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TWO-STEP INTEGRAL WITH RESPECT TO FUZZY MEASURE

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ABSTRACT. A two-step integration method based on the Choquet integral is introduced. In some special cases, this method gives the standard Choquet integral and then it coincides with the hierarchical decomposition of the Choquet integral introduced by Sugeno et al. A similar approach based on the Sugeno integral is considered, too.

1. Introduction

In the classical measure theory, the Bayes principle allows to reduce the global calculations into the local ones and then to deal with calculated output only.

More precisely, let $(X, \mathscr{S}, m)$ be a classical measure space, where m is a finite σ-additive measure. Let $\mathscr{C} = \{C_i\}_{i\in I}$ be a finite measurable partition of X and let $f : X \to \mathbb{R}$ be an integrable function. The Lebesgue integral $\int\limits_X f\,dm$ can be computed in two steps as follows:

1) for any $C_i \in \mathscr{C}$, $i \in I$, compute $F(i) = \int\limits_{C_i} f\,dm_{C_i}$, where

$$m_{C_i}(A) = \begin{cases} \frac{m(C_i\cap A)}{m(C_i)} & \text{if } m(C_i) > 0\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

is the corresponding conditional measure;

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2) compute $\int_I F\,dM$, where M is a measure on $(I, 2^I)$ defined by $M(J) = m\Big(\bigcup_{i\in J} C_i\Big)$, $J \subset I$:

$$\int_X f\,dm \equiv: \int_I \bigg(\int_{C_i} f\,dm_{C_i}\bigg)\,dM\,.$$

Vice-versa, given m_{C_i} and M, we can determine m, $m(A) = \sum_{i\in I} M(\{i\}) \times m_{C_i}(A)$, $A \subset \mathscr{S}$, so that the above two-step calculation leads just to the Lebesgue integral with respect to m.

Note that we can also start with an arbitrary finite measurable system $(C_i)_{i\in I}$ and to apply the above described procedures.

In the framework of fuzzy measures, several integration methods were developed. Recall, e.g., the Choquet integral ([3], [17]), Sugeno integral ([15], [13]), integrals based on pan-arithmetical operations of pseudo-addition $\oplus$ and pseudo-multiplication $\odot$ ([18], [1], [8]).

First attempts in a decomposition of the Choquet integral computation in two steps as indicated above (in the case of Lebesgue integral) were done in [16], where the first level was computed by means of the Choquet integral with respect to fuzzy measures m_{C_i}, while the second level was based on an additive measure M (for which the Choquet and the Lebesgue integral on finite universe coincide).

Continuation of [16] in [6], [10], [11] deals with generalizing the second level set function M to be a fuzzy measure.

However, we can always apply the indicated two step integration procedure to get either new types of functional and/or to simplify and decrease the necessary calculations. The use of other types of general fuzzy integral will be investigated, too.

We restrict ourselves to the case of finite universe X (and $\mathscr{S} = 2^X$). However, our considerations can be easily extended for infinite X.

In this paper we have applied the procedure given in [16] for Choquet and Sugeno integral and we have obtained some representation theorems.

2. Two-step Choquet integral

Let $X \neq \emptyset$ be a given finite space and let $(C_i)_{i\in I}$ be a system of non empty subsets of X (we admit even $C_i = C_j$ for $i \neq j$), $I = \{1, 2, \dots, n\}$. Let $(m_{C_i})_{i\in I}$ be a system of fuzzy measures on the corresponding spaces $(C_i, 2^{C_i})$, $i \in I$, and let M be a fuzzy measure on I. Two-step Choquet integral $C(f)$ is defined by means of the Choquet integral

$$C(f) = (C) \int_I F \, dM \,, \tag{1}$$

where $F(i) = (C) \int_{C_i} f_i \, dm_{C_i}$, $f_i = f|_{C_i}$.

PROPOSITION 1. *$C(\cdot)$ is a monotone, homogeneous functional.*

P r o o f . The monotonicity and homogeneity of the standard Choquet integral ensures the result. □

As X is finite, the fuzzy measures m_{C_i} are continuous from below and so the functional $C(f)$ is continuous from below.

The standard Choquet integral is comonotone additive, i.e., if two functions f and g are comonotone ($\big(f(x) - f(y)\big) \cdot \big(g(x) - g(y)\big) \geq 0$ for all x and $y \in X$), then for any fuzzy measure m, $(C) \int (f+g) dm = (C) \int f \, dm + (C) \int g \, dm$. Vice-versa, any monotone and comonotone additive (and hence homogeneous [4]) functional $\mathcal{I}$ on non-negative functions on X is a Choquet integral with respect to a fuzzy measure m given by $m(A) = \mathcal{I}(1_A)$, where 1_A is the indicator function of the set $A \in \mathscr{S}$ ([4], [14]).

THEOREM 1. (Representation Theorem). *If the two-step Choquet integral $C(f)$ defined by (1) is comonotone additive, then it is necessarily [4] the Choquet integral with respect to the fuzzy measure $m(A) = C(1_A)$, $\forall\, A \in \mathscr{S}$:*

$$C(f) = (C) \int_X f \, dm \,.$$

Note that $m(A) = (C) \int_I F \, dM$, where $F(i) = (C) \int_{C_i} 1_{A \cap C_i} \, dm_{C_i} = m_{C_i}(A \cap C_i)$.

THEOREM 2. (Representation Theorem). *Let M be an additive measure on I. Then the two-step Choquet integral coincides with the standard Choquet integral with respect to a fuzzy measure m given by*

$$m(A) = \sum_{i \in I} M\big(\{i\}\big) \, m_{C_i}(A \cap C_i) \,, \tag{2}$$

i.e.,

$$C(f) = (C) \int_X f \, dm \,.$$

P r o o f . The comonotone additivity of the standard Choquet integral ensures for any comonotone functions f_i and g_i that for their sum $h_i = f_i + g_i$, we have

$$H(i) = (C)\int_{C_i} h_i \, dm_{C_i} = (C)\int_{C_i} (f_i + g_i)\, dm_{C_i}$$

$$= (C)\int_{C_i} f_i \, dm_{C_i} + (C)\int_{C_i} g_i \, dm_{C_i} = F(i) + G(i)\,.$$

However, the comonotonicity of f_i and g_i for all $i \in I$ follows from the comonotonicity of f and g. Consequently, for comonotone functions f and g, the additivity of M ensures that

$$C(h) = (C)\int_I H \, dM = (C)\int_I (F + G)\, dM$$

$$= (C)\int_I F \, dM + (C)\int_I G \, dM = C(f) + C(g)\,,$$

i.e., the functional $C(f)$ is comonotone additive. The result follows, [4]. □

Vice-versa, the decomposition of a standard Choquet integral in two-step Choquet integrals is studied in [6].

We have already mentioned that the comonotonicity of f and g ensures the comonotonicity of f_i and g_i, $i \in I$. However, the comonotonicity of F and G may be violated, in general, as the following example shows.

EXAMPLES. Let X be the interval $[0,1]$, $C_1 = [0, 0.7]$, $C_2 =]\,0.7, 1]$ and let m_{C_1}, m_{C_2} be the Lebesgue measure on C_1 and C_2, respectively. We consider two functions: $f(x) = x$, $g(x) = \frac{x}{2} + \frac{1}{4}$ for every $x \in [0,1]$. It is easy to see that f and g are comonotone, but F and G are not comonotone: $F(1) = \frac{0.49}{2} < \frac{1}{4}$, $F(2) = \frac{0.51}{2} > \frac{1}{4}$, $G(1) = \frac{1.19}{4} > \frac{1}{4}$ and $G(2) = \frac{0.81}{4} < \frac{1}{4}$.

However, for special measures m_{C_i} we obtain that if f and g are comonotone, also F and G are comonotone. For example, fix two constants $k > 0$ and $h > 0$, and consider the following measures:

$$m^*_{C_i}(A) = \begin{cases} k\,, & A \neq \emptyset\,, \\ 0\,, & A = \emptyset\,, \end{cases} \quad \forall\, i\,, \tag{3}$$

$$\widetilde{m}_{C_i}(A) = \begin{cases} h\,, & A = C_i\,, \\ 0\,, & A \neq C_i\,, \end{cases} \quad \forall\, i\,. \tag{4}$$

We know that if f and g are comonotone, then $\max\limits_{x\in A} f(x) = f(x^*_A)$ and $\max\limits_{x\in A} g(x) = g(x^*_A)$, $\forall\, A \subset X$. Similarly, $\min\limits_{x\in A} f(x) = f(\tilde{x}_A)$ and $\min\limits_{x\in A} g(x) =$

$g(\tilde{x}_A)$, $\forall A \subset X$. So, in both cases if f and g are comonotone, F^* and G^* as well as $\widetilde{F}$ and $\widetilde{G}$ are comonotone, where $F^*(i) = (C)\int_I f_i\, dm^*_{C_i} = k \max_{x \in C_i} f(x)$ and $\widetilde{F}(i) = (C)\int_I f_i\, d\widetilde{m}_{C_i} = h \min_{x \in C_i} f(x)$.

The comonotonicity of F and G (for general comonotone f and g) ensures the comonotone additivity of the two-step Choquet integral.

Nevertheless, if the fuzzy measure M is additive, the two-step Choquet integral will be comonotone additive in any case and therefore it can be represented as a single Choquet integral.

If M is not additive, the two-step Choquet integral need not be comonotone and hence it defines a new type of functional.

The conditions under which the two-step Choquet integral is a decomposition of a standard Choquet integral are rather complicate and they are completely solved in [10, Theorem 4.1].

Due to the Theorem 2, in the case when M is an additive measure on I, all properties of the two step Choquet integral $C(\cdot)$ can be derived from the properties of fuzzy measure m given by (2).

3. Properties of the two-step Choquet integral

In some special cases, the two-step Choquet integral may be reduced to the single Choquet integral, as already mentioned and as the following examples will show. First we introduce some definitions.

We recall that if m is a fuzzy measure on $(X, 2^X)$, its *Möbius transform* $\mathcal{M}_m \colon 2^X \to \mathbb{R}$ ([12], [7], [9], [2]) is defined by

$$\mathcal{M}_m(A) = \sum_{B \subset A} (-1)^{|A \setminus B|} m(B).$$

Note that m is *k-order additive*, $k \in \mathbb{N}$, iff $\mathcal{M}_m(A) = 0$ whenever $|A| > k$ ([7]). Further note that m is *k-monotone* [2] iff $\sum_{C \subset B \subset A} \mathcal{M}_m(A) \geq 0$ for any A and C with $|C| \leq k$. Moreover, *∞-monotone fuzzy measure* are characterized by $\mathcal{M}_m \geq 0$ and they can be characterized by means of additive measures, $m(A) = \min\big(\mu(A) \colon m \leq \mu,\ \mu \text{ is additive fuzzy measure},\ \mu(X) = m(X)\big)$.

PROPOSITION 2. *Let I be a finite set of indexes, $I = \{1, \dots, n\}$. Let $(C_i)_{i \in I}$ be a system of non empty subsets of X, $(m_{C_i})_{i \in I}$ a system of fuzzy measures on C_i and M an additive measure on I.*

Define a fuzzy measure m on X by $m(A) = \sum_{i \in I} M(\{i\})\, m_{C_i}(A \cap C_i) = \sum_{i \in I} m_i(A \cap C_i)$, where $m_i = m_{C_i} \cdot M(\{i\})$, $i \in I$. Then:

1) $\mathcal{M}_m(A) = \sum\limits_{C_i \supset A} \mathcal{M}_{m_i}(A)$*;*

2) if m_{C_i} *are* k_i*-order additive,* m *is* k*-order additive, where* $k = \max\limits_{i \in I} k_i$*;*

3) m *is* k*-order additive, where* $k = \max\limits_{i \in I} |C_i|$*;*

4) if m_{C_i} *are* k_i*-monotone,* m *is* k*-monotone, where* $k = \min\limits_{i \in I} k_i$*;*

5) if m_{C_i} *are* ∞*-monotone, then* m *is* ∞*-monotone.*

Proof. We will prove only 1), as the other claims follows from 1). First we extend each fuzzy measure m_i to the whole space X, putting $\overline{m}_i(A) = m_i(A \cap C_i)$, $i \in I$. Then $m = \sum\limits_{i \in I} \overline{m}_i$ and consequently $\mathcal{M}_m = \sum\limits_{i \in I} \mathcal{M}_{\overline{m}_i}$. In fact, for $A \subset \Omega$, $i \in I$, we have

$$\mathcal{M}_{\overline{m}_i}(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \, \overline{m}_i(B) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \, m_i(B \cap C_i).$$

It is evident that $\mathcal{M}_{\overline{m}_i}(A) = \mathcal{M}_{m_i}(A)$ whenever $A \subseteq C_i$. Let $A \setminus C_i = D$ be non empty, then

$$\begin{aligned} \mathcal{M}_{\overline{m}_i}(A) &= \sum_{\substack{H \subset D \\ E \subset A \cap C_i}} (-1)^{|A \setminus H \cup E|} \, m_i(E) \\ &= \sum_{E \subset A \cap C_i} m_i(E) \cdot (-1)^{|A \cap C_i \setminus E|} \cdot \sum_{H \subset D} (-1)^{|D \setminus H|} = 0 \,, \end{aligned}$$

as for any finite set D, $\sum\limits_{H \subset D} (-1)^{|D \setminus H|} = 0$ ([12] Lemma 4.1). Consequently

$$\mathcal{M}_{\overline{m}_i}(A) = \begin{cases} \mathcal{M}_{m_i}(A) & \text{if } A \subseteq C_i \,, \\ 0 & \text{otherwise} \,, \end{cases}$$

hence $\mathcal{M}_m(A) = \sum\limits_{C_i \supset A} \mathcal{M}_{m_i}(A)$. □

Now, we are ready to apply these results to the two-step Choquet integral.

PROPOSITION 3. *By the hypothesis of the Proposition 2, for* m k*-order additive, we have*

$$C(f) = \sum_{\substack{A \subset X \\ |A| \le k}} \left(\bigwedge_{x \in A} f(x) \right) \cdot \sum_{C_i \supset A} \mathcal{M}_{\overline{m}_i}(A).$$

Now we can apply the Möbius transform in order to compute the two-step Choquet integral: in this case it may be reduced to the single Choquet integral.

Given a fuzzy measure M and $\mathcal{M}_M$ its Möbius transform, let $\widetilde{m}_{C_i}$ be the fuzzy measures defined by (4). So

$$C(f) = (C)\int_I \widetilde{F}\,dM = \sum_{J\subset I}\left(\bigwedge_{i\in J}\widetilde{F}(i)\right)\mathcal{M}_M(J)$$

$$= \sum_{J\subset I} h\left(\bigwedge_{x\in\bigcup\limits_{j\in J} C_i} f(x)\right)\mathcal{M}_M(J) = \sum_{A\subset X}\left(\bigwedge_{x\in A} f(x)\right)\cdot\mathcal{M}_m(A),$$

where

$$\mathcal{M}_m(A) = \begin{cases} h\cdot\mathcal{M}_M(J) & \text{if } A = \bigcup\limits_{i\in J} C_i\,,\\ 0 & \text{otherwise}. \end{cases}$$

In the similar way for the measures $m^*_{C_i}$ given by (3) we have

$$C(f) = (C)\int_I F\,dM = k\sum_{J\subset I}\left(\bigwedge_{j\in J}\bigvee_{x\in C_j} f(x)\right)\cdot\mathcal{M}_M(J).$$

4. Two-step Sugeno integral

We can repeat the construction of a functional over non-negative functions defined on X described in Section 2, with the only change that the Choquet integral is replaced by the Sugeno integral ([15], [13]). Now the two-step Sugeno integral is defined by

$$S(f) = (S)\int_I F\,dM\,, \tag{5}$$

where $F(i) = (S)\int\limits_{C_i} f_i\,dm_{C_i}$.

Proposition 4. *$S(\cdot)$ is monotone and $\wedge$-homogeneous.*

Theorem 3. (Representation Theorem). *The functional $S(\cdot)$ defined by (5) is a Sugeno integral with respect to the fuzzy measure m on X defined by $m(A) = S(\gamma 1_A)$, where $\gamma = m(\Omega) = M(I)\vee\left(\bigvee\limits_{i\in I} m_{C_i}(C_i)\right)$ if and only if S is comonotone maxitive, i.e., $S(f\vee g) = S(f)\vee S(g)$ whenever f and g are comonotone* [8].

Theorem 4. (Representation Theorem). *Let M be a maxitive measure, i.e., $M(J\cup K) = M(J)\vee M(K)$ for any subsets J, K of I. Then the two-step*

Sugeno integral $S(\cdot)$ coincides with the standard Sugeno integral with respect to the fuzzy measure $m\colon 2^X \to \mathbb{R}^+$ defined by

$$m(A) = \bigvee_{i\in I} \big(M(\{i\}) \wedge m_{C_i}(A \cap C_i)\big)\,.$$

P r o o f . Let $f, g\colon X \to \mathbb{R}^+$ be two comonotone functions and let $h = f \vee g$. The comonotone maxitivity of the standard Sugeno integrals results to $H = F \vee G$ (note that F and G need not more be comonotone!).

However, the maxitivity of M ensures that

$$S(f\vee g) = (S)\int_I (F \vee G)\,dM = (S)\int_I F\,dM \vee (S)\int_I G\,dM = S(f) \vee S(g)\,.$$

Consequently $S(\cdot)$ is a Sugeno integral with respect to fuzzy measure m given by $m(A) = S(\gamma 1_A)$, $\gamma = M(I) \vee \Big(\bigvee\limits_{i\in I} m_{C_i}(C_i)\Big)$. Then, (see [5]),

$$m(A) = S(\gamma 1_A) = (S)\int_i m_{C_i}(A\cap C_i)\,dM = \bigvee_{i\in I}\Big(M\big(\{i\}\big) \wedge m_{C_i}(A\cap C_i)\Big)\,.$$

$\square$

Note that in the situation described in Theorem 4, $m(A) = \bigvee\limits_{i\in I} m_i(A\cap C_i)$, where $m_i = M\big(\{i\}\big) \wedge m_{C_i}$, $i \in I$.

The complete characterization of all two-step Sugeno integrals which are representable by means of a standard Sugeno integral (and vice-versa the hierarchical decomposition of a Sugeno integral into two-step Sugeno integral) can be an interesting topic for further investigations.

5. Concluding remarks

When using some other types of general fuzzy integral in a two-step construction of a functional over non-negative functions on X, similar results as those described in Sections 2 and 3 are expected. So, e.g., in the case of g-Choquet integral, when $g : [0,\infty] \to [0,\infty]$ is some increasing bijection, $(g-C)\int\limits_X f dm = g^{-1}\big((C)\int\limits_X g\circ f d g\circ m\big)$, ([1], [8]), we can repeat all results from Section 2 applying, where necessary, the transformation by g. So, for example, if fuzzy measure M is g-additive, i.e., $J \cap K = \emptyset$ implies $M(J\cup K) = g^{-1}\Big(g\big(M(J)\big) + g\big(M(K)\big)\Big)$, then the two-step g-Choquet integral is just a g-Choquet integral with respect to a fuzzy measure m on X given by $m(A) = (g-C)\int\limits_X g^{-1}(1)1_A dm$.

Similarly if we will work with a fuzzy integral based on $\vee$ as a pseudo-addition $\oplus$ and on some pseudo-multiplication $\odot$ (e.g., the standard multiplication) [8], maxitivity of M results to the standard integral representation of the proposed two-step procedure.

Obviously, we can introduce a new type of functionals based on several steps using some fuzzy integral. However, we do not expect essentially new results from such generalization. On the other hand, we can combine different types of integrals on different levels. Such new functionals can be interesting as new aggregation operators in multi-criteria decision making.

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