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SEARCHING THE FRONTIER OF THE PYTHAGOREAN SYSTEM

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ABSTRACT. It is known that the perfect fifth and octave can be expressed as X^4Y^3 and X^7Y^5 respectively, where $X = 256/243$ is the minor and $Y = 9/8\colon X = 3^7/2^{11}$ the major Pythagorean semitone. In this paper, the Pythagorean system is studied for the case when couples of semitones need not be considered in rational numbers; we deal with semitone couples (X, Y) in algebraic numbers expressed by dth roots, $d = 1, 2, 3, \ldots$.

1. Anatomy of the whole Pythagorean tone

The Pythagorean system was developed in antiquity and satisfactorily formalized by Boethius (480 – 524). Musically, this system is based on the perfect fifths (or, equivalently, on perfect fourths) and octaves.

EXAMPLES. Usually, under the Pythagorean Tuning (Scale) (cf., e.g., [1], [2], [5]), we mean one of the following number sequences (physically: the relative frequencies of tones within the octave $[1/1, 2/1)$): $P_2, P_3, P_5, P_7, P_{12}, P_{17}, P_{22}, P_{27}$, $P_{31} \subset \mathcal{P} = \{2^\alpha 3^\beta \colon \alpha, \beta \in \mathbb{Z}\}$, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

$P_2 = \langle 1/1\,, 3/2\rangle\,,$

$P_3 = \langle 1/1, 4/3, 3/2\rangle\,,$

$P_5 = \langle 1/1, 9/8, 81/64, 3/2, 27/16\rangle\,,$

$P_7 = \langle 1/1, 9/8, 81/64, 4/3, 3/2, 27/16, 16/9\rangle\,,$

$P_{12} = \langle 1/1, 2\,187/2\,048, 9/8, 32/27, 81/64, 4/3, 729/512, 3/2, 6\,561/4\,096, 27/16, 16/9, 243/128\rangle\,,$

$P_{17} = \langle 1/1, 256/243, 2\,187/2\,048, 9/8, 32/27, 19\,683/16\,384, 81/64, 4/3, 1\,024/729, 729/512, 3/2, 128/81, 6\,561/4\,096, 27/16, 16/9, 59\,049/32\,768, 243/128\rangle\,,$

$P_{22} = \langle 1/1, 256/243, 2\,187/2\,048, 65\,536/59\,049, 9/8, 32/27, 19\,683/16\,384, 8\,192/6\,561,$

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81/64, 4/3, 177 147/131 072, 1 024/729, 729/512, 3/2, 128/81, 6 561/4 096, 32 768/19 683, 27/16, 16/9, 59 049/32 768, 4 096/2 187, 243/128⟩ ,

P_{27} = ⟨1/1, 531 441/524 288, 256/243, 2 187/2 048, 65 536/59 049, 9/8, 4 782 969/4 194 304, 32/27, 19 683/16 384, 8 192/6 561, 81/64, 4/3, 177 147/131 072, 1 024/729, 729/512, 262 144/177 147, 3/2, 1 594 323/1 048 576, 128/81, 6 561/4 096, 32 768/19 683, 27/16, 14 348 907/8 388 608, 16/9, 59 049/32 768, 4 096/2 187, 243/128⟩ ,

P_{31} = ⟨1/1, 531 441/524 288, 256/243, 2 187/2 048, 1 162 261 467/1 073 741 824, 9/8, 4 782 969/4 194 304, 32/27, 19 683/16 384, $3^{21}/2^{33}$, 8 192/6 561, 81/64, 43 046 721/33 554 432, 4/3, 177 147/131 072, 1 024/729, 729/512, 387 420 489/268 435 456, 3/2, 1 594 323/1 048 576, 128/81, 6 561/4 096, $3^{20}/2^{31}$, 27/16, 14 348 907/8 388 608, 16/9, 59 049/32 768, $3^{22}/2^{34}$, 4 096/2 187, 243/128, 129 140 163/67 108 864⟩ .

We shall deal with the usual multiplicative group $\mathcal{L}$, $\big((0,\infty),\cdot,1,\leq\big)$, equipped with the usual order. In Figure 1, we see the Boethius scheme, the skeleton of the Pythagorean whole tone, cf. [5]. If we denote by $Q = 4/3$ the *perfect fourth* and by $O = 2$ the *octave*, then $R = O/Q = 3/2$ is the *perfect fifth*, $T = R/Q = 9/8$ the *Pythagorean whole tone, tonos*; $U = Q/T^2 = 256/243$ the *minor Pythagorean semitone, limma*; $V = T/U = 2\,187/2\,048$ the *major Pythagorean semitone, apotome*; $Z = \sqrt{U} = 3\sqrt{2}/4$ *diaschizma*, $K = V/U = 531\,441/524\,288$ *comat*; $W = \sqrt{K} = \sqrt{531\,441/524\,288}$ *schizma. We see that the Boethius theory of music does not avoid irrational numbers and it can be considered as a microtonal theory in the present modern sense.* Recall that one octave has the length 1 200 cents ($x \mapsto 1\,200\log_2 x$, $1 \leq x \leq 2$). So, $Z \approx 1.026400479 \approx 45.11$ cents, $K \approx 1.013643265 \approx 23.46$ cents; $W \approx 1.006798522 \approx 11.73$ cents. For comparison: $81/80 = 1.0125 \approx 21.51$ cents.

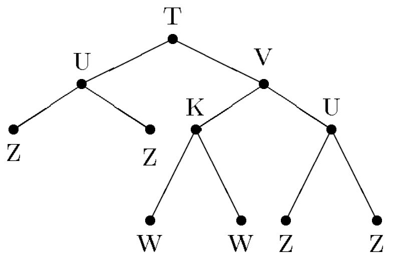

FIGURE 1. Boethius scheme of the Pythagorean whole tone

By a *generalized sequence* (*net*) with values in $\mathcal{L}$ we mean a function from I to $\mathcal{L}$, where I is a directed partially ordered set. For the generalized sequences in general, cf. [3]

The semitones U and V are unique rational numbers which enables to Pythagorean Tuning P_{31} (and also $P_{12}, P_{17}, P_{22}, P_{28}$; for P_{17}, cf. [1], [2], [4]) to be expressed via a geometrical generalized sequence (which we abbreviate as GGS) in rational numbers with some natural conditions, more precisely, in the form

$$\{\Gamma_i = \alpha \cdot X^{m_i} Y^{n_i} : i \in \mathbb{Z}\} \tag{1}$$

where X, Y are positive rational numbers, α a positive real number, and (possibly not unique for every $i \in \mathbb{Z}$) $(m_i, n_i) \in I \subset \mathbb{Z}^2$ are such that

$$m_i + n_i = i\,, \quad m_i \le m_{i+1}\,, \qquad n_i \le n_{i+1}\,, \quad i \in \mathbb{Z}\,. \tag{2}$$

The additional conditions are

$$\alpha = 1\,, \tag{3}$$

$$X^{m_{12}} Y^{n_{12}} = 2\,, \tag{4}$$

$$X^{m_7} Y^{n_7} = 3/2\,, \tag{5}$$

$$\Gamma_{j+12k} = 2^k \cdot \Gamma_j\,, \qquad j = 0, 1, 2, \ldots, 11\,, \quad k \in \mathbb{Z}\,. \tag{6}$$

Condition (3) says that we consider *no shift of the fundamental tone.* Condition (4) shows that *the 12th degree is the octave.* Condition (5) asserts that the seventh degree is the *perfect fifth (3/2).* Condition (6) describes the so called *octave equivalency.*

Twelve qualitative musical degrees are the characteristic attribute of the European tone systems and culture. They are: the unison (octave), minor second, major second, minor third, major third, fourth, tritone, fifth, minor sixth, major sixth, minor seventh, major seventh. Each degree may contain one or more tuning values The GGSs $P_{17}, P_{22}, P_{28}, P_{31}$ have $17, 22, 28, 31$ values for 12 degrees within the octave $[1, 2)$, respectively. There are different I for $P_{12}, P_{17}, P_{22}, P_{28}, P_{31}$. In this paper, we shall consider GGSs only for pairs (X, Y) (two different semitones, not three or more). We can also see examples of index sets I for GGGs in Figure 2 (Table 3; a sequence) and in Figure 2 (Table 4). We deal with index sets directed both forward and backward (two sided GGS).

The Pythagorean Tunings $P_{12}, P_{17}, P_{22}, P_{28}, P_{31}$ are GGSs, the solutions of the system of equations (1), (2), (3), (4), (5) with rational (X, Y). Therefore, the question arises a satisfying: *does there exist* GGS, *the system of equations* (1), (2), (3), (4), (5) *with irrational* (X, Y)? In this paper, we answer this question positively for semitone couples (X, Y) expressed with algebraic numbers.

The Pythagorean tunings $P_{17}, P_{22}, P_{28}, P_{31}$ *are* many valued (fuzzy) coding information systems in music. For more explanations concerning the philosophy of fuzziness of tunings in music see [2]. The important and obvious fact is that the theoretical tunings (and operations among them) never occur in real live music because there are always at least two different tone systems in the sound interaction during the performance: the tone subset A of the tuned system itself, and physically, the system of overtones B of tones in A. Clearly, always $A \neq$ B.

2. The second level: algebraic numbers

We note, [1], that P_i, $i = 2, 3, 5, 7, 12, 17, 22, 28, 31$, can be obtained as the solution of the following Problem 1 by using the enlargement algorithm based on the fifth $X^{m_7} Y^{n_7} = 3/2$ and the octave $X^{m_{12}} Y^{n_{12}} = 2$, i.e.:

$$Q_{[k_1,k_2]} = \left\{(3/2)^k \pmod 2) \colon k \in [k_1, k_2]\,,\ k_1 < k_2\,,\ k_1, k_2 \in \mathbb{Z}\right\}.$$

For instance, $k = 1, 2, 3, 4, 5$ for P_5.

Problem 1. Find rational numbers X, Y such that (3), (4), (5) hold for some nonnegative integers m_i, n_i such that

$$m_i + n_i = i\,, \quad i = 0, 7, 12\,, \quad 0 \le m_7 \le m_{12}\,, \quad 0 \le n_7 \le n_{12}\,.$$

Problem 2. To find all the triplets (m_{12}, m_7, d) of nonnegative integers such that

$$7m_{12} - 12m_7 = d\,, \quad 0 \le 7 - m_7 \le 12 - m_{12}\,, \quad 0 \le m_7 \le m_{12}\,.$$

The solution of Problem 1 is given by the solution of Problem 2. Indeed, for $d = 0$, we have $m_{12} = 12, m_7 = 7$, i.e., Equal Temperament ($X = Y = \sqrt[12]{2}$) which does not contain any perfect fifth 3/2, (analogously for $d = 12$). If $d \neq 0$ and (m_{12}, m_7, d) is a solution of Problem 2, then we obtain the solution of (3), (4), (5) as follows:

$$X = 2^{E_{2,X}} 3^{E_{3,X}}\,, \qquad Y = 2^{E_{2,Y}} 3^{E_{3,Y}}\,, \tag{7}$$

where $E_{2,X} = (19 - m_7 - m_{12})/d$, $E_{3,X} = (m_{12} - 12)/d$, $E_{2,Y} = (-m_{12} - m_7)/d$, $E_{3,Y} = m_{12}/d$. The numbers X, Y are rational solutions of Problem 1 if and only if $d = 1$ (or $d = -1$, the symmetrical case). For $d = 1$, the values are $X = U$, $Y = V$.

Now it is possible to extend Pythagorean Tuning (in any form: P_2 or $\ldots$ or P_{31}) to algebraic numbers: to solve Problem 1a, where

Problem 1a. Find algebraic numbers X, Y such that (4), (5) hold for some nonnegative integers m_i, n_i such that

$$m_i + n_i = i\,, \quad i = 0, 7, 12\,, \quad 0 \le m_7 \le m_{12}\,, \quad 0 \le n_7 \le n_{12}\,.$$

THEOREM 1. *For every $d \in \mathbb{Z}$, there exists either a unique (X_d, Y_d) or no solution of Problem 1a such that X_d, Y_d are expressed via dth roots, multiplication, and/or dividing by the numbers 2 and 3.*

P r o o f . The solutions of the Diophantine equation

$$7m_{12} - 12m_7 = d, \quad 0 \le m_7 \le m_{12},$$

where $d, m_7, m_{12} = 0, 1, 2, 3, 4 \ldots$ are collected in Table 1. From (2), we obtain the values n_{12}, n_7. Take only the solutions such that $n_{12} \ge n_7$. Then the solutions of Problem 2 are collected in Table 2. From (7) we obtain the solutions of Problem 1a (the case $d < 0$ is to be found by symmetry). ☐

TABLE 1. Solutions of $7m_{12} - 12m_7 = d$

m_7	d	
0	7, 14, 21, 28, 35,	42, 49, 56, 63, 70, 77
1	2, 9, 16, 23, 30,	37, 44, 51, 58, 65
2	4, 11, 18, 25,	32, 39, 46, 53
3	6, 13, 20,	27, 34, 41
4	1, 8, 15,	22, 29
5	3, 10,	17
6	5	

TABLE 2. Solutions of Problem 2

d	1	2	3	4	5	6	7	8	9	10	11	
m_{12}	7	2	9	4	11	6	1	8	3	10	5	
n_{12}	5	10	3	8	1	6	11	4	9	2	7	
m_7	4	1	5	2	6	3	0	4	1	5	2	
n_7	3	6	2	5	1	4	7	3	6	2	5	
d	13	14	15	16	18	20	21	23	25	28	30	35
m_{12}	7	2	9	4	6	8	3	5	7	4	6	5
n_{12}	5	10	3	8	6	4	9	7	5	8	6	7
m_7	3	0	4	1	2	3	0	1	2	0	1	0
n_7	4	7	3	6	5	4	7	6	5	7	6	7

TABLE 3. $d = 2$, $P_7(\bigcirc)$, $(\sqrt{2}) \cdot P_5$ $(\bullet)$

C	$X_2^0 Y_2^0$	1	0.00	$U^0 K^0$	$\bigcirc$
$C_\sharp$	$X_2^0 Y_2^1$	$3/\sqrt{8}$	101.96	$U^1 K^{1/2}$	$\bullet$
D	$X_2^0 Y_2^2$	$9/8$	203.91	$U^2 K^1$	$\bigcirc$
$E_\flat$	$X_2^0 Y_2^3$	$27/\sqrt{512}$	305.86	$U^3 K^{3/2}$	$\bullet$
E	$X_2^0 Y_2^4$	$81/64$	407.82	$U^4 K^2$	$\bigcirc$
F	$X_2^1 Y_2^4$	$4/3$	498.04	$U^5 K^2$	$\bigcirc$
$F_\sharp$	$X_2^1 Y_2^5$	$\sqrt{2}$	600.00	$U^6 K^{5/2}$	$\bullet$
G	$X_2^1 Y_2^6$	$3/2$	701.96	$U^7 K^3$	$\bigcirc$
$G_\sharp$	$X_2^1 Y_2^7$	$9/\sqrt{32}$	803.91	$U^8 K^{7/2}$	$\bullet$
A	$X_2^1 Y_2^8$	$27/16$	905.86	$U^9 K^4$	$\bigcirc$
$B_\flat$	$X_2^1 Y_2^9$	$81/\sqrt{2048}$	1 007.82	$U^{10} K^{9/2}$	$\bullet$
B	$X_2^1 Y_2^{10}$	$243/128$	1 109.78	$U^{11} K^5$	$\bigcirc$
C'	$X_2^2 Y_2^{10}$	2	1 200.00	$U^{12} K^5$	

THEOREM 2. *Denote by $\mathcal{G}_d$ the set theoretic union of all GGSs which are generated by the solution (X_d, Y_d) of Problem 1a, d in Table 2. Then*

$$\mathcal{G}_1 = \mathcal{P} = \{2^\alpha 3^\beta : \alpha, \beta \in \mathbb{Z}\},$$
$$\mathcal{G}_2 = \mathcal{P} \cup (\sqrt{2}) \cdot \mathcal{P},$$
$$\mathcal{G}_3 = \mathcal{P} \cup (\sqrt[3]{4}) \cdot \mathcal{P} \cup (3\sqrt[3]{2}) \cdot \mathcal{P},$$
$$\dots$$
$$\mathcal{G}_d = \bigcup_{i=1}^{d} \beta_{i,d} \cdot \mathcal{P},$$
$$\dots$$
$$\mathcal{G}_{35} = \bigcup_{i=1}^{35} \beta_{i,35} \cdot \mathcal{P},$$

where $\beta_{1,d}, \dots, \beta_{d,d}$ are pairwise incommensurable algebraic numbers for every d.

Proof. Consider the set

$$Q_{\mathbb{Z}} = \{(3/2)^k \pmod 2 \colon k \in \mathbb{Z}\} = \{(X^{m_7} Y^{n_7})^k \pmod{X^{m_{12}} Y^{n_{12}}} \colon k \in \mathbb{Z}\}.$$

Clearly, $Q_{\mathbb{Z}} = \mathcal{P}$ and $Q_{\mathbb{Z}} \supset Q_{[k_1,k_2]}$ for every $k_1, k_2 \in \mathbb{Z}$, $k_1 < k_2$.

TABLE 4. $d = 3$, P_5 (○), $(\sqrt[3]{32}/3) \cdot P_5$ (•), $(3/\sqrt[3]{4}) \cdot P_5$ (■)

C	$X_3^0 Y_3^0$	1	0.00	$U^0 K^0$	○
$D_\flat$	$X_3^1 Y_3^0$	$\sqrt[3]{32}/3$	98.04	$U^1 K^{1/3}$	•
$C_\sharp$	$X_3^0 Y_3^1$	$27/\sqrt[3]{16384}$	105.86	$U^1 K^{2/3}$	■
D	$X_3^1 Y_3^1$	$9/8$	203.91	$U^2 K^1$	○
$E_\flat$	$X_3^2 Y_3^1$	$3/\sqrt[3]{16}$	301.96	$U^3 K^{4/3}$	•
$D_\sharp$	$X_3^1 Y_3^2$	$243/\sqrt[3]{8388608}$	309.78	$U^3 K^{5/3}$	■
E	$X_3^2 Y_3^2$	$81/64$	407.82	$U^4 K^2$	○
F	$X_3^3 Y_3^2$	$27/\sqrt[3]{8192}$	505.86	$U^5 K^{7/3}$	•
$F_\sharp$	$X_3^4 Y_3^2$	$9/\sqrt[3]{256}$	603.91	$U^6 K^{8/3}$	■
G	$X_3^5 Y_3^2$	$3/2$	701.96	$U^7 K^3$	○
$A_\flat$	$X_3^6 Y_3^2$	$\sqrt[3]{4}$	800.00	$U^8 K^{10/3}$	•
$G_\sharp$	$X_3^5 Y_3^3$	$81/\sqrt[3]{931072}$	807.82	$U^8 K^{11/3}$	■
A	$X_3^6 Y_3^3$	$27/16$	905.86	$U^9 K^4$	○
$B_\flat$	$X_3^7 Y_3^3$	$9/\sqrt[3]{128}$	1 003.91	$U^{10} K^{13/3}$	•
B	$X_3^8 Y_3^3$	$3/\sqrt[3]{4}$	1 101.96	$U^{11} K^{14/3}$	■
C'	$X_3^9 Y_3^3$	2	1 200.00	$U^{12} K^5$	

The assertion $\mathcal{G}_1 = \mathcal{P}$ is obvious for $d = 1$, $X_1 = X = 256/243$, $Y_1 = Y = (9/8) : X_1$.

For $d = 2$, $X_2 = 256/243$, $Y_2 = 3/\sqrt{8}$. We have (for example, cf. Figure 2):

$$Q_{\mathbb{Z}} = \mathcal{P} = \bigcup_{p \in \mathbb{Z}} \bigcup_{q \in \mathbb{Z}} X_2^p Y_2^{-2q} .$$

Then

$$\mathcal{G}_2 = \mathcal{P} \cup (Y_2) \cdot \mathcal{P} = \mathcal{P} \cup \left(\sqrt{2}\right) \cdot \mathcal{P} .$$

For $d = 3$, $X_3 = \sqrt[3]{32}/3$, $Y_3 = 27/\sqrt[3]{16384}$. We have (for example, cf. Figure 3):

$$Q_{\mathbb{Z}} = \mathcal{P} = \bigcup_{p \in \mathbb{Z}} \bigcup_{q \in \mathbb{Z}} X_3^p Y_3^p \cdot X_3^{3q} .$$

Then

$$\mathcal{G}_3 = \mathcal{P} \cup \left(X_3\right) \cdot \mathcal{P} \cup \left(X_3^2\right) \cdot \mathcal{P} = \mathcal{P} \cup \left(\sqrt[3]{4}\right) \mathcal{P} \cup \left(3\,\sqrt[3]{2}\right) \cdot \mathcal{P} .$$

Similarly, for every $d = 4$, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 18, 20, 21, 23, 25, 28, 30, and 35. □

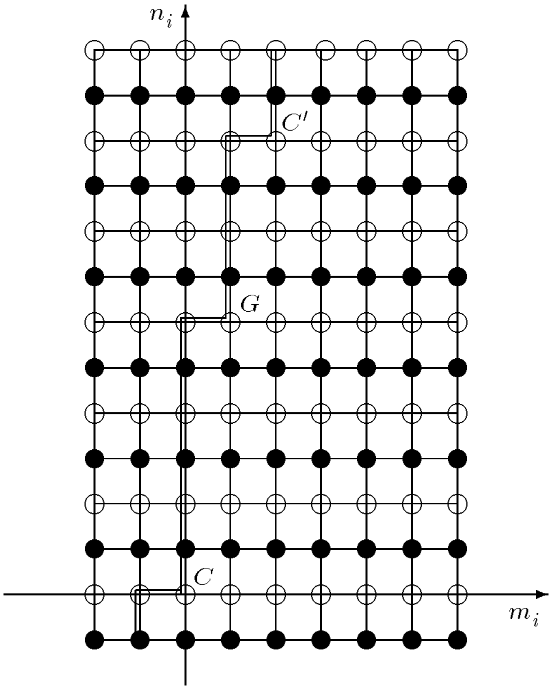

FIGURE 2. $\mathcal{G}_2 = \mathcal{P} \cup \left(\sqrt{2}\right) \cdot \mathcal{P}$

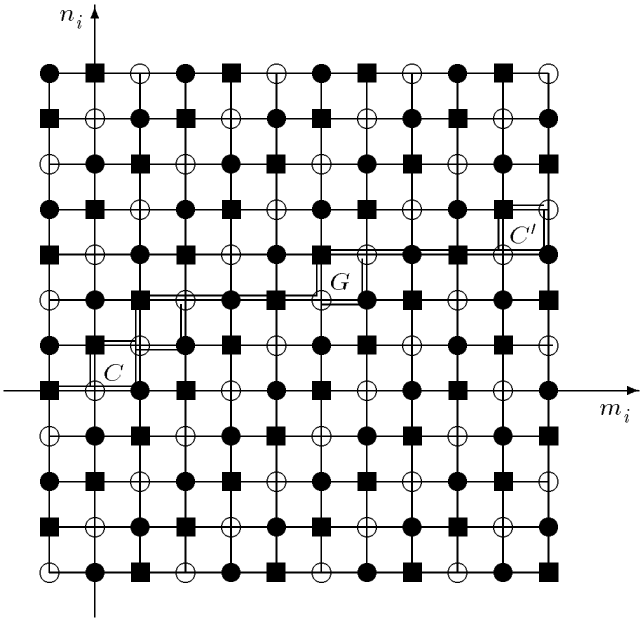

FIGURE 3. $\mathcal{G}_3 = \mathcal{P} \cup \left(\sqrt[3]{4}\right) \cdot \mathcal{P} \cup \left(3\sqrt[3]{2}\right) \cdot \mathcal{P}$

3. Semitone metric space

We now show that Pythagorean tone systems expressed via GGSs are a certain closer in sense to Equal Temperament the greater the positive integer d.

Theorem 3. *For $K = V/U$ (i.e., comat), set $\rho(X,Y) = \left|\log_K(X) - \log_K(Y)\right|$ $X, Y \in \mathcal{L}$. Let $(X_{d_1}, Y_{d_1}), (X_{d_2}, Y_{d_2})$ be solutions of Problem 1a corresponding d_1, d_2 and $d_1 < d_2$, $d_1, d_2 = 1, 2, \dots$. Then $\rho(X_1, Y_1) > \rho(X_2, Y_2)$.*

P r o o f . It can be verified directly that $(\mathcal{L}, \rho)$ is a metric space.

If (X_d, Y_d) is a pair of numbers which is a solution of Problem 1a corresponding to d, $d = 0, 1, 2, 3, \dots$, then by (7),

$$\begin{aligned}\left|\log_K(Y_d) - \log_K(X_d)\right| &= \log_K 2^{\frac{-m_{12}-m_7}{d}} 3^{\frac{m_{12}}{d}} - \log_K 2^{\frac{19-m_{12}-m_7}{d}} 3^{\frac{m_{12}-12}{d}} \\ &= \frac{12}{d}\log_K 3 - \frac{19}{d}\log_K 2 = \frac{1}{d}\log_K \frac{3^{12}}{2^{19}} = \frac{1}{d}\log_K K = \frac{1}{d}.\end{aligned}$$

If $d_1 < d_2$, then

$$\rho(X_{d_1}, Y_{d_1}) = \frac{1}{d_1} > \frac{1}{d_2} = \rho(X_{d_2}, Y_{d_2}).$$

□

Corollary 1. *If $Y_d > X_d$, then $Y_d/X_d = \sqrt[d]{K}$.*

Definition. Denote by $\mathcal{S}$ the set of all numbers $X \in \mathcal{L}$ such that (X, Y) is a solution of Problem 1a for some integer m_7, m_{12}, d and $Y \in \mathcal{L}$. We will call the metric space $(\mathcal{S}, \rho)$ the *metric space of semitones generating the perfect fifths.*

4. Applications

The perception sensitivity of the human ear is 5–6 cents. The psychoacoustical boundary (to which the listener or player tends) is 2–3 cents. This is the reason why there is no practical purpose in using GGSs for $d > 12$, we hear no difference between Equal Temperament and the constructed GGS (we do not suppose too many values within 12 qualitative degrees). From the acoustical point of view, the most interesting cases are those of low values of $d = 1, 2, 3, \dots, 11$ for which $Y_d/X_d \approx 23.46$, 11.73, 7.82, 5.86, 4.69, 3.91, $3.35, \dots, 2.13$ cents, respectively. From the point of view of modulation, the ordering is reversed: $d = 11, 10, \dots 3, 2, 1$. So, we have to choose an appropriate compromise of d depending on the kind of modulations within a given musical composition (set of compositions). Some of GGSs in this paper, new from the point of view of musicology, may be considered for the tuning of musical instruments.

For $d = 2$, cf. Table 3 (Figure 2). The set P_7 is the Pythagorean heptatonic $F \to C \to G \to D \to A \to E \to B$, the ("white keys"). The set $\left(\sqrt{2}\right) \cdot P_5$, is a Pythagorean pentatonic $F_\sharp \to C_\sharp \to G_\sharp \to D_\sharp \to A_\sharp$, (the "black keys").

For $d = 3$, cf. Table 4 (Figure 3, many valued tuning). The sets P_5 (i.e., $C \to G \to D \to A \to E$), $(3/\sqrt[3]{4}) \cdot P_5$, (i.e., $B \to F_\sharp \to C_\sharp \to G_\sharp \to D_\sharp$), and $\left(\sqrt[3]{32}/3\right) \cdot P_5$ (i.e., $D_\flat \to A_\flat \to E_\flat \to B_\flat \to F$) are three Pythagorean pentatonics.

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