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PERIODS OF SPECIAL FUZZY MATRICES

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ABSTRACT. An algorithm for computing the periods of circulant and some Toeplitz matrices in fuzzy algebra is described. For matrices of order n, the algorithm works in $O(n)$ time, or in $O(k + \log n)$ time (if the maximal element of the matrix occurs on k places in each row and the set of the occurences is given).

1. Introduction

Circulant matrices represent a special class of matrices: every row contains the same elements as the previous row, but the elements are shifted one position to the right and the last element is on the first position. For circulant matrices, many difficult problems on matrices have more simple solutions than in the general case. E.g., the traveling salesman problem is polynomially solvable if the corresponding distance matrix is circulant.

Toeplitz matrices are of a slightly more general form. Such a matrix contains the same element on any diagonal parallel to the main diagonal. Circulant matrices are a special case of Toeplitz matrices.

In this paper we study the periodicity of circulant and some Toeplitz fuzzy matrices. The periodic properties of fuzzy matrices are used for investigating the behaviour of fuzzy discrete dynamic systems. The convergence and periodicity in special classes of matrices were studied by M. G. Thomason [10], and subsequently by many other authors. In [6], an algorithm for computing the period of a fuzzy matrix in $O(n^3)$ time is described. Periods of circulant matrices were studied in [1]. Matrix periods in max-plus algebra were studied in [8], for matrices with zero-weight cycles.

For simpler notation of index sets we shall use the convention by which any natural number n is considered as the set of all smaller natural numbers, i.e.,

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$n = \{0, 1, \ldots, n-1\}$. By $\mathbb{N}$, $\mathbb{N}^+$ we denote the set of all non-negative integers and the set of all positive integers, respectively, $\mathbb{Z}$ denotes the set of all integers. The greatest common divisor and the least common multiple of a set $S \subset \mathbb{N}$ are denoted by the abbreviations $\gcd S$ and $\operatorname{lcm} S$, respectively.

If $\mathscr{G} = (V, E)$ is a digraph (directed graph), then by strongly connected component of $\mathscr{G}$ we mean a subdigraph $\mathscr{K} = (K, E \cap K^2)$ generated by a non-empty subset $K \subseteq V$ such that any two vertices $x, y \in K$ are contained in a common cycle, and K is a maximal subset with this property. $\mathscr{K}$ is called trivial, if K contains only one vertex x, and there is no loop from x to x in $\mathscr{K}$. If $\mathscr{K}_1 = (K_1, E \cap K_1^2)$, $\mathscr{K}_2 = (K_2, E \cap K_2^2)$ are strongly connected components, we write $\mathscr{K}_1 \subseteq \mathscr{K}_2$, if $K_1 \subseteq K_2$.

DEFINITION 1.1. The *fuzzy algebra* $\mathscr{B}$ is a triple $(\mathscr{B}, \oplus, \otimes)$, where $\mathscr{B}$ is a linearly ordered set and $\oplus, \otimes$ are the binary operations of maximum or minimum, respectively, on $\mathscr{B}$. For any natural $n > 0$, $\mathscr{B}(n)$ denotes the set of all n-dimensional column vectors over $\mathscr{B}$, and $\mathscr{B}(n, n)$ denotes the set of all square matrices of order n over $\mathscr{B}$. The matrix operations over $\mathscr{B}$ are defined formally in the same manner (with respect to $\oplus, \otimes$) as the matrix operations over any field.

DEFINITION 1.2. Let $A \in \mathscr{B}(n, n)$, $h \in \mathscr{B}$.

(i) The *threshold digraph* $\mathscr{G}(A, h)$ is the digraph $\mathscr{G} = (n, E)$, with the vertex set $n = \{0, 1, \ldots, n-1\}$ and with the arc set $E = \{(i, j) \colon i, j \in n,\ a_{ij} \geq h\}$.

(ii) For any natural r and for any $i, j \in n$, we denote by $W_{\mathscr{G}}^{(r)}(i, j)$ the set of all walks in $\mathscr{G}$, of length r, beginning in i and ending in j.

Several authors studied the graph properties using power sequences of associated matrices (R. A. Cuninghame-Green [5], M. Gondran and M. Minoux [7], U. Zimmermann [11]). Other connections between eigenvectors of a max-min matrix and its associated digraphs are described in [3, 4, 9]. Conversely, powers of a matrix over $(\mathscr{B}, \oplus, \otimes)$ can be characterized by walks in the corresponding threshold graphs. The following formulation is due to K. Cechlárová [2].

LEMMA 1.1. *Let $A \in \mathscr{B}(n, n)$, $x \in \mathscr{B}(n)$, $h \in \mathscr{B}$, $r \in \mathbb{N}^+$, $i, j \in n$. Then*

(i) $(A^r)_{ij} \geq h \iff W_{\mathscr{G}(A,h)}^{(r)}(i, j) \neq \emptyset$,

(ii) $(A^r x)_i \geq h \iff (\exists\, j \in n)\big[\, x_j \geq h,\ W_{\mathscr{G}(A,h)}^{(r)}(i, j) \neq \emptyset \big]$.

Proof. By induction on r. $\square$

2. Matrix periods

The necessary definitions and facts from [6] concerning matrix periods in fuzzy algebra are presented in this section.

Definition 2.1. Let $A \in \mathscr{B}(n,n)$, $x \in \mathscr{B}(n)$. The *matrix period*, in notation: $\mathrm{per}(A)$, is defined as the minimal natural p for which there is R such that for all $r > R$ the equation $A^r = A^{r+p}$ holds true.

Remark 2.1. Since the ordering of $\mathscr{B}$ is linear, any element of any power of the matrix A is equal to some element of A. Therefore, the sequence of powers of A contains only finitely many different matrices. As a consequence, the period of A is always well-defined.

Definition 2.2. Let $A \in \mathscr{B}(n,n)$, $h \in \mathscr{B}$, let $\mathscr{G}(A,h)$ be a threshold digraph of A. By $\mathrm{SCC}^* \mathscr{G}(A,h)$ we denote the set of all non-trivial strongly connected components of $\mathscr{G}(A,h)$. For any $\mathscr{K} \in \mathrm{SCC}^* \mathscr{G}(A,h)$, we define

(i) $\mathrm{per}(\mathscr{K}) := \gcd\{|C|\colon |C| \text{ is the length of a cycle } C \text{ in } \mathscr{K}\}$,

(ii) $\bar{A} = \{a_{ij}\colon i,j \in n\}$,

(iii) $\mathrm{SCC}^*(A) = \bigcup\{\mathrm{SCC}^* \mathscr{G}(A,h)\colon h \in \bar{A}\}$,

(iv) $\mathrm{SCC}^{\min}(A) = \{\mathscr{K} \in \mathrm{SCC}^*(A)\colon \mathscr{K} \text{ is minimal in } (\mathrm{SCC}^*(A), \subseteq)\}$.

In [6] the period of a fuzzy matrix was characterized by the periods of the non-trivial strongly connected components in the threshold graphs of the matrix. The following theorem was proved in [6].

Theorem 2.1. *Let $A \in \mathscr{B}(n,n)$. Then*

(i) $\mathrm{per}(A) = \mathrm{lcm}\{\mathrm{per}(A,x)\colon x \in \mathscr{B}(n)\}$,

(ii) $\mathrm{per}(A) = \mathrm{lcm}\{\mathrm{per}(\mathscr{K})\colon \mathscr{K} \in \mathrm{SCC}^*(A)\}$,

(iii) $\mathrm{per}(A) = \mathrm{lcm}\{\mathrm{per}(\mathscr{K})\colon \mathscr{K} \in \mathrm{SCC}^{\min}(A)\}$.

3. Circulant matrices

Definition 3.1. Let $(a_0, a_1, \dots, a_{n-1}) \in \mathscr{B}(n)$. A matrix $A \in \mathscr{B}(n,n)$ of the form

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix} = A(a_0, a_1, \dots, a_{n-1})$$

is called a *circulant matrix* (generated by elements $a_0, a_1, \dots, a_{n-1}$). We denote

(i) $h(A) := \max \{a_i : i \in n\}$,

(ii) $I(A) := \{i \in n : a_i = h(A)\} \cup \{n\}$,

(iii) $M(A) := \Big\{m \in \mathbb{Z}^{I(A)} : \sum\limits_{i \in I(A)} m_i\, i = 0\Big\}$.

Further, we define a function $s : M(A) \to \mathbb{Z}$ by setting

(iv) $s(m) := \sum\limits_{i \in I(A) - \{n\}} m_i$ for any $m \in M(A)$.

Theorem 3.1. *Let $A \in \mathscr{B}(n,n)$ be a circulant matrix, then for any component $\mathscr{K} \in \mathrm{SCC}^* \mathscr{G}(A, h(A))$*

(i) $\mathrm{per}(A) = \mathrm{per}(\mathscr{K})$,

(ii) $\mathrm{per}(A) = \gcd \{s(m) : m \in M(A)\}$.

Proof. (i) Any vertex in the threshold graph $\mathscr{G} = \mathscr{G}(A, h(A))$ is contained in some cycle and in some $\mathscr{K} \in \mathrm{SCC}^* \mathscr{G}$. As the matrix A is circulant, all strongly connected components of $\mathscr{G}$ are isomorphic and have the same period. For any $h' \le h(A)$ and for any $\mathscr{K}' \in \mathrm{SCC}^* \mathscr{G}(A, h')$, there is $\mathscr{K} \in \mathrm{SCC}^* \mathscr{G}(A, h(A))$, which is a subgraph of $\mathscr{K}'$. Therefore $\mathrm{per}(\mathscr{K}') | \mathrm{per}\, \mathscr{K}$ and the statement follows by Theorem 2.1(iii).

(ii) By (i), it is sufficient to prove that $\mathrm{per}(\mathscr{K}) = \gcd \{s(m) : m \in M(A)\}$.

Let $0 \in I(A)$. Then $\mathscr{K}$ contains a loop on each node and $\mathrm{per}(\mathscr{K}) = 1$. On the other hand, for any integer k we can find $m \in M(A)$ such that $s(m) = k$. It is sufficient to put $m_0 = k$ and $m_i = 0$ for all $0 \ne i \in I(A)$. Thus, $\gcd \{s(m) : m \in M(A)\} = 1$.

Let $0 \notin I(A)$. Then $\mathscr{K}$ contains arcs of span i for all $i \in I(A) - \{n\}$. To obtain a cycle, we must return to the starting point traversing a path containing the given types of arcs. In other words, adding the spans of these arcs we must get a multiple of n. Therefore, the length of any cycle in $\mathscr{K}$ can be expressed in the form $s(m)$, for some $m \in M(A)$ such that $m_i \ge 0$ for all $i \in I(A) - \{n\}$. Conversely, for any $m \in M(A)$ such that $m_i \ge 0$ for all $i \in I(A) - \{n\}$, the value $s(m)$ is the length of some cycle in $\mathscr{K}$. We have proved that

$$\mathrm{per}(\mathscr{K}) = \gcd \{s(m) : m \in M(A) \wedge (\forall i \in I(A) - \{n\})\ m_i \ge 0\}.$$

On the other hand, we can easily show that n can be expressed in the form $s(m)$ for some $m \in M(A)$. It is sufficient to take arbitrary $0 \ne i \in I(A) - \{n\}$ and put $m_i = n$, $m_n = -i$ and $m_j = 0$ for any $j \ne i, n$.

Finally, let $m \in M(A)$. Then there is $m' \in M(A)$ such that $m'_i \ge 0$ and $m_i \equiv m'_i \mod n$ for all $i \in I(A) - \{n\}$. Clearly, $s(m) \equiv s(m') \mod n$. By the well-known fact that $\gcd(a,b) = \gcd(a-b,b)$, we have $\gcd\bigl(s(m),n\bigr) = \gcd\bigl(s(m'),n\bigr)$. Thus, we have

$$\begin{aligned}
\operatorname{per}(\mathscr{K}) &= \gcd\bigl\{s(m')\colon m' \in M(A) \wedge \bigl(\forall i \in I(A)-\{n\}\bigr)\, m'_i \geq 0\bigr\} \\
&= \gcd\bigl\{\gcd\bigl(s(m'),n\bigr)\colon m' \in M(A) \wedge \bigl(\forall i \in I(A)-\{n\}\bigr)\, m'_i \geq 0\bigr\} \\
&= \gcd\bigl\{\gcd\bigl(s(m),n\bigr)\colon m \in M(A)\bigr\} \\
&= \gcd\bigl\{s(m)\colon m \in M(A)\bigr\}\,.
\end{aligned}$$

The assertion (ii) is proved. □

Definition 3.2. Let $J \subseteq I(A)$. We define

(i) $M(J) := \bigl\{m \in M(A)\colon \bigl(\forall i \in I(A)-J\bigr)\ m_i = 0\bigr\}$,

(ii) $S(J) := \bigl\{s(m)\colon m \in M(J)\bigr\}$,

(iii) for any $i,j \in I(A)$, $i \neq j$, we define $m^{(i,j)} \in M\bigl(\{i,j\}\bigr)$ such that

$$m_i^{(i,j)} = -j\,, \qquad m_j^{(i,j)} = i\,.$$

Lemma 3.2. *Let $J \subseteq I(A)$, $|J| > 1$, $j \in J$, let $d := \frac{\gcd(J-\{j\})}{\gcd J}$. Then*

(i) $\bigl(\forall m \in M(J)\bigr) \quad d \,|\, m_j$,

(ii) $\bigl(\forall m \in M(J)\bigr)$

$$\Bigl(\exists\, m' \in M\bigl(J-\{j\}\bigr)\Bigr)\ \Bigl(\exists\, b \in \mathbb{Z}^{J-\{j\}}\Bigr) \quad m = m' + \sum_{i \in J-\{j\}} b_i\, \frac{m^{(i,j)}}{\gcd J}\,.$$

P r o o f. (i) Let $m \in M(J)$. We have $\sum\limits_{i\in J} m_i\, i = 0$, i.e.,

$$m_j\, j = -\sum_{i \in J-\{j\}} m_i\, i\,.$$

Dividing by $\gcd J$ we get

$$m_j\, \frac{j}{\gcd J} = -\sum_{i \in J-\{j\}} m_i\, \frac{i}{\gcd J}\,.$$

Every summand on the right side of the equation is divisible by the integer d, therefore

$$d \,\Big|\, m_j\, \frac{j}{\gcd J}\,.$$

As $\gcd J$ is the greatest common divisor of integers $\gcd\bigl(J - \{j\}\bigr)$ and j, the integers

$$d = \frac{\gcd\bigl(J - \{j\}\bigr)}{\gcd J} \qquad \text{and} \qquad \frac{j}{\gcd J}$$

have no non-trivial common divisor. Therefore $d \,|\, m_j$.

(ii) There are integer coefficients $\bigl(c_i \colon i \in J - \{j\}\bigr)$ such that

$$\gcd\bigl(J - \{j\}\bigr) = \sum_{i \in J - \{j\}} c_i \, i \,.$$

We denote for $i \in J - \{j\}$,

$$b_i := c_i \frac{m_j}{d} \in \mathbb{Z} \qquad \text{and} \qquad m'' := \sum_{i \in J - \{j\}} b_i \frac{m^{(i,j)}}{\gcd J} \,.$$

Then we have

$$\begin{aligned} m''_j &= \sum_{i \in J - \{j\}} b_i \frac{m_j^{(i,j)}}{\gcd J} = \sum_{i \in J - \{j\}} c_i \frac{m_j}{d} \frac{i}{\gcd J} \\ &= \frac{m_j}{d \gcd J} \sum_{i \in J - \{j\}} c_i \, i = \frac{m_j}{d \gcd J} \gcd\bigl(J - \{j\}\bigr) \\ &= \frac{m_j}{d} d = m_j \,. \end{aligned}$$

Therefore $m' = m - m'' \in M\bigl(J - \{j\}\bigr)$. $\square$

THEOREM 3.3. *Let $A \in \mathscr{B}(n,n)$ be a circulant matrix, let $I(A) := \bigl\{n, i_0, i_1, \dots, i_{k-1}\bigr\}$, $\bigl|I(A)\bigr| = k+1$. Then $\operatorname{per}(A)$ is equal to any of the following expressions*

(i) $\gcd\left(\dfrac{n}{\gcd(n,i_0)}, \dfrac{i_0 - i_1}{\gcd(i_0,i_1)}, \dfrac{i_0 - i_2}{\gcd(i_0,i_2)}, \dots, \dfrac{i_0 - i_{k-1}}{\gcd(i_0,i_{k-1})}\right)$,

(ii) $\gcd\left(\dfrac{n}{\gcd(n,i_0)}, \dfrac{i_0 - i_1}{\gcd(i_0,i_1)}, \dfrac{i_1 - i_2}{\gcd(i_1,i_2)}, \dots, \dfrac{i_{k-2} - i_{k-1}}{\gcd(i_{k-2},i_{k-1})}\right)$,

(iii) $\gcd\left(\dfrac{n}{\gcd(n,i_0)}, \dfrac{i_0 - i_1}{\gcd(n,i_0,i_1)}, \dfrac{i_0 - i_2}{\gcd(n,i_0,i_1,i_2)}, \dots, \dfrac{i_0 - i_{k-1}}{\gcd(n,i_0,i_1,\dots,i_{k-1})}\right)$.

Proof. By Definition 3.2(ii), the assertion of Theorem 3.1(ii) can be written in the form $\operatorname{per}(A) = \gcd S\bigl(I(A)\bigr)$.

For any $J \subseteq I(A)$, $i,j \in J$, $i \neq j$ we have

$$\frac{m^{(i,j)}}{\gcd J} \in M\bigl(\{i,j\}\bigr) \subseteq M\bigl(I(A)\bigr)\,, \quad \text{i.e.,} \quad \frac{s\bigl(m^{(i,j)}\bigr)}{\gcd J} \in S\bigl(I(A)\bigr)\,.$$

If $i, j \in J - \{n\}$, then

$$\frac{i-j}{\gcd J} = \frac{s\big(m^{(i,j)}\big)}{\gcd J} \in S\big(I(A)\big)\,,$$

if $i = n$, then

$$\frac{n}{\gcd J} = \frac{s\big(m^{(n,j)}\big)}{\gcd J} \in S\big(I(A)\big)\,.$$

Thus, by Theorem 3.1(ii), $\operatorname{per}(A)$ divides any of the integers

$$\begin{gathered}
\frac{n}{\gcd(n,i_0)}, \frac{i_0-i_1}{\gcd(i_0,i_1)}, \frac{i_0-i_2}{\gcd(i_0,i_2)}, \dots, \frac{i_0-i_{k-1}}{\gcd(i_0,i_{k-1})}, \\
\frac{n}{\gcd(n,i_0)}, \frac{i_0-i_1}{\gcd(i_0,i_1)}, \frac{i_1-i_2}{\gcd(i_1,i_2)}, \dots, \frac{i_{k-2}-i_{k-1}}{\gcd(i_{k-2},i_{k-1})}, \\
\frac{n}{\gcd(n,i_0)}, \frac{i_0-i_1}{\gcd(n,i_0,i_1)}, \frac{i_0-i_2}{\gcd(n,i_0,i_1,i_2)}, \dots, \frac{i_0-i_{k-1}}{\gcd(n,i_0,i_1,\dots,i_{k-1})},
\end{gathered}$$

and divides also their greatest common divisors which are expressed in (i), (ii) and (iii).

On the other hand, by repeated use of Lemma 3.2(ii) with sets

$$J_{k-1} = \{n, i_0, i_1, \dots, i_{k-1}\}, \dots, J_1 = \{n, i_0, i_1\},\ J_0 = \{n, i_0\}\,,$$

and with integers

$$j_{k-1} = i_{k-1},\ \dots,\ j_1 = i_1,\ j_0 = i_0$$

we can show that any element $s(m) \in S\big(I(A)\big)$ can be expressed as an integer linear combination of the values

$$\begin{aligned}
&\frac{s\big(m^{(n,i_{k-1})}\big)}{\gcd J_{k-1}}, \frac{s\big(m^{(i_0,i_{k-1})}\big)}{\gcd J_{k-1}}, \frac{s\big(m^{(i_1,i_{k-1})}\big)}{\gcd J_{k-1}}, \dots, \frac{s\big(m^{(i_{k-2},i_{k-1})}\big)}{\gcd J_{k-1}}, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\frac{s\big(m^{(n,i_2)}\big)}{\gcd J_2}, \frac{s\big(m^{(i_0,i_2)}\big)}{\gcd J_2}, \frac{s\big(m^{(i_1,i_2)}\big)}{\gcd J_2}, \\
&\frac{s\big(m^{(n,i_1)}\big)}{\gcd J_1}, \frac{s\big(m^{(i_0,i_1)}\big)}{\gcd J_1}, \\
&\frac{s\big(m^{(n,i_0)}\big)}{\gcd J_0},
\end{aligned}$$

which are equal to

$$\begin{aligned}
&\frac{n}{\gcd J_{k-1}}\,, \frac{i_0 - i_{k-1}}{\gcd J_{k-1}}\,, \frac{i_1 - i_{k-1}}{\gcd J_{k-1}}\,, \dots \frac{i_{k-2} - i_{k-1}}{\gcd J_{k-1}}\,,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&\frac{n}{\gcd J_2}\,, \frac{i_0 - i_2}{\gcd J_2}\,, \frac{i_1 - i_2}{\gcd J_2}\,,\\
&\frac{n}{\gcd J_1}\,, \frac{i_0 - i_1}{\gcd J_1}\,,\\
&\frac{n}{\gcd J_0}\,.
\end{aligned}$$

For any $0 \le r < s < k$ we have

$$\begin{aligned}
\frac{n}{\gcd J_s} &= \frac{n}{\gcd(n, i_0)}\,\frac{\gcd(n, i_0)}{\gcd J_s}\,,\\
\frac{i_r - i_s}{\gcd J_s} &= \frac{(i_0 - i_s) - (i_0 - i_r)}{\gcd J_s}\\
&= \frac{(i_0 - i_s)}{\gcd(i_0, i_s)}\,\frac{\gcd(i_0, i_s)}{\gcd J_s} - \frac{(i_0 - i_r)}{\gcd(i_0, i_r)}\,\frac{\gcd(i_0, i_r)}{\gcd J_s}\,.
\end{aligned}$$

Thus, any element $s \in S\big(I(A)\big)$ can be expressed as an integer linear combination of the integers

$$\frac{n}{\gcd(n, i_0)}\,, \frac{i_0 - i_1}{\gcd(i_0, i_1)}\,, \frac{i_0 - i_2}{\gcd(i_0, i_2)}\,, \dots, \frac{i_0 - i_{k-1}}{\gcd(i_0, i_{k-1})}\,.$$

As a consequence, the gcd of these integers divides all elements in $S\big(I(A)\big)$, i.e., it divides also $\operatorname{per}(A)$. We have proved the statement (i).

The statements (ii), (iii) are proved analogously. Namely,

$$\begin{aligned}
\frac{i_r - i_s}{\gcd J_s} &= \frac{(i_r - i_{r+1} + i_{r+1} - \dots - i_{s-1} + i_{s-1} i_s)}{\gcd J_s}\\
&= \frac{(i_r - i_{r+1})}{\gcd(i_r, i_{r+1})}\,\frac{\gcd(i_r, i_{r+1})}{\gcd J_s} + \frac{(i_{r+1} - i_{r+2})}{\gcd(i_{r+1}, i_{r+2})}\,\frac{\gcd(i_{r+1}, i_{r+2})}{\gcd J_s} + \dots\\
&\quad \dots + \frac{(i_{s-1} - i_s)}{\gcd(i_{s-1}, i_s)}\,\frac{\gcd(i_{s-1}, i_s)}{\gcd J_s}\,,\\
\frac{i_r - i_s}{\gcd J_s} &= \frac{(i_0 - i_s)}{\gcd J_s} - \frac{(i_0 - i_r)}{\gcd J_r}\,\frac{\gcd J_r}{\gcd J_s}\,.
\end{aligned}$$

The proof is complete.

$\square$

THEOREM 3.4. *There is an algorithm by which, for a given circulant matrix* $A = A(a_0, a_1, \ldots, a_{n-1}) \in \mathscr{B}(n,n)$,

(i) *the period* $\operatorname{per}(A)$ *can be computed in* $O(n)$ *time,*

(ii) *if* $I(A)$ *is known and* $\left|I(A)\right| = k$, *then* $\operatorname{per}(A)$ *can be computed in* $O(k + \log n)$ *time.*

P r o o f . (i) If the set $I(A)$ is not known, we can compute $I(A)$ from the elements of the first row $(a_0, a_1, \ldots, a_{n-1})$ in $O(n)$ time. Then the computation goes on by (ii).

(ii) We use the expression given in Theorem 3.3.(iii). The gcd's in the denominators are computed by Euclidean algorithm. It is a trivial fact that for integers $a > b$, the remainder $r = a - bq$ fulfills the inequality $r < \frac{a}{2}$. Therefore, in the course of the computation, the consecutive remainders are descending geometrically, and the computation can be done in $O(k + \log n)$ time.

The differences in the nominators and the total gcd can also be computed in $O(k + \log n)$ time. □

Remark 3.1. Using permutations of $I(A)$, we can get further expressions for $\operatorname{per}(A)$, similar to those in Theorem 3.3. It is a consequence of the fact that the proof of the theorem does not use substantially any special properties of the ordering of $I(A)$.

Remark 3.2. The formula given for circulant matrix period in [1] has the computational complexity $O(k^2 + k \log n)$ if $I(A)$ is known, and $O(n + k^2)$ in the general case.

4. Toeplitz matrices

The arguments shown in the previous part for circulant matrices can be analogously applied also to Toeplitz matrices, a special case of which are the circulant matrices. However, the situation with Toeplitz matrices is not completely the same, and our results concern only Toeplitz matrices with a certain restrictive condition.

DEFINITION 4.1. Let $a_{-n+1}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{n-1} \in \mathscr{B}$. A matrix $A \in$

$\mathscr{B}(n,n)$ of the form

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ a_{-1} & a_0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\ a_{-2} & a_{-1} & a_0 & \cdots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{-n+2} & a_{-n+3} & a_{-n+4} & \cdots & a_0 & a_1 \\ a_{-n+1} & a_{-n+2} & a_{-n+3} & \cdots & a_{-1} & a_0 \end{bmatrix}$$

is called a *Toeplitz matrix* $A = T(a_{-n+1},\dots,a_{-1},a_0,a_1,\dots,a_{n-1})$. We denote

(i) $h(A) := \min\bigl(\max\{a_i \colon 0 < i < n\}, \max\{a_i \colon -n < i < 0\}\bigr)$,

(ii) $I^+(A) := \bigl\{i \colon 0 < i < n \wedge a_i \geq h(A)\bigr\}$,

$I^-(A) := \bigl\{i \colon -n < i < 0 \wedge a_i \geq h(A)\bigr\}$,

$I(A) := \bigl\{i \colon -n < i < n \wedge a_i \geq h(A)\bigr\}$,

(iii) $M(A) := \Bigl\{m \in \mathbb{Z}^{I(A)} \colon \sum\limits_{i \in I(A)} m_i\, i = 0\Bigr\}$,

$M^+(A) := \bigl\{m \in M(A) \colon \bigl(\forall i \in I(A)\bigr)\, m_i \geq 0\bigr\}$.

Moreover, we define a function $s \colon M(A) \to \mathbb{Z}$ by setting

(iv) $s(m) := \sum\limits_{i \in I(A)} m_i$ for any $m \in M(A)$.

Remark 4.1. We note that $h(A)$ is the maximal threshold level for which both the sets $I^+(A)$, $I^-(A)$ are non-empty. If $a_0 \leq h(A)$, then for any $h' > h(A)$, the threshold digraph $\mathscr{G}(A,h')$ contains no cycle, i.e., all strongly connected components in $\mathscr{G}(A,h')$ are trivial.

LEMMA 4.1. *Let $A \in \mathscr{B}(n,n)$ be a Toeplitz matrix fulfilling the following condition*

($\mathcal{C}^*$) *for any $i \in I^+(A)$ and $j \in I^-(A)$, the inequality $i - j \leq n$ holds true. Then for any $m \in M^+(A)$ and for any $v \in n$, there is a cycle C of length $s(m)$ in the threshold digraph $\mathscr{G} = \mathscr{G}\bigl(A, h(A)\bigr)$ such that $v \in C$ and C contains exactly m_i of arcs of span i, for any $i \in I(A)$.*

Proof. Let the matrix $A \in \mathscr{B}(n,n)$ fulfil the condition ($\mathcal{C}^*$), let $m \in M^+(A)$, $v \in n$. We denote $s := s(m)$ and we choose a sequence of integers, S, of length s, containing any number $i \in I(A)$ exactly m_i times. By C we denote the sequence of all cumulative sums of S enlarged by the value v, i.e., C consists of the sums of all initial segments in S, with added value v, beginning with the empty segment and ending with the sum $v + \sum\limits_{i \in I(A)} m_i$. It is easy to see that

(1) the length of the sequence C is $s + 1$ and its first and last value are equal to v.

Moreover, we can choose the sequence S in such a way that

(2) all the cumulative sums in C are non-negative (this can be easily achieved, e.g., by taking all the positive values $i \in I^+(A)$ in S before the negative values $j \in I^-(A)$).

Now, we modify the sequence S so that the corresponding C will become cycle. The modifications will preserve the conditions (1) and (2).

If no value in C exceeds $n-1$, then C is a cycle in the threshold digraph $\mathscr{G}(A, h(A))$ and C contains exactly m_i of arcs with span i, for any $i \in I(A)$.

If some of the values in C exceeds $n-1$, then we take the first local maximum in C of that kind, i.e., we take $0 < k < s$ such that $C(k) > n-1$ and $C(k) > C(k-1), C(k+1)$. By definition of C, there are $i \in I^+(A)$, $j \in I^-(A)$ such that $S(k) = i$, $S(k+1) = j$ and $C(k) = C(k-1)+i$, $C(k+1) = C(k)+j$. We modify S to S' by transposing the values i, j on positions $k, k+1$, i.e., we put $S'(k) = j$ and $S'(k+1) = i$. The transposition preserves all the cumulative sums but $C(k)$, which is changed to $C'(k) = C(k-1) + j < C(k-1)$ (the value j is negative). By assumption, we have $i - j \le n \le C(k)$, i.e., $0 \le C(k) - i + j = C'(k)$. Thus, the conditions (1), (2) are preserved in C'. On the other hand, $C'(k)$ is not a local maximum in C'. By repeating the procedure finitely many times, we come to the desired cycle. □

THEOREM 4.2. *Let $A = T(a_{-n+1}, \dots, a_{-1}, a_0, a_1, \dots, a_{n-1}) \in \mathscr{B}(n,n)$ be a Toeplitz matrix, fulfilling the condition $\mathcal{C}^*$ from Lemma 4.1. Then for any component $\mathscr{K} \in \mathrm{SCC}^* \mathscr{G}(A, h(A))$*

(i) $\mathrm{per}(A) = \mathrm{per}(\mathscr{K})$,

(ii) $\mathrm{per}(A) = \gcd\{s(m)\colon m \in M(A)\}$.

P r o o f. (i) If $a_0 \ge h(A)$, then there is a loop at any vertex in the threshold digraph $\mathscr{G}(A, h(A))$. Therefore, any component $\mathscr{K}$ contains cycles of length 1. Then we have $\mathrm{per}(\mathscr{K}) = 1$ and $\mathrm{per}(A) = 1$, in view of Theorem 2.1(iii).

If $a_0 < h(A)$ then, by Lemma 4.1, any vertex v of the threshold digraph $\mathscr{G}(A, h(A))$ is contained in a non-trivial strongly connected component $\mathscr{K}$, which is minimal, in view of Remark 4.1. Moreover, all such components are isomorphic and have the same period. By Theorem 2.1(iii), we get $\mathrm{per}(A) = \mathrm{per}(\mathscr{K})$.

(ii) By (i), it is sufficient to prove that $\mathrm{per}(\mathscr{K}) = \gcd\{s(m)\colon m \in M(A)\}$. The length of any cycle in $\mathscr{K}$ is of the form $s(m)$ for some $m \in M^+(A)$. By Lemma 4.1, the converse implication holds true, as well. Therefore

$$\mathrm{per}(\mathscr{K}) = \gcd\{s(m)\colon m \in M^+(A)\}.$$

On the other hand, for any $m \in M(A)$, there is $m' \in M^+(A)$ such that the sum $m'' := m + m'$ belongs to $M^+(A)$. Namely, if there is $i \in I^+(A)$ such that $m_i < 0$, then we choose an index $j \in I^-(A)$ and define $m^{(i)} \in M^+(A)$ as follows

$$\begin{aligned} m_i^{(i)} &= m_i\, j\,, \\ m_j^{(i)} &= -m_i\, i\,, \\ m_k^{(i)} &= 0 \quad \text{for} \quad k \neq i,\, j\,. \end{aligned}$$

Then we have

$$\begin{aligned} m_i + m_i^{(i)} &= m_i + m_i\, j = m_i(1+j) \geq 0\,, \\ m_j + m_j^{(i)} &= m_j - m_i\, i \geq m_j\,, \\ m_k + m_k^{(i)} &= m_k\,, \end{aligned}$$

because $j + 1 \leq 0$, $m_i < 0$ hold true.

Analogously we define elements $m^{(j)} \in M^+(A)$ for any $j \in I^-(A)$ with $m_j < 0$. Finally, we put

$$m' = \sum_{\substack{m_i<0 \\ i\in I^+(A)}} m^{(i)} + \sum_{\substack{m_j<0 \\ j\in I^-(A)}} m^{(j)}\,.$$

Clearly, we have $m'' = m + m' \in M^+(A)$, i.e., $m = m'' - m'$. This implies $s(m) = s(m'') - s(m')$. Therefore $\gcd\big(s(m), s(m'')\big) = \gcd\big(s(m'), s(m'')\big)$ and we have

$$\begin{aligned} \operatorname{per}(A) = \operatorname{per}(\mathscr{K}) &= \gcd\{s(m) \colon m \in M^+(A)\} \\ &= \gcd\{\gcd\big(s(m'), s(m'')\big) \colon m', m'' \in M^+(A)\} \\ &= \gcd\big\{\gcd\big(s(m), s(m'')\big) \colon m \in M(A)\,,\ m'' \in M^+(A)\big\} \\ &= \gcd\big\{s(m) \colon m \in M(A)\big\}\,. \end{aligned}$$

□

The analogy of Theorem 3.1 and Theorem 4.2 allows us to formulate the following result.

THEOREM 4.3. *Let $A \in \mathscr{B}(n,n)$ be a Toeplitz matrix, fulfilling the condition $\mathcal{C}^*$ from Lemma 4.1, let $I(A) := \{i_0, i_1, \dots, i_{k-1}\}$. Then $\operatorname{per}(A)$ is equal to any of the expressions*

$$\text{(i)} \quad \gcd\left(\frac{i_0 - i_1}{\gcd(i_0, i_1)}, \frac{i_0 - i_2}{\gcd(i_0, i_2)}, \dots, \frac{i_0 - i_{k-1}}{\gcd(i_0, i_{k-1})}\right),$$

$$\text{(ii)} \quad \gcd\left(\frac{i_0 - i_1}{\gcd(i_0, i_1)}, \frac{i_1 - i_2}{\gcd(i_1, i_2)}, \dots, \frac{i_{k-2} - i_{k-1}}{\gcd(i_{k-2}, i_{k-1})}\right),$$

$$\text{(iii)} \quad \gcd\left(\frac{i_0 - i_1}{\gcd(n, i_0, i_1)}, \frac{i_0 - i_2}{\gcd(n, i_0, i_1, i_2)}, \dots, \frac{i_0 - i_{k-1}}{\gcd(n, i_0, i_1, \dots, i_{k-1})}\right).$$

P r o o f . The assertion is proved quite analogously as in Theorem 3.3, only the case $i = n$ is left out. Lemma 3.2 can be used without any change. □

THEOREM 4.4. *There is an algorithm by which, for a given Toeplitz matrix* $A = T(a_{-n+1}, \dots, a_{-1}, a_0, a_1, \dots, a_{n-1}) \in B(n,n)$ *fulfilling the condition* $(\mathcal{C}^*)$ *from Lemma 4.1,*

(i) *the period* $\operatorname{per}(A)$ *can be computed in* $O(n)$ *time,*

(ii) *if* $I(A)$ *is known and* $|I(A)| = k$, *then* $\operatorname{per}(A)$ *can be computed in* $O(k + \log n)$ *time.*

P r o o f . The proof is quite analogous as in Theorem 3.4. □

The next example shows that the condition $\mathcal{C}^*$ cannot be left out of Theorem 4.2.

EXAMPLE 1. Let $\mathscr{B} = \{0,1\}$, $n = 7$. Consider the Toeplitz matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} = T(1,0,1,0,0,0,0,0,1,0,0,0,0).$$

According to Definition 4.1, we have $h(A) = 1$, $I^+(A) = \{2\}$, $I^-(A) = \{-6,-4\}$. The condition $\mathcal{C}^*$ is not fulfilled, because for $i = 2$, $j = -6$ we have $i - j = 8 > n$.

The threshold digraph $\mathscr{G}(A, h(A))$ consists of two strongly connected components $\mathscr{K}_0 = \{0,2,4,6\}$ and $\mathscr{K}_1 = \{1,3,5\}$. The component $\mathscr{K}_0$ contains cycles $(0,2,4,0)$ and $(0,2,4,6,0)$ of length 3 and 4, respectively, therefore $\operatorname{per}(\mathscr{K}_0) = \gcd(3,4) = 1$. On the other hand, the component $\mathscr{K}_1$ itself is a cycle $(1,3,5,1)$ of length 3, therefore $\operatorname{per}(K_1) = 3$. Thus, the components $\mathscr{K}_0, \mathscr{K}_1$ are not isomorphic and have different periods. By Theorem 2.1(iii), $\operatorname{per}(A) = \operatorname{lcm}(1,3) = 3$. The example shows that without the condition $\mathcal{C}^*$, the assertions of Theorem 4.2 may not be satisfied.

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