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A NOTE ON IDEALS IN GENERALIZED EFFECT ALGEBRAS

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ABSTRACT. In [3], the notion of a *central ideal* of an effect algebra was introduced. We present a generalization of this notion for the class of positive cancellative partial abelian monoids.

1. Preliminaries

Definition 1.1. Let $\mathscr{P} = (P, \oplus, 0)$ be a partial algebra with a nullary operation 0 and a binary partial operation $\oplus$. Denote the domain of $\oplus$ by $\perp$. $\mathscr{P}$ is called a *positive cancellative partial abelian monoid* (CPAM) iff for all $a, b, c \in P$ the following conditions are satisfied:

(P1) $a \perp b$ implies $b \perp a$, $a \oplus b = b \oplus a$.

(P2) $b \perp c$ and $a \perp b \oplus c$ imply $a \perp b$, $a \oplus b \perp c$, $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

(P3) $a \perp 0$ and $a \oplus 0 = a$.

(P4) $a \oplus b = a \oplus c$ implies $b = c$.

(P5) $a \oplus b = 0$ implies $a = 0$.

In a CPAM $(P, \oplus, 0)$, we denote $a \leq b$ iff $a \oplus c = b$ for some $c \in P$. We denote $c = a \ominus b$ iff $a = b \oplus c$. It is easy to see that $\ominus$ is a well-defined partial operation and that $\leq$ is a partial order on P. In [5], an equivalent class of partial $\ominus$-structures was introduced (*abelian RI-posets*).

An upper-bounded CPAM is called an *effect algebra* (cf. [1], [2]). A *D-poset* (cf. [6]) is an algebraic structure equivalent to an effect algebra, but it is defined using $\ominus$ as a primary operation.

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Definition 1.2. An *ideal* of a CPAM $(P, \oplus, 0)$ is a nonempty subset I of P having the property $x \oplus y \in I$ if and only if $x \perp y$, $x, y \in I$.

We denote the system of all ideals of a CPAM $(P, \oplus, 0)$ by $I(P)$. It it easy to show that $\big(I(P), \subseteq\big)$ is a complete lattice. The meet of two arbitrary ideals is their intersection. The join has a more complex structure.

Proposition 1.2. *Let $\mathscr{P}$ be a CPAM. Let $A \subseteq P$. Let $A_0 = A \neq \emptyset$. For every $i \in \mathbb{N}$, let*

$$\begin{aligned} A_i' &= \{y \colon \exists x \in A_i\,,\ y \leq x\}\,, \\ A_{i+1} &= \{x \oplus y \colon x \perp y \quad \text{and} \quad x, y \in A_i'\}\,. \end{aligned}$$

Then $\bigcup\limits_{i \in \mathbb{N}} A_i$ is the smallest ideal containing A.

P r o o f . The proof is left to the reader. □

2. Subcentral ideals

In [3], a notion of central ideal of an effect algebra was studied. There is a natural one-to-one correspondence between direct decompositions of an effect algebra and central ideals. The lattice of all central ideals is a Boolean algebra.

Definition 2.1. We say that an ideal I of a CPAM $(P, \oplus, 0)$ is *subcentral* if and only if there is an ideal J such that for every $x \in P$ there is a unique decomposition $x = x_1 \oplus x_2$ with $x_1 \in I$, $x_2 \in J$.

If I is a subcentral ideal, then J is called a *complement* of I.

Clearly, if J is a complement of I, then J is subcentral and I is a complement of J. We denote the set of all subcentral ideals of $(P, \oplus, 0)$ by $\mathrm{SCI}(P)$.

It is easy to see that in a direct product of two CPAMs, say $P \times Q$, the set $\{(x, 0) \colon x \in P\}$ is a subcentral ideal with complement $\{(0, x) \colon x \in Q\}$. Thus, every central ideal of an effect algebra is subcentral.

Proposition 2.1. *Every subcentral ideal has a unique complement.*

P r o o f . Let I be a subcentral ideal. Let J and K be complements of I. Let $x \in J$. There is a unique pair of elements x_I, x_K such that $x = x_I \oplus x_K$, where $x_I \in I$ and $x_K \in K$. Similarly, there is a unique pair x_K^I, x_K^J such that $x_K = x_K^I \oplus x_K^J$ and $x_K^I \in I$, $x_K^J \in J$. Observe that $x_K^I \in K$. It follows that $x = (x_I \oplus x_K^I) \oplus x_K^J$ is a decomposition of x into I and its complement J. Since J is a complement of I, this decomposition is unique. Evidently, $x_I \oplus x_K^I = 0$ because $x = 0 \oplus x$, $0 \in I$, $x \in J$ and the decomposition is unique. Thus $x = x_K^J \in K$. Analogously, $x \in K$ implies $x \in J$. □

The above proposition allows us to define a bijective unary operation $'$: $\mathrm{SCI}(P) \mapsto \mathrm{SCI}(P)$ such that for $I \in \mathrm{SCI}(P)$, I' is a complement of I. Note that $I'' = I$ and that $P, \{0\} \in \mathrm{SCI}(P)$. Further, $I \wedge I' = \{0\}$ and $I \vee I' = P$.

Moreover, for each $I \in \mathrm{SCI}(P)$ there is an idempotent map $\pi_I \colon P \mapsto I$ with the property $\forall x \in P \colon x = \pi_I(x) \oplus \pi_{I'}(x)$.

Proposition 2.2. *For any $I, J \in \mathrm{SCI}(P)$ $I = J$ if and only if $\pi_I = \pi_J$.*

Proof. Trivially, it suffices to observe that $x \in I$ iff $\pi_I(x) = x$. □

Proposition 2.3. *Let I be a subcentral ideal. Then π_I is a morphism.*

Proof.

$$\begin{aligned} x_1 \oplus x_2 &= \pi_I(x_1) \oplus \pi_{I'}(x_1) \oplus \pi_I(x_2) \oplus \pi_{I'}(x_2) \\ &= \big(\pi_I(x_1) \oplus \pi_I(x_2)\big) \oplus \big(\pi_{I'}(x_1) \oplus \pi_{I'}(x_2)\big) . \end{aligned}$$

Since I and I' are ideals, $\pi_I(x_1) \oplus \pi_I(x_2) \in I$ and $\pi_{I'}(x_1) \oplus \pi_{I'}(x_2) \in I'$. Obviously, $\pi_I(x_1 \oplus x_2) = \pi_I(x_1) \oplus \pi_I(x_2)$ by subcentrality of I. □

Now, let us turn our attention to the structure of $\big(\mathrm{SCI}(P), \subseteq\big)$. In order to prove the main result of this paper, the following proposition will be helpful.

Proposition 2.4. *Let $I, J \in \mathrm{SCI}(P)$. For all $x \in I' \vee J'$, $\pi_I\big(\pi_J(x)\big) = 0$.*

Proof. Consider Proposition 1.1. Put $A = I' \cup J'$. For every $x \in I' \vee J' = \langle I' \cup J' \rangle$, there is $n \in \mathbb{N}$ such that $x \in A_n$. We will proceed using induction over n.

(1) Let $x \in A_0 = I' \cup J'$. If $x \in J'$, then $\pi_J(x) = 0$ and, consequently, $\pi_I\big(\pi_J(x)\big) = \pi_I(0) = 0$. On the other hand, assume $x \in I'$. We know that $x = \pi_J(x) \oplus \pi_{J'}(x)$, because J is a subcentral ideal. It follows that $\pi_I(x) = \pi_I\big(\pi_J(x) \oplus \pi_{J'}(x)\big) = \pi_I\big(\pi_J(x)\big) \oplus \pi_I\big(\pi_{J'}(x)\big)$. But $\pi_I(x) = 0$, which implies $\pi_I\big(\pi_J(x)\big) = 0$.

(2) Assume that for all $y \in A_n$, $\pi_I\big(\pi_J(y)\big) = 0$. If $x \in A'_n$, then $x \leq y$ for some $y \in A_n$. That means $y = x \oplus a$, where a is some element of P. π_I and π_J are morphisms; $\pi_J\big(\pi_I(x \oplus a)\big) = \pi_J\big(\pi_I(x)\big) \oplus \pi_J\big(\pi_I(a)\big) = 0$. Obviously, $\pi_J\big(\pi_I(x)\big) = 0$.

Assume $x \in A_{n+1}$. Then there are $x_1, x_2 \in A'_n$ such that $x_1 \oplus x_2 = x$. We already know that $\pi_I\big(\pi_J(x_1)\big) = \pi_I\big(\pi_J(x_2)\big) = 0$. Again, by Proposition 2.3, $\pi_I\big(\pi_J(x)\big) = \pi_I\big(\pi_J(x_1)\big) \oplus \pi_I\big(\pi_J(x_2)\big) = 0 + 0 = 0$.

□

Recall that an *ortholattice* is defined as an algebra $\mathscr{L} = (L, \wedge, \vee, 0, 1, ')$ satisfying

(i) $(L, \wedge, \vee, 0, 1)$ is a lattice bounded by 0 and 1.

(i) if $x \le y$ then $y' \le x'$.

(iii) $x'' = x$.

(iv) $x \vee x' = 1$ and $x \wedge x' = 0$.

Condition (ii) can be replaced by any of the De Morgan laws. If the additional condition

(v) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

holds, then $\mathscr{L}$ is a *Boolean algebra.*

Proposition 2.5. *Let* $(P, \oplus, 0)$ *be a* CPAM. *Then*

$$\big(\mathrm{SCI}(P), \wedge, \vee, {}', \{0\}, P\big)$$

is an ortholattice.

Proof. Conditions (iii) and (iv) are evidently satisfied. Let I, J be subcentral ideals. We will prove that $I \wedge J$ is subcentral. For every $x \in P$,

$$x = \pi_I(x) \oplus \pi_{I'}(x) = \pi_J\big(\pi_I(x)\big) \oplus \pi_{J'}\big(\pi_I(x)\big) \oplus \pi_{I'}(x)\,.$$

By definition, $\pi_J\big(\pi_I(x)\big) \in J$. $\pi_J\big(\pi_I(x)\big) \le \pi_I(x) \in I$; thus $\pi_J\big(\pi_I(x)\big) \in I$. This implies $\pi_J\big(\pi_I(x)\big) \in I \wedge J$. On the other hand, $\pi_{J'}\big(\pi_I(x)\big) \oplus \pi_{I'}(x) \in I' \vee J'$.

Let us prove that this decomposition of x is unique. Let $x = x_1 \oplus x_2$, where $x_1 \in I \wedge J$ and $x_2 \in I' \vee J'$. We see that $\pi_I(x) = \pi_I(x_1) \oplus \pi_I(x_2) = x_1 \oplus \pi_I(x_2)$ and $\pi_J\big(\pi_I(x)\big) = x_1 \oplus \pi_J\big(\pi_I(x_2)\big)$. By Proposition 2.4, $\pi_J\big(\pi_I(x_2)\big) = 0$.

Thus $I' \vee J'$ is a complement of $I \wedge J$. We have proved that for $I, J \in \mathrm{SCI}(P)$, $I \wedge J \in \mathrm{SCI}(P)$, $I' \vee J' \in \mathrm{SCI}(P)$.

Let $K, L \in \mathrm{SCI}(P)$. Then there are $I, J \in \mathrm{SCI}(P)$ such that $I' = K$, $J' = L$. Consider previous part of the proof: the complement of $K \vee L = I' \vee J'$ exists in $\mathrm{SCI}(P)$ and is equal to $I \wedge J = K' \wedge L'$. This shows that $K \vee L$ is subcentral.

Thus, $\mathrm{SCI}(P)$ is sublattice of $I(P)$ and De Morgan laws hold. This completes the proof. □

Corollary 2.1. *Let* $A, B \in \mathrm{SCI}(P)$. *Then for every* $x \in P$, $\pi_{A \wedge B}(x) = \pi_A\big(\pi_B(x)\big)$ *and* $\pi_{A \vee B}(x) = \pi_B\big(\pi_{A'}(x)\big) \oplus \pi_A(x)$.

Proof. Consider the proof of the previous proposition. Put $A = I$, $B = J$, $A = I'$, $B = J'$, respectively. □

Theorem 2.1. (Main result) $\big(\mathrm{SCI}(P), \wedge, \vee, {}', \{0\}, P\big)$ *is a Boolean algebra.*

Proof. Let $x \in P$. Then, by Corollary 2.1 and Proposition 2.3,

$$\begin{aligned}
\pi_{(I\wedge J)\vee(I\wedge K)}(x) &= \pi_{I\wedge K}\bigl(\pi_{I'\vee J'}(x)\bigr) \oplus \pi_{I\wedge J}(x) \\
&= \pi_{I\wedge K}\bigl(\pi_{J'}\bigl(\pi_I(x)\bigr) \oplus \pi_{I'}(x)\bigr) \oplus \pi_{I\wedge J}(x) \\
&= \pi_{I\wedge K\wedge J'\wedge I}(x) \oplus \pi_{I\wedge K\wedge I'}(x) \oplus \pi_{I\wedge J}(x) \\
&= \pi_{I\wedge K\wedge J'}(x) \oplus \pi_{I\wedge J}(x) = \pi_I\bigl(\pi_{K\wedge J'}(x)\bigr) \oplus \pi_I\bigl(\pi_J(x)\bigr) \\
&= \pi_I\bigl(\pi_{K\wedge J'}(x) \oplus \pi_J(x)\bigr) = \pi_I\bigl(\pi_{K\vee J}(x)\bigr) \\
&= \pi_{I\wedge(K\vee J)}(x)\,.
\end{aligned}$$

By Proposition 2.2, $I \wedge (K \vee J) = (I \wedge J) \vee (I \wedge K)$. □

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