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# TRIANGULAR NORMS WITH CONTINUOUS DIAGONALS

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ABSTRACT. It is an old open question whether a t-norm with a continuous diagonal must be continuous [7]. We give a partial positive answer for t-norms with noncontinuous additive generators (constructed by the technique of [1, 3]). Besides this, we give necessary and sufficient conditions for a function to be the diagonal of a continuous t-norm, and we characterize all continuous t-norms with a given diagonal (according to [6]).

## 1. Basic notions

We shall give basic definitions and introduce some notations.

**DEFINITION 1.1.** A *t-norm* is a mapping $T\colon [0,1]^2 \to [0,1]$ such that (for every $x, y, z \in [0,1]$):

(1) $T(x,y) = T(y,x)$ (commutativity);
(2) $T\big(T(x,y),z\big) = T\big(x,T(y,z)\big)$ (associativity);
(3) $x \le y$ implies $T(x,z) \le T(y,z)$ (monotonicity);
(4) $T(1,x) = x$ (boundary conditions).

A *diagonal* $d\colon [0,1] \to [0,1]$ of a mapping $T\colon [0,1]^2 \to [0,1]$ is the mapping $d(x) = T(x,x)$.

We shall denote by id the identity function on $[0,1]$ and by $F(d)$ the set of all fixed points of the diagonal $d$, i.e.,

$$F(d) = \big\{x \in [0,1]\colon d(x) = x\big\}\,.$$

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It can be easily checked that a diagonal is a nondecreasing function $d\colon [0,1] \to [0,1]$ bounded by the identity ($d \leq \mathrm{id}$) such that $d(0) = 0$ and $d(1) = 1$ (i.e., $\{0,1\} \subset F(d)$).

For a function $f$, we shall denote by $f^n$ the composition of $n$ copies of $f$.

**Definition 1.2.** A continuous t-norm $T$ with the diagonal $d$ is called

- *Archimedean*, if $\lim\limits_{n\to\infty} d^n(x) = 0$ for every $x \in [0,1)$;
- *nilpotent*, if for every $x \in [0,1)$ there is a natural number $n$ such that $d^n(x) = 0$;
- *strict*, if $T(x,z) < T(y,z)$ for every $x,y,z \in [0,1]$ with $x < y$ and $z > 0$ (i.e., $T$ is strictly increasing at each variable on $(0,1]^2$).

Let us review some basic properties of diagonals for various classes of t-norms.

1) The diagonal of a continuous t-norm is a continuous nondecreasing function $[0,1] \xrightarrow{\text{onto}} [0,1]$ such that $d \leq \mathrm{id}$.
2) The set of fixed points of the diagonal of an Archimedean t-norm is trivial, i.e., $F(d) = \{0,1\}$.
3) The diagonal of a nilpotent t-norm is constant on a neighbourhood of $0$.
4) The diagonal of a strict t-norm is strictly increasing.

**Definition 1.3.** An *additive generator* of a t-norm $T$ is a strictly decreasing function $f\colon [0,1] \to [0,\infty]$ such that (for every $x,y,z \in [0,1]$)

$$T(x,y) = f^{[-1]}\big(f(x) + f(y)\big), \qquad \text{where} \quad f^{[-1]}(z) = \inf f^{-1}[0,z]\,.$$

From the boundary conditions of t-norms it follows that $f(1) = 0$ for every additive generator. It is easy to see that a positive multiple of an additive generator is an additive generator of the same t-norm and that the value of the generator at $0$ is not essential. It is known that a continuous Archimedean t-norm has a continuous additive generator and that a continuous strictly decreasing function $f\colon [0,1] \to [0,\infty]$ with $f(1) = 0$ is a generator of a continuous t-norm [7, 3]. For a continuous generator $f$ we have either $T(x,y) = f^{-1}\big(f(x) + f(y)\big)$ (if $f(x) + f(y) \leq f(0)$) or $T(x,y) = 0$ (otherwise). As a special case, $f(x) = f\big(d(x)\big)/2$ whenever $d(x) > 0$.

## 2. Characterization of diagonals of continuous t-norms

In this section we give characterizations of functions which are diagonals of continuous (strict, nilpotent, Archimedean, resp.) t-norms. Moreover, we will

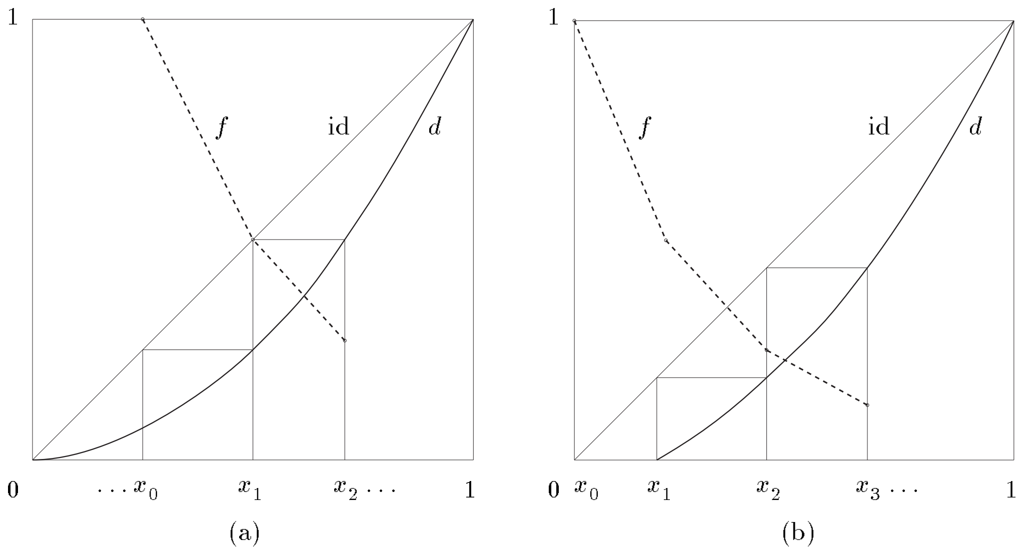


FIGURE 1. Examples of diagonals of a strict (a) and a nilpotent (b) t-norm and constructions of respective additive generators.

characterize all continuous t-norms with a given diagonal. The proofs and more details can be found in [6].

Let us start with strict t-norms (see Fig. 1(a)).

**THEOREM 2.1.** *A continuous nondecreasing function* $d\colon [0,1] \xrightarrow{\text{onto}} [0,1]$ *with* $d \le \mathrm{id}$ *is a diagonal of a strict t-norm iff* $F(d) = \{0,1\}$ *and* $d$ *is strictly increasing.* (*In this case,* $d$ *is an automorphism of* $([0,1],\le)$ *with* $d(x) < x$ *for* $x \in (0,1)$.)

The proof and the characterization of all t-norms with a given diagonal follows from the construction of an additive generator. We can choose an arbitrary positive value of the additive generator $f$ for an arbitrary point $x \in (0,1)$. Put $x_1 = 1/2$ and $f(x_1) = 1/2$ (see Fig. 1(a)). Define inductively $x_n$ ($n = 0, \pm 1, \pm 2, \dots$) by $x_n = d(x_{n+1})$. Since $f(x) = f\big(d(x)\big)/2$ whenever $d(x) > 0$, we obtain $f(x_n) = 1/2^n$ for every integer $n$. We can define $f$ on $(x_0, x_1)$ as an arbitrary strictly decreasing function continuous on $[x_0, x_1]$ and by the condition $f(x) = f\big(d(x)\big)/2$ we consecutively obtain the definition of $f$ on $(0,1)$.

The following proposition shows that for a large class of strict t-norms the behavior of the diagonal is restricted also in outer points. For the formulation we need other type of a generator of a t-norm.

**DEFINITION 2.2.** A *multiplicative generator* of a t-norm $T$ is a strictly increasing function $f\colon [0,1] \to [a,1]$ for some $a \in [0,1)$ such that (for every $x, y, z \in [0,1]$)

$$T(x,y) = f^{[-1]}\big(f(x)\cdot f(y)\big), \qquad \text{where} \quad f^{[-1]}(z) = \inf f^{-1}[z,1]\,.$$

**Proposition 2.3.** *If the multiplicative generator of a strict t-norm $T$ has nonzero finite derivatives at $0, 1$, then $d'_+(0) = 0$, $d'_-(1) = 2$ (i.e., the same as for the product t-norm $T(x, y) = xy$).*

Now, let us proceed to nilpotent t-norms (see Fig. 1(b)).

**Theorem 2.4.** *A continuous nondecreasing function $d\colon [0,1] \xrightarrow{\text{onto}} [0,1]$ with $d \le \mathrm{id}$ is a diagonal of a nilpotent t-norm iff $F(d) = \{0,1\}$, $d^{-1}(0) \neq \{0\}$ and $d$ is strictly increasing on $d^{-1}(0,1)$.*

The proof and the characterization of all t-norms with a given diagonal is similar to the case of strict t-norms. (The only difference is that we start with $x_1 = \max d^{-1}(0)$ and we obtain the sequence $x_n$ for $n = 0, 1, \ldots$ with $x_0 = 0$.)

Since an Archimedean t-norm is either strict or nilpotent [7, 3], we obtain the following theorem.

**Theorem 2.5.** *A continuous nondecreasing function $d\colon [0,1] \xrightarrow{\text{onto}} [0,1]$ with $d \le \mathrm{id}$ is a diagonal of an Archimedean t-norm iff $F(d) = \{0,1\}$ and $d$ is strictly increasing on $d^{-1}(0,1)$.*

Since every continuous t-norm is a subdirect product of Archimedean t-norms [4], we obtain the following characterization (see Fig. 2).

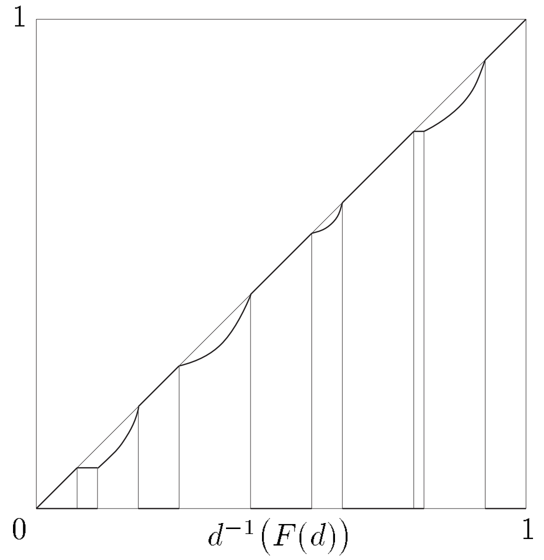


Figure 2. An example of the diagonal of a continuous t-norm.

**Theorem 2.6.** *A continuous nondecreasing function $d\colon [0,1] \xrightarrow{\text{onto}} [0,1]$ with $d \le \mathrm{id}$ is a diagonal of a continuous t-norm iff $d$ is strictly increasing on $[0,1] \setminus d^{-1}\big(F(d)\big)$.*

## 3. Noncontinuous t-norms with (almost) continuous diagonals

In this section we will give a partial answer to the problem whether there is a noncontinuous t-norm with a continuous diagonal. We solve this problem positively in case the boundary points of the domain are not considered, i.e., if we accept that the diagonal is not continuous at $1$.

Denote by $R(f)$ the range of the function $f$ and let us start with basic lemma (see also [2]).

**LEMMA 3.1.** *For every strictly decreasing function $f\colon [0,1] \to [0,\infty]$ with $f(1)=0$ the function $T\colon [0,1]^2 \to [0,1]$ defined by*

$$T(x,y) = f^{[-1]}\big(f(x)+f(y)\big), \qquad \text{where} \quad f^{[-1]}(z) = \inf f^{-1}[0,z]\,,$$

*is commutative, nondecreasing in each variable and $T(1,x)=x$ for every $x \in [0,1]$. It is a t-norm (i.e., $f$ is an additive generator of a t-norm) if the following range condition is fulfilled:*

$$R(f)+R(f) \subset R(f) \cup \big[f(0_+),\infty\big]\,.$$

*The diagonal of $T$ is continuous at $x \in [0,1]$ iff*

$$2\cdot\big(f(x_+),f(x_-)\big) \cap R(f)$$

*is at most one-element.*

P r o o f. It can be easily checked that $T$ is commutative, nondecreasing at each variable and that $T(1,x)=x$ for every $x\in[0,1]$.

Now, suppose that $R(f)+R(f) \subset R(f) \cup \big[f(0_+),\infty\big]$ and we shall prove the associativity of $T$. Since the value $f(0)$ is not essential, we may (and will) suppose without any loss of generality that $f(0)=f(0_+)$. For every $x,y\in[0,1]$ either $f(x)+f(y)\in R(f)$ and therefore $T(x,y)=f^{-1}\big(f(x)+f(y)\big)$ or $f(x)+f(y)>f(0)$ and therefore $T(x,y)=0=f^{-1}\big(f(0)\big)$. In both cases, $T(x,y)=f^{-1}\big(\min\{f(0),f(x)+f(y)\}\big)$. Hence, $T\big(T(x,y),z\big) = f^{-1}\big(\min\{f(0),f(x)+f(y)+f(z)\}\big) = T\big(x,T(y,z)\big)$.

Finally, denote by $d$ the diagonal of $T$ and suppose that $2\cdot\big(f(x_+),f(x_-)\big)\cap R(f)$ is at least two-element. This is equivalent to the statement that either $f\big(d(x)_+\big) > 2f(x_+)$ or $f\big(d(x)_-\big) < 2f(x_-)$. The first condition is equivalent to $d(x) < \inf f^{-1}\big[0,2f(x_+)\big]$ and therefore to $d(x)<d(y)$ for every $y\in[0,1]$ with $y>x$; this means that $d$ is not continuous at $x$ from the right (if $x<1$). In an analogous way, $f\big(d(x)_-\big) < 2f(x_-)$ is equivalent to discontinuity of $d$ at $x$ from the left. □

As the following example shows, the range condition in Lemma 3.1. is not necessary to obtain a t-norm (see also [8]).

EXAMPLE 3.2. Put

$$f(x) = \begin{cases} 1 - \frac{x}{2} & \text{if} \quad x \in \left[0, \frac{1}{2}\right), \\ \frac{1}{2} - \frac{x}{2} & \text{if} \quad x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Then $f$ is an additive generator of a t-norm and $R(f) + R(f) \not\subset R(f) \cup \left[f(0_+), \infty\right]$.

We have shown that the range $R(f)$ plays an important role. If we want a continuous diagonal, then "twice a hole should be a hole" (or "half of an interval should be a part of the range"). On the other side, we can ensure to obtain a t-norm if the range is closed to sums and therefore "half of a hole is a hole". Hence we may try some invariance of the range of the additive generator with respect to going to halves. We will study noncontinuous generators with ranges to be unions of a sequence of open intervals and of a sequence of points (in the "holes") such that every subsequent interval is "the half" of the proceeding one.

We shall start with one interval in the range.

EXAMPLE 3.3. Let $f$ be the function given in Fig. 3. It is an additive generator of the t-norm

$$T(x,y) = \begin{cases} 0 & \text{if} \quad \max\{x,y\} < 1 \text{ and } x + y \leq \frac{3}{2}, \\ x + y - \frac{3}{2} & \text{if} \quad \max\{x,y\} < 1 \text{ and } x + y > \frac{3}{2}, \\ \min\{x,y\} & \text{if} \quad \max\{x,y\} = 1, \end{cases} \qquad x, y \in [0,1],$$

discontinuous at the points of $(0,1] \times \{1\} \cup \{1\} \times (0,1]$.

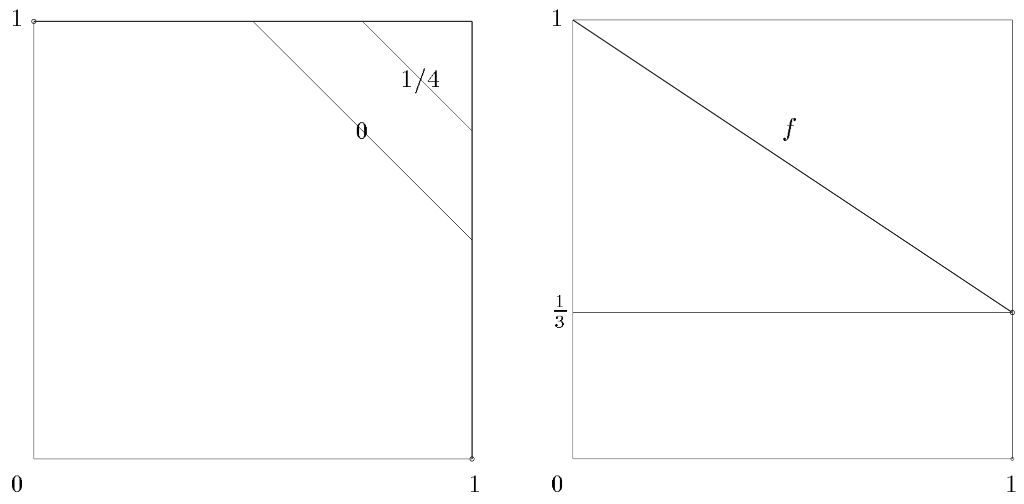


FIGURE 3. An example of a noncontinuous t-norm with the diagonal continuous on $[0,1)$ and its additive generator.

We note that an arbitrary strictly decreasing function $f\colon [0,1] \to [0,\infty]$ with the range $R(f) = \{0\} \cup (a,b]$ for some $0 < a < b \leq \infty$ is an additive generator of a t-norm with the set of points of discontinuity equal to $(0,1] \times \{1\} \cup \{1\} \times (0,1]$.

It should be noted that such t-norms are contractions of continuous t-norms: For every $c > 1$ we can find a strictly decreasing continuous function $\bar{f}$ on $[0,c]$ such that $\bar{f}|_{(0,1)} = f|_{(0,1)}$ and $\bar{f}(c) = 0$; if we put $f_2(x) = \bar{f}(cx)$ for every $x \in [0,1]$, then $f_2$ is an additive generator of a continuous t-norm $T_2$ and (see [5])

$$T(x,y) = \begin{cases} c \cdot T_2\left(\frac{x}{c}, \frac{y}{c}\right) & \text{if} \quad \max\{x,y\} < 1\,, \\ \min\{x,y\} & \text{if} \quad \max\{x,y\} = 1\,. \end{cases}$$

Using one interval in the range of the additive generator, we obtain a t-norm with points of discontinuity only in the boundary. If we would like to find a t-norm not continuous on $(0,1)^2$, we should take a generator not continuous on $(0,1)$.

The following example uses two intervals in the range of the additive generator.

EXAMPLE 3.4. Let $f$ be the function given in Fig. 4. It is an additive generator of the t-norm

$$T(x,y) = \begin{cases} 0 & \text{if} \quad \max\{x,y\} < 1 \text{ and } \min\{x,y\} < \frac{1}{2}\,, \\ \frac{1}{2}(x+y-1) & \text{if} \quad \max\{x,y\} < 1 \text{ and } \min\{x,y\} \geq \frac{1}{2}\,, \\ \min\{x,y\} & \text{if} \quad \max\{x,y\} = 1\,, \end{cases} \qquad x,y \in [0,1]\,,$$

discontinuous at the points of $(0,1] \times \{1\} \cup \{1\} \times (0,1] \cup \left(\frac{1}{2},1\right) \times \left\{\frac{1}{2}\right\} \cup \left\{\frac{1}{2}\right\} \times \left(\frac{1}{2},1\right)$.

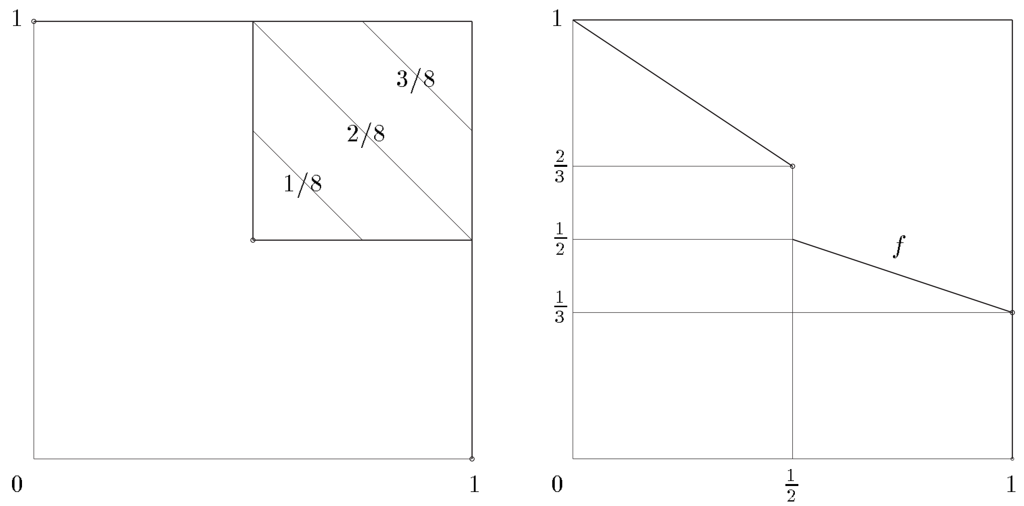


FIGURE 4. An example of a noncontinuous t-norm with continuous diagonal (considering $(0,1)^2$) and its additive generator.

Hence, we have an example of a t-norm that is not continuous on $(0,1)^2$ while its diagonal is continuous on $(0,1)$ (moreover, this t-norm is continuous at the points of $\{(x,x) \colon x \in [0,1)^2\}$).

The continuity of the diagonal at $1$ needs in the range of the additive generator at least an infinite sequence of intervals converging to $0$. Unfortunately, the following lemma shows that our construction cannot be used already for three intervals.

**LEMMA 3.5.** *Let $r \in (0,\infty)$ and let $f \colon [0,1] \to [0,\infty]$ with $f(1) = 0$ be a strictly decreasing function such that $r, 2r, 4r \in R(f)$, $3r \notin R(f)$ and $4r$ is a bilateral limit point of $R(f)$ (i.e., $f$ is continuous at $f^{-1}(4r)$). Then $f$ is not an additive generator of a t-norm.*

Proof. We shall show that the function $T$ defined in Lemma 3.1 is not associative. Indeed, for $x = f^{-1}(r)$ and $y = f^{-1}(2r)$ we obtain $T\big(T(x,x),y\big) = f^{-1}(4r) \neq f^{[-1]}\big(r + f\big(T(x,y)\big)\big) = T\big(x, T(x,y)\big)$. □

It should be noted that there are constructions of an additive generator with three (or more) intervals in the range, as the following lemma shows.

**LEMMA 3.6.** *Let $n$ be a positive integer, $A \in \big(1, 1+\frac{1}{n-1}\big)$ and let $f \colon [0,1] \to [0,\infty]$ be a strictly decreasing function such that $R(f) = \{0\} \cup \bigcup\limits_{k=1}^{n} k(1,A]$. Then $f$ is an additive generator of a t-norm.*

Proof. It is clear that $R(f) + R(f) = \{0\} \cup \bigcup\limits_{k=1}^{2n} k(1,A] \subset R(f) \cup \big[f(0),\infty\big]$. Hence, according to Lemma 3.1, $f$ is an additive generator. □

It seems to be an open problem, whether our construction (using the invariance to "going to halves for holes") can be improved by considering the range of an additive generator to be a more general set.

On the other side, there are some negative results. In [3] it is stated that an additive generator fulfilling the range condition in Lemma 3.1 generates a t-norm with the diagonal continuous at $1$ iff it is continuous. Moreover, in [8] it is claimed that an additive generator continuous from the left should fulfill the range condition in Lemma 3.1.

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