, \dots, U_{n-1})$ in $\mathscr{P}$, and for some reals $a_0 \geq a_1 \geq a_2 \geq \dots \geq a_n$ in $[0,1]$.

One can prove this in the same way as for fuzzy vector spaces (see Theorem 3.2 in [5]).

3. The classical case

We shall start this section with a short explanation of how in the classical case, a flag-transitive geometry can be constructed from its automorphism group. For simplicity, we construct the smallest possible projective space: the Fano plane.

The Fano plane $PG(2,2)$ is the projective plane over the finite field $GF(2)$. Throughout this article we shall denote it by $\mathscr{F}$. It consists of 7 points and 7 lines and is the smallest non-trivial projective plane. Every point of $\mathscr{F}$ is incident with exactly three lines of $\mathscr{F}$ and every line of $\mathscr{F}$ contains exactly three points of $\mathscr{F}$. Its automorphism group is $L_3(2)$, and consists of 168 elements. The subgroups of $L_3(2)$ that stabilize points are symmetric groups of order 24, as are the subgroups of $L_3(2)$ that stabilize a line.

Now, suppose that we are given the group $L_3(2)$ (the type preserving automorphisms); how do we recover the corresponding Fano plane? We shall construct a geometry $\mathscr{F}'$ based on $L_3(2)$, and show that $\mathscr{F}' = \mathscr{F}$. For this, we select two subgroups S_4 and S_4' (symmetric groups of order 24) of $L_3(2)$, intersecting in a group isomorphic to D_8. The latter is a group that stabilizes a line and fixes a point on that line; hence a group stabilizing a flag. So we see that D_8 is the maximal possible intersection of S_4 and S_4'. Both S_4 and S_4' have 7 left cosets. We shall denote a left coset by gS_4, with $g \in L_3(2)$. Since we shall only consider left cosets, we shall from now on write 'coset' instead of left coset, since no confusion is possible.

We define the points and lines of $\mathscr{F}'$ as follows:

points: the (7) cosets of S_4,

lines: the (7) cosets of S_4',

incidence: $gS_4 \ \mathrm{I} \ hS_4' \iff gS_4 \cap hS_4' \neq \emptyset$, so a point is incident with a line if the cosets that define them have a nonzero intersection (in fact, if they intersect in a coset of the group D_8, then the flags are the (21) cosets of $S_4 \cap S_4' = D_8$).

One can show that this geometry $\mathscr{F}'$ is indeed the Fano configuration $\mathscr{F}$ (see [2], 1.2.17).

4. Fuzzy groups from fuzzy projective spaces

For the sake of clarification, before we turn to the general case, we start with a construction of the fuzzy group corresponding to a fuzzy projective geometry on the Fano plane $\mathscr{F} = GF(2,2)$.

4.1 The fuzzy group corresponding to the fuzzy Fano plane.

Suppose $[\lambda, \mathscr{F}]$ is a fuzzy projective space, then for a certain flag (q, L) in $\mathscr{F}$ and reals $a_0 \geq a_1 \geq a_2 \in [0,1]$ we have:

$$\begin{aligned} \lambda \colon \mathscr{F} &\longrightarrow [0,1]\,, \\ p \mapsto a_0 & \quad \text{if} \quad p = q\,, \\ p \mapsto a_1 & \quad \text{if} \quad p \in L \backslash \{q\}\,, \\ p \mapsto a_2 & \quad \text{if} \quad p \in \mathscr{F} \backslash \{L\}\,. \end{aligned}$$

Since we have to construct a fuzzy group, we know, by Proposition 2.2, that we have to find a chain of subgroups of $L_3(2)$. We do this as follows.

The stabilizer of $\mathscr{F}$ is simply its automorphism group $L_3(2)$. We can consider $L_3(2)$ to be the group stabilizing all points of $[\lambda, \mathscr{F}]$ with membership degree at least a_2. We now seek the subgroup of $L_3(2)$ which stabilizes the points with membership degree at least a_1. Since all these points are on the line L, this subgroup is simply the stabilizer of L, hence a symmetric group of order 24: S_4'. Finally, we seek for the subgroup of S_4' that fixes the unique point p with membership degree a_0 on L. This is a dihedral group of order 8: D_8. (Note that this is not the group that stabilizes the point p, since the latter is a symmetric group of order 24: S_4; but we have $S_4 \cap S_4' = D_8$.)

This reasoning yields the chain $D_8 \leq S_4' \leq L_3(2)$. With this chain we construct the following fuzzy set on $L_3(2)$ in a natural way

$$\begin{aligned} \mu \colon L_3(2) &\longrightarrow [0,1]\,, \\ x \mapsto a_0 & \quad \text{if} \quad x \in D_8\,, \\ x \mapsto a_1 & \quad \text{if} \quad x \in S_4' \backslash D_8\,, \\ x \mapsto a_2 & \quad \text{if} \quad x \in L_3(2) \backslash S_4'. \end{aligned}$$

We have to check that this fuzzy set is indeed a fuzzy group. This is straightforward by using the multiplication in the group $L_3(2)$ and since $a_0 \geq a_1 \geq a_2$.

4.2 General case.

Suppose $\mathscr{P}$ is an n-dimensional projective space, and $[\lambda, \mathscr{P}]$ is a fuzzy n-dimensional projective space on $\mathscr{P}$. As in the previous section, it is possible to define a fuzzy subgroup on the automorphism group of $\mathscr{P}$.

Suppose $[\lambda, \mathscr{P}]$ is of the following form:

$$\begin{aligned}
\lambda\colon \mathscr{P} &\longrightarrow [0,1]\,,\\
p \mapsto a_0 &\qquad \text{if} \quad p = q\,,\\
p \mapsto a_1 &\qquad \text{for} \quad p \in U_1\backslash\{q\}\,,\\
p \mapsto a_2 &\qquad \text{for} \quad p \in U_2\backslash U_1\,,\\
\dots &\\
p \mapsto a_{n-1} &\qquad \text{for} \quad p \in U_{n-1}\backslash U_{n-2}\,,\\
p \mapsto a_n &\qquad \text{for} \quad p \in \mathscr{P}\backslash U_{n-1}\,,
\end{aligned}$$

for a certain maximal flag $(q, U_1, \dots, U_{n-1})$ in $\mathscr{P}$, and for some reals $a_0 \geq a_1 \geq a_2 \geq \dots \geq a_n$ in [0,1].

We now seek a chain of subgroups of G, in the same way as we did in the previous section. So we first seek the stabilizer group St_{n-1} stabilizing all points in $[\lambda, \mathscr{P}]$ with membership degrees at least a_{n-1}, that is the points in the hyperplane U_{n-1} in $\mathscr{P}$. Then we look for the subgroup St_{n-2} of St_{n-1}, stabilizing the $(n-2)$-dimensional subspace U_{n-2}, and so on, creating a chain of subgroup $(St_0, St_1, \dots, St_{n-2}, St_{n-1}, G)$. Using this chain we define the following fuzzy set on G:

$$\begin{aligned}
\mu\colon G &\longrightarrow [0,1]\,,\\
x \mapsto a_n &\qquad \text{if} \quad x \in G\backslash St_{n-1}\,,\\
x \mapsto a_{n-1} &\qquad \text{if} \quad x \in St_{n-1}\backslash St_{n-2}\,,\\
\dots &\\
x \mapsto a_1 &\qquad \text{if} \quad x \in St_1\backslash St_0\,,\\
x \mapsto a_0 &\qquad \text{if} \quad x \in St_0\,.
\end{aligned}$$

One can easily prove that μ is a fuzzy subgroup of $\mathscr{P}$. We define this μ to be the fuzzy subgroup corresponding to the fuzzy projective space $[\lambda, \mathscr{P}]$.

5. Fuzzy projective spaces from fuzzy groups

We start this section with a concrete case: the construction of the fuzzy Fano plane from a particular fuzzy subgroup on the automorphism group of $\mathscr{F}$. Afterwards, we give the construction in the n-dimensional case.

5.1 A small fuzzy example.

We want to define a fuzzy projective Fano plane starting from a particular fuzzy subgroup μ of $L_3(2)$. Since the base plane will be the geometry deduced from $L_3(2)$, the base plane of this fuzzy projective plane will be the Fano plane.

To agree with [5], the resulting fuzzy projective plane $[\lambda, \mathscr{F}]$ has to be of the following form:

$$\begin{aligned} \lambda\colon \mathscr{F} &\longrightarrow [0,1]\,, & & \\ p \mapsto \alpha_0 & & \text{if} \quad & p = q\,, \\ p \mapsto \alpha_1 & & \text{if} \quad & p \in L\backslash\{q\}\,, \\ p \mapsto \alpha_2 & & \text{if} \quad & p \in \mathscr{F}\backslash\{L\}\,, \end{aligned} \tag{1}$$

for some $\alpha_0 \geq \alpha_1 \geq \alpha_2 \in [0,1]$ and (q, L) a flag in $\mathscr{F}$.

Of what form must μ be to obtain such a fuzzy projective geometry? From Proposition 2.2 we know that to μ there corresponds a chain of subgroups of $L_3(2)$.

We prove that if we choose this chain of subgroups as follows:

$$D_8 \leq S_4' \leq L_3(2)\,,$$

S_4' being the stabilizer of a line, and if we take the point stabilizer S_4 such that $S_4 \cap S_4' = D_8$ (this S_4 is indeed unique), that we can recover the fuzzy Fano plane from the fuzzy subgroup μ. (Note that this chain is not maximal.) This chain allows us to write μ as the following fuzzy subgroup on $L_3(2)$:

$$\begin{aligned} \mu\colon L_3(2) &\longrightarrow [0,1]\,, & & \\ x \mapsto a_0 & & \text{if} \quad & x \in D_8\,, \\ x \mapsto a_1 & & \text{if} \quad & x \in S_4'\backslash D_8\,, \\ x \mapsto a_2 & & \text{if} \quad & x \in L_3(2)\backslash S_4', \end{aligned}$$

with $a_0 \geq a_1 \geq a_2 \in [0,1]$. In the sequel, however, we shall suppose that the real numbers a_0, a_1 and a_2 are different: this is the generic situation. When some of these values coincide, similar reasoning can be used.

We wish to obtain the fuzzy projective plane $[\lambda, \mathscr{F}]$ for some α_i's, so we restrict our attention to the fuzzy points, since the shape of the fuzzy lines is deduced completely from the fuzzy points.

We define the base points to be the classical cosets of S_4. Every classical point lies in exactly 3 flags, i.e., in 3 cosets of D_8 in $L_3(2)$. Now we explain how the membership degrees of the fuzzy points are given.

Consider the group $D_8 = S_4 \cap S_4'$. It has 21 cosets, one of them is simply D_8 itself, 2 of them are disjoint subsets of $S_4'\backslash D_8$ and the 18 others are disjoint subsets of $L_3(2)\backslash S_4'$.

We define the following fuzzy set on $\mathscr{K}$, the set of all cosets of D_8 in $L_3(2)$:

$$\begin{aligned} \nu\colon \mathscr{K} &\longrightarrow [0,1]\,, \\ gD_8 &\mapsto \mu(g)\,, \end{aligned}$$

for all $g \in L_3(2)$. Is this well–defined, i.e., is the membership degree of a coset independent of the chosen representative $g \in L_3(2)$ of that coset? It is, since from elementary group theory we know that

$$\begin{aligned} gD_8 = D_8 &\iff g \in D_8\,, \\ gD_8 \subseteq S_4' \backslash D_8 &\iff g \in S_4' \backslash D_8\,, \\ gD_8 \subseteq L_3(2) \backslash S_4' &\iff g \in L_3(2) \backslash S_4'\,. \end{aligned}$$

So the fuzzy set ν on $\mathscr{K}$ is of the following form:

$$\begin{aligned} \nu\colon \mathscr{K} &\longrightarrow [0,1]\,, \\ X &\mapsto a_0 \qquad \text{if } \ X = D_8\,, \\ X &\mapsto a_1 \qquad \text{if } \ X \subseteq S_4' \backslash D_8\,, \\ X &\mapsto a_2 \qquad \text{if } \ X \subseteq L_3(2) \backslash S_4', \end{aligned}$$

with $a_0 \geq a_1 \geq a_2 \in [0,1]$. Note that this fuzzy set has exactly the same structure as μ. The only difference is the base set ($L_3(2)$ for μ and $\mathscr{K}$ for ν).

Each base point lies in 3 flags. Consider a base point p, and the 3 flags with which it is incident: gD_8, hD_8 and jD_8, where g, h and j are different elements of $L_3(2)$. How do we determine the membership degree $\lambda(p)$ of the fuzzy point $\bigl[\lambda(p), p\bigr]$? We define the following fuzzy set λ on the set P of all base points of $[\lambda, \mathscr{F}]$:

$$\begin{aligned} \lambda\colon P &\longrightarrow [0,1]\,, \\ p &\mapsto \max\bigl(\nu(gD_8)\,,\ \nu(hD_8)\,,\ \nu(jD_8)\bigr)\,, \end{aligned} \tag{2}$$

where gD_8, hD_8 and jD_8 are the 3 flags through the point p.

We claim that this defines a suitable membership degree. There are 7 base points, every point lies in 3 flags and every flag contains exactly one point. The membership degree of each flag in the fuzzy set ν will be used only once, since there are 21 flags, and three flags are needed for the determination of the membership degree of one point (see (2)).

This means that exactly one fuzzy point will have the membership degree a_0. There are two flags with membership degree a_1 in the fuzzy set ν, let us call them nD_8 and mD_8. There can only be two fuzzy points with membership degree a_1, if these two flags do not contain the same point, and if no flag of membership degree a_1 contains the base point of the fuzzy point with membership degree a_0. This is not the case, since $D_8 \cup nD_8 \cup mD_8 = S_4'$, the group that stabilizes a line L. Since D_8, nD_8 and mD_8 are mutually disjoint, and since they all stabilize the same line L, they all have to fix another point on that line, because they have to be different. Hence it is impossible for two of these flags to appear in the determination of the membership degree (by the maximum in

(2)) of the same fuzzy point. Thus we find 3 points on a line with values a_0, a_1 and a_1. All the other points will have membership degree a_2, since only flags with value a_2 are left. So we find a fuzzy projective plane of the form:

$$\begin{aligned} \lambda\colon \mathscr{F} &\longrightarrow [0,1]\,, \\ p \mapsto a_0 &\qquad \text{if} \quad p = q\,, \\ p \mapsto a_1 &\qquad \text{if} \quad p \in L\backslash\{q\}\,, \\ p \mapsto a_2 &\qquad \text{if} \quad p \in \mathscr{F}\backslash\{L\}\,, \end{aligned}$$

with $a_0 \geq a_1 \geq a_2 \in [0,1]$, which proves our claim. Thus we have proved the following theorem:

Theorem 5.1. *The fuzzy projective plane $[\lambda, \mathscr{F}]$:*

$$\begin{aligned} \lambda\colon \mathscr{F} &\longrightarrow [0,1]\,, \\ p \mapsto a_0 &\qquad \textit{if} \quad p = q\,, \\ p \mapsto a_1 &\qquad \textit{for} \quad p \in L\backslash\{q\}\,, \\ p \mapsto a_2 &\qquad \textit{for} \quad p \in \mathscr{F}\backslash\{L\}\,, \end{aligned}$$

where $\mathscr{F}$ is the Fano plane, (p, L) is a flag in $\mathscr{F}$ and $a_0 \geq a_1 \geq a_2$ are reals in $[0,1]$, can be constructed from the following fuzzy subgroup μ on the automorphism group $L_3(2)$ of $\mathscr{F}$:

$$\begin{aligned} \mu\colon L_3(2) &\longrightarrow [0,1]\,, \\ x \mapsto a_0 &\qquad \textit{if} \quad x \in D_8\,, \\ x \mapsto a_1 &\qquad \textit{if} \quad x \in S_4'\backslash D_8\,, \\ x \mapsto a_2 &\qquad \textit{if} \quad x \in L_3(2)\backslash S_4'\,, \end{aligned}$$

where S_4' is the stabilizer group of the line L and D_8 is the stabilizer group of the point p on the line L.

5.2 General case.

Suppose $PG(n,q) = \mathscr{P}$ is an n-dimensional projective space (over the finite field $GF(q)$ of order q), with automorphism group G. We choose a flag $F = (U_0, U_1, U_2, \ldots, U_{n-1})$ in $\mathscr{P}$, where U_i is an i-dimensional subspace of $\mathscr{P}$, so U_0 is a point and U_{n-1} is a hyperplane, and construct a fuzzy projective space $[\lambda, \mathscr{P}]$ on $\mathscr{P}$ (see Property 2.3), based on this flag. We will now construct a fuzzy subgroup μ on G that allows us to recover $[\lambda, \mathscr{P}]$. For this, we construct the following chain of subgroups of G.

First, we seek the group that stabilizes U_{n-1}, we call it St_{n-1}. Next, we consider the $[n-2, n-1]$-flag (U_{n-2}, U_{n-1}). We call its stabilizer group

St_{n-2}. In general, we call St_{n-i} the group that stabilizes the $[n-i, n-1]$-flag $(U_{n-i}, U_{n-i+1}, \dots, U_{n-1})$, so the group stabilizing the $[1, n-1]$-flag $(U_1, U_2, \dots, U_{n-1})$ is called St_1, and the stabilizer of the whole maximal flag is called St_0.

These stabilizers form the following chain of subgroups of G:

$$St_0 \subseteq St_1 \subseteq \cdots \subseteq St_{n-1} \subseteq G.$$

Using this chain we construct the following fuzzy subgroup μ on G:

$$\begin{aligned}
\mu\colon G &\longrightarrow [0,1], \\
x \mapsto a_n & \qquad \text{if} \quad x \in G \backslash St_{n-1}, \\
x \mapsto a_{n-1} & \qquad \text{if} \quad x \in St_{n-1} \backslash St_{n-2}, \\
\dots & \\
x \mapsto a_1 & \qquad \text{if} \quad x \in St_1 \backslash St_0, \\
x \mapsto a_0 & \qquad \text{if} \quad x \in St_0,
\end{aligned}$$

where the real numbers a_i are the same as in λ. Again we suppose that the values a_i are distinct, as above. A similar reasoning can be made if some values are the same. In fact, the requirement that $\mathscr{P}$ has to be a projective space over a *finite* field is also unnecessary. However, the finite case is much easier to explain and to understand, so we focus on finite fields. The same construction can be made for base projective spaces over infinite fields.

We now take the subgroup St' of G stabilizing the point U_0, such that $St_1 \cap St' = St_0$. Then we know that the base points of the fuzzy points will be given by the classical cosets of St'. We shall now determine the membership degree of the fuzzy points with these base points.

As in Section 5.1, we define the following fuzzy set on $\mathscr{K}$, the set of all cosets of St_0 in G:

$$\begin{aligned}
\nu\colon \mathscr{K} &\longrightarrow [0,1], \\
gSt_0 &\mapsto \mu(g).
\end{aligned}$$

This fuzzy set is well-defined, since:

$$\begin{aligned}
gSt_0 = St_0 &\iff g \in St_0, \\
gSt_0 \subseteq St_1 \backslash St_0 &\iff g \in St_1 \backslash St_0, \\
\dots & \\
gSt_0 \subseteq St_{n-1} \backslash St_{n-2} &\iff g \in St_{n-1} \backslash St_{n-2}, \\
gSt_0 \subseteq St_n \backslash St_{n-1} &\iff g \in St_n \backslash St_{n-1}.
\end{aligned}$$

So we can write the fuzzy set ν on $\mathscr{K}$ as follows:

$$
\begin{aligned}
\nu\colon \mathscr{K} &\longrightarrow [0,1]\,,\\
X \mapsto a_0 &\qquad \text{if} \quad X = St_0\,,\\
X \mapsto a_1 &\qquad \text{if} \quad X \subseteq St_1 \backslash St_0\,,\\
\dots\\
X \mapsto a_{n-1} &\qquad \text{if} \quad X \subseteq St_{n-1} \backslash St_{n-2}\,,\\
X \mapsto a_n &\qquad \text{if} \quad X \subseteq G \backslash St_{n-1}\,.
\end{aligned}
$$

Since $\mathscr{P} = PG(n,q)$ is a projective space of order q, there are $q+1$ points on a line. Furthermore, there are $q^i + q^{i-1} + q^{i-2} + \cdots + q^2 + q + 1$ points in every subspace U_i, for all $i \in \{1, 2, \dots n\}$ (of course there is only one point 'in' U_0, since it is itself a point). From now on, we denote the number $q^i + q^{i-1} + q^{i-2} + \cdots + q^2 + q + 1$ by N_i for $i \in \{1, 2, \dots, n\}$. There are $q + 1 = N_1$ hyperplanes passing through a fixed $(n-2)$-dimensional space. In general, there are N_{n-i} i-dimensional subspaces of $\mathscr{P}$ passing through a fixed $(i-1)$-dimensional subspace of $\mathscr{P}$. The total number of maximal flags in $\mathscr{P}$ is thus $N_1 \cdot N_2 \cdots N_{n-1} \cdot N_n$. From this we conclude that there are $N_1 \cdot N_2 \cdots N_{n-1}$ flags through a fixed point of $\mathscr{P}$. Since St_0 stabilizes $(U_0, \dots, U_n)$ and St_1 stabilizes $(U_1, U_2, \dots, U_n)$ and since there are $q + 1 = N_1$ points on U_1, we have:

$$|St_1| = (q+1)|St_0|\,,$$

which means that there are $q+1$ cosets of St_0 that are contained in St_1 (including St_0 itself). In the same way, since St_2 stabilizes $(U_2, U_3, \dots, U_n)$, and since in U_2 we can choose a pair (p, L) such that p I L, p a point and L a line in $(q+1)(q^2+q+1)$ different ways, we find that:

$$|St_2| = (q+1)(q^2+q+1)|St_0|\,,$$

which means that there are $(q+1)(q^2+q+1) = N_1 \cdot N_2$ cosets of St_0 in St_2.

In general, we will find that there are $N_1 \cdot N_2 \cdots N_i$ cosets of St_0 in St_i, thus there are equally many $[0, i]$-flags in U_i. This means, since there are N_i points in U_i, that there are $N_1 \cdot N_2 \cdots N_{i-1}$ flags through every point in U_i, since the number of flags through a point is constant in every subspace of $\mathscr{P}$.

We define the fuzzy projective space derived from the fuzzy subgroup μ as the following fuzzy set on P, the set of all base points of $[\lambda, \mathscr{P}]$:

$$
\begin{aligned}
\lambda\colon P &\longrightarrow [0,1]\,,\\
p &\mapsto \max_{i=1}^{k} \nu(x_i St_0)\,,
\end{aligned}
\tag{3}
$$

where $x_i St_0$ are the $k = N_1 \cdot N_2 \cdots N_{n-1}$ flags through the point p. Since every flag contains just one point, the membership degree of a flag in ν will appear in the determination of the membership degree of just one fuzzy point in λ.

Since there is one coset of St_0 (St_0 itself) that has membership degree a_0 in ν, the point p this flag contains (i.e., U_0) will be given the membership degree a_0, and p will be unique with this membership degree.

St_1 contains $q+1$ cosets of St_0, this means that there will be exactly q flags with membership degree a_1 in ν. Since all these flags together with the flag with membership degree a_0 contain the line U_1 which in turn contains $q+1$ points, these flags all contain a different point. Since subsequently we shall only find other flags through these points with a lower membership degree, this means that all points on $U_1 \backslash U_0$ have membership degree a_1 in λ.

St_2 contains $N_2 \cdot N_1$ cosets of St_0, thus there are equally many flags in U_2, of which $(N_2 \cdot N_1) - N_1 = (q^2+q)(q+1)$ with membership degree a_2 (this is the number of flags in $U_2 \backslash U_1$). There are N_2 points in U_2, so that there are $N_1 = q+1$ flags through every point. The membership degree of the points of U_1 does not change, and since the flags that are added have a smaller membership degree in the fuzzy set ν, they can be deleted from the right hand side of (3) without changing the resulting membership degree, since the maximum operator is used. All points of $U_2 \backslash U_1$ get membership degree a_2.

St_i contains $N_i \cdots N_1$ cosets of St_0, thus U_i contains $N_i \cdots N_1$ flags. This means that there are $N_{i-1} \cdots N_1$ flags through every point, of which $(N_i \cdots N_1) - (N_{i-1} \cdots N_1)$ with membership degree a_i (this is the number of flags in $U_i \backslash U_{i-1}$). The new flags that go through points in U_{i-1} that have already been given a membership degree before, do not change these membership degrees, since the membership degrees of the new flags are lower than those of the flags already used, so they will not contribute to the determination of the membership degree of the points, since for this, the maximum of the membership degrees of all flags is taken (see (3)). The points in $U_i \backslash U_{i-1}$ will all have membership degree a_i in λ.

We end up with the situation where every point of $\mathscr{P}$ is given a membership degree in the fuzzy set λ, so that there is 1 point having the (highest) membership degree a_0, there are q points with membership degree a_1, q^2 points with membership degree a_2, ..., in general, there are q^i points of $\mathscr{P}$ with membership degree a_i. Moreover, since the point with membership degree a_0 is U_0, the line defined by the points with membership degree a_1 and a_0 is U_1, and in general, since U_i is the space determined by the points of $\mathscr{P}$ with membership degree a_0 or a_1 or ... or a_i in λ, we have proved that λ is a fuzzy n-dimensional projective space of the form described in Property 2.3.

THEOREM 5.2. *Let $\mathscr{P}$ be an n-dimensional projective space, over an arbitrary field. The fuzzy projective space $[\lambda, \mathscr{P}]$:*

$$\begin{array}{llll}
\lambda\colon \mathscr{P} & \longrightarrow & [0,1]\,, & \\
p \mapsto a_0 & & \textit{if} & p = q\,, \\
p \mapsto a_1 & & \text{for} & p \in U_1 \backslash \{q\}\,, \\
p \mapsto a_2 & & \text{for} & p \in U_2 \backslash U_1\,, \\
\dots & & & \\
p \mapsto a_{n-1} & & \text{for} & p \in U_{n-1} \backslash U_{n-2}\,, \\
p \mapsto a_n & & \text{for} & p \in \mathscr{P} \backslash U_{n-1}\,,
\end{array}$$

with $(q = U_0, U_1, \dots, U_{n-1})$ *a maximal flag in* $\mathscr{P}$ *and* $a_0 \ge a_1 \ge a_2 \ge \dots \ge a_n$ *reals in* $[0,1]$, *can be constructed from the following fuzzy subgroup* μ *on the automorphism group* G *of* $\mathscr{P}$:

$$\begin{array}{llll}
\mu\colon G & \longrightarrow & [0,1]\,, & \\
x \mapsto a_n & & \textit{if} & x \in G \backslash St_{n-1}\,, \\
x \mapsto a_{n-1} & & \textit{if} & x \in St_{n-1} \backslash St_{n-2}\,, \\
\dots & & & \\
x \mapsto a_1 & & \textit{if} & x \in St_1 \backslash St_0\,, \\
x \mapsto a_0 & & \textit{if} & x \in St_0\,,
\end{array}$$

where $St_0 \subseteq St_1 \subseteq \dots \subseteq St_{n-1}$ *is a chain of subgroups of* G *such that* St_i *stabilizes the* $[i, n-1]$*-flag* $(U_i, U_{i+1}, \dots, U_{n-1}, U_n)$, *for all* $i \in \{1, 2, \dots n\}$.

We note that the stabilizer groups St_i, are in fact parabolic subgroups of G. Hence it is clear how to generalize the procedure explained in the present paper to geometries belonging to (almost) simple classical groups, or to exceptional groups of Lie type. Also, one can do exactly the same for semi-simple algebraic groups, or, more generally, for all groups with a BN-pair (or Tits system). The parabolic subgroups are the subgroups containing a Borel subgroup. It follows that the base geometry is just the associated *building*, as defined by Tits [8]. The level subgroups form (in the general case) a maximal chain of subgroups between a Borel subgroup and the whole group. Basically, the arguments of the present paper can be used, but for non-linear diagrams there are some additional choices to make owing to the fact that a flag is not linearly ordered by inclusion! This will be investigated in a forthcoming paper. Also, one is tempted to apply the same ideas to (simple) Lie algebras where a Borel subalgebra must replace the role of the Borel subgroup used above.

REFERENCES

[1] SIVARAMAKRISHNA DAS, P.: *Fuzzy groups and level subgroups*, J. Math. Anal. Appl. **84** (1981), 264–269.

[2] DEMBOWSKI, P.: *Finite Geometries*, Ergeb. Math. Grenzgeb, Vol. 44, Springer-Verlag, Berlin, 1986.

[3] HUGHES, D. R.—PIPER, F. C.: *Projective Planes*, Springer-Verlag, Berlin, 1973.

[4] KATSARAS, A. K.—LIU, D. B.: *Fuzzy vector spaces and fuzzy topological vector spaces*, J. Math. Anal. Appl. **58** (1977), 135–146.

[5] KUIJKEN, L.—Van MALDEGHEM, H.—KERRE, E.: *Fuzzy projective geometries from fuzzy vector spaces*, in: Proceedings of IPMU '98 (to appear).

[6] KUMAR, I. J.—SAXENA, P. K.—PRATIBHA, YADAV: *Fuzzy normal subgroups and fuzzy quotients*, Fuzzy Sets and Systems **46** (1992), 121–132.

[7] ROSENFELD, A.: *Fuzzy groups*, J. Math. Anal. Appl. **35** (1971), 512–517.

[8] TITS, J.: *Buildings of Spherical Type and Finite BN-pairs*, in: Lecture Notes in Math., Vol. 386, Springer-Verlag, Berlin, 1974.

[9] ZHANG, YUN JIE—ZOU, KAIQI: *Normal fuzzy subgroups and conjugate fuzzy subgroups*, J. Fuzzy Math. **1** (1993), 571–585.

[10] ZADEH, L.: *Fuzzy sets*, Information and Control **8** (1965), 338–358.

Received April 24, 1998

Leen Kuijken & Hendrik Van Maldeghem
Department of Pure Mathematics
and Computer Algebra
Galglaan 2
9000 Gent
BELGIUM
E-mail: lk@cage.rug.ac.be
hvm@cage.rug.ac.be

Etienne Kerre
Department of Applied Mathematics
and Computer Science
Krijgslaan 281
9000 Gent
BELGIUM
E-mail: Etienne.Kerre@rug.ac.be