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ON FUZZY PREDICTIVE DENSITIES

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ABSTRACT. In Bayesian statistics predictive distributions may be calculated using the updated distribution of the parameter. This distribution is obtained by integration of the joint density of the observed quantity and the parameter. In case of fuzzy data a fuzzy a-posteriori density of the parameter is obtained. Therefore the integration is an integration of a fuzzy function. The corresponding procedures are described in this paper.

1. Introduction

For a stochastic model $X \sim f(\cdot\,|\,\theta)$, $\theta \in \Theta$ with observation space M_X, i.e., the set of possible values for X, and the a-priori density $\pi(\theta)$ of the parameter, the density $p(\cdot\,|\,x_1,\dots,x_n)$ of the predictive distribution of X based on the data $x_1,\dots,x_n$ is given via the a-posteriori density $\pi(\theta\,|\,x_1,\dots,x_n)$ as follows

$$p(x\,|\,x_1,\dots,x_n) = \int\limits_{\Theta} f(x\,|\,\theta)\,\pi(\theta\,|\,x_1,\dots,x_n)\,d\theta \qquad \forall x \in M_X .$$

The a-posteriori density $\pi(\theta\,|\,x_1,\dots,x_n)$ is obtained by Bayes' theorem from the a-priori density $\pi(\theta)$ and the likelihood function $\ell(\theta; x_1,\dots,x_n)$. For details compare [1].

2. Fuzzy observed data

Observations of one-dimensional continuous stochastic quantities X yield non-precise values which are best described by fuzzy numbers x_i^*. The characterizing function of x_i^* is denoted by $\xi_i(\cdot)$. For a sample of size n of fuzzy observations $x_1^*,\dots,x_n^*$ there are n characterizing functions $\xi_1(\cdot),\dots,\xi_n(\cdot)$. There are generalizations based on fuzzy data of the equation of Section 1 using

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the extension principle. First it is necessary to generalize Bayes' theorem to the fuzzy data situation. In order to do this the fuzzy sample has to be combined with a fuzzy element $\underline{x}^*$ of the sample space M_X^n where M_X denotes the observations space of the stochastic quantity X. This combination is done by a t-norm in n variables. The fuzzy vector $\underline{x}^*$ obtained has a vector-characterizing function $\xi(x_1, \ldots, x_n)$ of n real variables. Using this vector-characterizing function Bayes' theorem is generalized in the following way: The classical form of Bayes' theorem is

$$\pi(\theta \,|\, x_1, \ldots, x_n) \propto \pi(\theta)\,\ell(\theta; x_1, \ldots, x_n)$$

where $\propto$ denotes "proportionality" up to a multiplicative constant.

For fuzzy data with fuzzy combined sample element $\underline{x}^*$ Bayes' theorem is generalized to obtain a fuzzy a-posteriori density $\pi^*(\cdot \,|\, \underline{x}^*)$ which is a family of fuzzy values $\big(\pi^*(\theta \,|\, \underline{x}^*); \theta \in \Theta\big)$ with corresponding characterizing functions

$$\big(\psi_\theta(\cdot); \theta \in \Theta\big)\,.$$

For details compare the monograph [2].

3. Fuzzy predictive densities

We need a generalization of the predictive density $p(x \,|\, x_1, \ldots, x_n)$ to the situation of fuzzy data $x_1^*, \ldots, x_n^*$ based on the fuzzy a-posteriori density $\pi^*(\theta \,|\, \underline{x}^*)$. Therefore the following integral

$$\int_\Theta f(x \,|\, \theta)\, \pi^*(\theta \,|\, \underline{x}^*)\, d\theta$$

has to be solved. This integral is an integral of the fuzzy valued function

$$h^*(\theta; x) = f(x \,|\, \theta)\, \pi^*(\theta \,|\, \underline{x}^*)\,.$$

Using the so-called α-level curves $\overline{h}_\alpha(\theta; x)$ and $\underline{h}_\alpha(\theta; x)$ which are assumed to be integrable, i.e.,

$$\overline{I}_\alpha(x) = \int_\Theta \overline{h}_\alpha(\theta; x)\, d\theta$$

and

$$\underline{I}_\alpha(x) = \int_\Theta \underline{h}_\alpha(\theta; x)\, d\theta$$

for all $\alpha \in (0,1]$, the fuzzy value $I^*(x)$ of the integral $\int_\Theta h^*(\theta; x)\, d\theta$ is given by its α-cuts

$$B_\alpha(x) = \big[\underline{I}_\alpha(x),\, \overline{I}_\alpha(x)\big]\,, \qquad \forall\, \alpha \in (0,1]$$

which yields the characterizing function $\chi_x(\cdot)$ of this fuzzy integral $I^*(x)$ using the representation

$$\chi_x(y) = \sup_{\alpha \in (0,1]} \alpha \cdot I_{B_\alpha(x)}(y) \qquad \forall\, y \in \mathbb{R}\,.$$

Now the generalized fuzzy predictive density is the fuzzy density whose values are

$$p^*(x \,|\, \underline{x}^*) = \int_\Theta f(x \,|\, \theta)\, \pi^*(\theta \,|\, \underline{x}^*)\, d\theta \qquad \forall\, x \in M_X$$

which is the family of fuzzy values $p^*(x \,|\, \underline{x}^*)$ whose characterizing functions are $\chi_x(\cdot)$, i.e.,

$$p^*(\cdot \,|\, \underline{x}^*) = \big(\chi_x(\cdot);\, x \in M_X\big)\,.$$

Graphically such fuzzy densities are best represented by so-called α-level curves $\overline{p}_\alpha(\cdot)$ and $\underline{p}_\alpha(\cdot)$. The values $\overline{p}_\alpha(x)$ and $\underline{p}_\alpha(x)$ are the endpoints of the α-cut of $\chi_x(\cdot)$, i.e.,

$$\big[\,\underline{p}_\alpha(x),\, \overline{p}_\alpha(x)\big] = \big\{y \in \mathbb{R} \colon \chi_x(y) \ge \alpha\big\}\,.$$

In Figure 1 an example of a fuzzy predictive density is given.

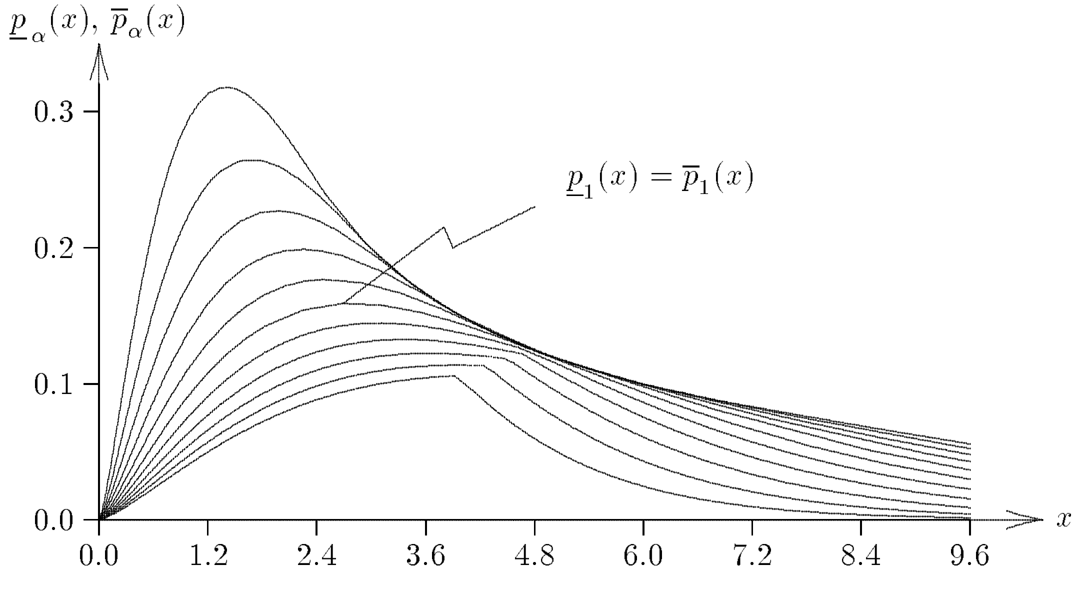

FIGURE 1. Fuzzy density. α-level curves for $\alpha = 0+,\ 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1$

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