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PERIODS OF MATRICES WITH ZERO-WEIGHT CYCLES IN MAX-ALGEBRA

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ABSTRACT. Periodicity of a matrix with zero-weight cycles in max-algebra is studied. The period of a matrix A is shown to be the least common multiple of the periods of all non-trivial strongly connected components in the corresponding digraph of A. $O(n^3)$ algorithms for verifying the mentioned condition for a given matrix and for computing the exact value of the matrix period in the positive case are described.

1. Introduction

A fuzzy discrete dynamic system is described by a transition matrix in extremal algebras such as *max-min algebra* (with operations max and min) and *max-algebra* (with operations max and plus). The properties of such systems were studied in [2]. Polynomial algorithms for computing the matrix and orbit periods in *max-min algebra* are described in [3, 4, 7], NP-completeness results connected with periodicity problems in *max-min algebra* are presented in [5, 6, 8].

The aim of this paper is to compute the exact value of the period for a given matrix $A \in \mathscr{W}(n,n)$ with zero-weight cycles in *max-algebra* $\mathscr{W}$. It is proved that the period of A is the least common multiple of the periods of all non-trivial strongly connected components in the digraph $\mathscr{G}(A)$. The main result is formulated in Theorem 3.2. Based on this result, an algorithm is proposed in Theorem 5.1, which enables to compute the matrix period in $O(n^3)$ time, if the matrix fulfils the condition of zero-weight cycles. In Theorem 4.1 another $O(n^3)$ algorithm for verifying this condition is described.

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2. Definitions

For simpler notation of index sets we shall use the convention by which any natural number n is considered as the set of all smaller natural numbers, i.e., $n = \{0, 1, \dots, n-1\}$. The set of all natural numbers is denoted by $\mathbb{N}$. If S is a finite subset of $\mathbb{N}$, then the greatest common divisor and the least common multiple of S are denoted by the abbreviations $\gcd S$ and $\operatorname{lcm} S$, respectively. For natural numbers a, b, if a divides b, we use the notation $a|b$.

If $\mathscr{G} = (V, E)$ is a digraph (directed graph), then by strongly connected component of $\mathscr{G}$ we mean a subdigraph $\mathscr{K} = (K, E \cap K^2)$ generated by a non-empty subset $K \subseteq V$ such that any two nodes $i, j \in K$ are contained in a common cycle, and K is a maximal subset with this property. $\mathscr{K}$ is called non-trivial if there is a cycle of a positive length in $\mathscr{K}$. In the opposite case, $\mathscr{K}$ is called trivial. By $\mathrm{SCC}^*(\mathscr{G})$ we denote the set of all non-trivial strongly connected components of $\mathscr{G}$. For any $\mathscr{K} \in \mathrm{SCC}^*(\mathscr{G})$ the period $\mathrm{per}(\mathscr{K})$ is defined as gcd of the lengths of all cycles in $\mathscr{K}$. A path in a digraph $\mathscr{G} = (V, E)$ is a sequence of nodes $(i_0, i_1, \dots, i_p)$ such that $(i_t, i_{t+1}) \in E$ for any $t \in p$. An elementary path is a path without repetitions. If a weight function w is defined on ares, then the weight of a path (or cycle), denoted by $w(p)$ (or w(c)), respectively, is the sum of the weights of its arcs.

Definition 2.1. The *max-algebra* $\mathscr{W}$ is a triple $(\mathscr{W}, \oplus, \otimes)$, where $\mathscr{W}$ denotes the set of all real numbers with added element $\varepsilon = -\infty$ and $\oplus, \otimes$ are the binary operations of maximum and plus, respectively, on $\mathscr{W}$. For any natural $n > 0$, $\mathscr{W}(n)$ denotes the set of all n-dimensional column vectors over $\mathscr{W}$, and $\mathscr{W}(n, n)$ denotes the set of all square matrices of order n over $\mathscr{W}$. The matrix operations over $\mathscr{W}$ are defined formally in the same way (with respect to $\oplus, \otimes$) as the matrix operations over any field.

For any $A \in \mathscr{W}(n,n)$, $x \in \mathscr{W}(n)$, the orbit of x generated by A is defined as the sequence $\left(A^{(r)}x \colon r \in \mathbb{N}\right)$. Let $A \in \mathscr{W}(n,n)$, $x \in \mathscr{W}(n)$, $i \in n$. The *period* of A, the *orbit period* and *coordinate-orbit period* of x with respect to A, in notation: $\mathrm{per}(A)$, $\mathrm{per}(Ax)$ and $\mathrm{per}\big((Ax)_i\big)$, are defined as the minimal natural p with the property $(\exists R)(\forall r > R)\ A^{(r)} = A^{(r+p)}$, or $x^{(r)} = x^{(r+p)}$, or $x_i^{(r)} = x_i^{(r+p)}$, respectively.

Definition 2.2. For a given matrix $A \in \mathscr{W}(n,n)$ we define the evaluated digraph $\mathscr{G}(A) = (V, E, w)$ with n nodes and with arcs (i,j) with the finite weights $w(i,j) = a_{ij}$. For any pair (i,j) with $a_{ij} = \varepsilon$, there is no arc from i to j in $\mathscr{G}(A)$.

Definition 2.3. We say that a matrix $A \in \mathscr{W}(n,n)$ fulfils the cycle-condition, if the weight of any cycle in the digraph $\mathscr{G}(A)$ is equal to zero, in notation $w(c) = 0$.

In this paper, we assume that the considered matrix fulfils the cycle-condition.

3. Period of the matrix

In this section we present the main result which is an explicit formula for the matrix period.

Lemma 3.1. *Let $A \in \mathscr{W}(n,n)$ fulfil the cycle-condition. Then the set of the weights of all paths in the digraph $\mathscr{G}$ is finite.*

P r o o f . The weight of any path is equal to the weight of some elementary path, obtained by cycle deletions [2]. The assertion of the lemma follows as a consequence of the fact that the set of all elementary paths in the digraph $\mathscr{G}$ is finite. □

Theorem 3.2. *Let $A \in \mathscr{W}(n,n)$ fulfil the cycle-condition. Let $d \in \mathbb{N}$. Then the following assertions are equivalent*

(i) $\operatorname{per}(A) \,|\, d$,

(ii) $\big(\forall \mathscr{K} \in \mathrm{SCC}^*(\mathscr{G})\big)\ \operatorname{per}(\mathscr{K}) \,|\, d$.

P r o o f . We prove the implication (ii)⇒(i). Let

$$\big(\forall \mathscr{K} \in \mathrm{SCC}^*(\mathscr{G})\big)\ \operatorname{per}(\mathscr{K}) \,|\, d\,.$$

We shall prove that

$$(\forall i,j \in n)\,(\exists R_{ij})\,(\forall r > R_{ij}) \quad a_{ij}^{(r)} = a_{ij}^{(r+d)}\,.$$

It means that, for any $i, j \in n$, the period $\operatorname{per}(a_{ij})$ divides d. Then

$$\big(\exists R = \max_{i,j \in n} R_{ij}\big)\big(\forall r > R\big)\ \big(\forall i,j \in n\big) \quad a_{ij}^{(r)} = a_{ij}^{(r+d)}$$

what implies that $\operatorname{per}(A) \,|\, d$.

Let $d \in \mathbb{N}$ be fixed, let $i, j \in n$ and let $k \in d = \{0, 1, \ldots, d-1\}$ be fixed. There are two possible cases.

Case 1. There are only elementary paths from i to j of length r, $r \equiv k \mod d$. Then

$$(\exists R_{ijk})(\forall r > R_{ijk}, r \equiv k \mod d) \quad a_{ij}^{(r)} = \varepsilon$$

and therefore

$$(\exists R_{ijk})(\forall r > R_{ijk}, r \equiv k \mod d) \quad a_{ij}^{(r)} = a_{ij}^{(r+d)}\,.$$

Case 2. There is a non-elementary path p_k from i to j of the length r_k, $r_k \equiv k \mod d$. Using Lemma 3.1 we can take such a path p_k that its weight is the maximum of the weights of all paths from i to j of the length r, $r \equiv k \mod d$. As p_k is non-elementary, it contains a cysle. The cycle must lie in a non-trivial strongly connected component $\mathscr{K}_k \in \mathrm{SCC}^*(\mathscr{G})$ with the period $\mathrm{per}(\mathscr{K}_k) = d' \mid d$. Since d' is the greatest common divisor of the lengths of all elementary cycles in $\mathscr{K}_k$, in view of the well known fact from number theory, there exists $m_k \in \mathbb{N}$ such that for any $m > m_k$, the multiple md' can be expressed as a linear combination of the lengths of all elementary cycles in $\mathscr{K}_k$, with coefficients from $\mathbb{N}$. By adding proper combinations of cycles in $\mathscr{K}_k$ to the path p_k we can obtain paths with lengths $r_k + md'$ and $r_k + md$, for any $m > m_k$, respectively. Since the weight of any cycle in $\mathscr{G}$ equals zero, the weights of these paths are the same as the weight of the path p_k, which is equal to $a_{ij}^{(r_k)}$.
Therefore

$$(\exists R_{ijk} = r_k + m_k d)(\forall r > R_{ijk}; r \equiv k \mod d) \quad a_{ij}^{(r)} = a_{ij}^{(r+d)}\,.$$

Denoting $R_{ij} = \max\limits_{k \in d} R_{ijk}$, we have

$$(\exists R_{ij})(\forall r > R_{ij}) \quad a_{ij}^{(r)} = a_{ij}^{(r+d)}\,.$$

As described in the beginning of the proof, it implies that $\mathrm{per}(A) \mid d$.

Now, we prove the converse implication (i) $\Rightarrow$ (ii).
Let d be fixed. Let

$$\mathrm{per}(A) \mid d\,.$$

We shall prove that

$$\big(\forall \mathscr{K} \in \mathrm{SCC}^*(\mathscr{G})\big) \ \mathrm{per}(\mathscr{K}) \mid d\,.$$

Let $\mathscr{K} \in \mathrm{SCC}^*(\mathscr{G})$. Let us denote $d' = \mathrm{per}(\mathscr{K})$. We shall prove that there exists i from $\mathscr{K}$ and x from $\mathscr{W}(n)$ such that $\mathrm{per}\big((Ax)_i\big) = d'$.
Then

$$\mathrm{per}\big((Ax)_i\big) \mid \mathrm{per}(Ax) \mid \mathrm{per}(A)$$

implies that $d' \mid \mathrm{per}(A)$ and using the assumption $\mathrm{per}(A) \mid d$ we get $\mathrm{per}(\mathscr{K}) = d' \mid d$.

Let $i \in \mathscr{K}$ be fixed. We define a vector $x \in \mathscr{W}(n)$ as follows

$$\begin{aligned} x_i &:= 0\,, \\ x_j &:= \varepsilon \qquad \text{for} \quad j \neq i\,. \end{aligned}$$

$(A^{(r)}x)_i$ equals zero if and only if there exists a cycle from i to i of the length r, otherwise it equals to ε. Using the fact from number theory we get

$$(\exists R)(\forall r > R)\big[(A^{(r)}x)_i = 0 \Leftrightarrow d' \,|\, r\,\big]$$

what is equivalent with

$$(\exists R)(\forall r > R)\big[a_{ii}^{(r)} = 0 \Leftrightarrow d' \,|\, r\,\big]\,.$$

The weight of any path from i to i with a length r, which is not a multiple of d', is equal to ε. Therefore $(A^{(r)}x)_i = \varepsilon$. Hence

$$\operatorname{per}\big((Ax)_i\big) = d'\,.$$

Then, as we have shown in the beginning of the proof,

$$\operatorname{per}(\mathscr{K}) = d' \,|\, d\,.$$

□

The main result can be expressed in the following exact way.

THEOREM 3.3. *Let $A \in \mathscr{W}(n,n)$ fulfil the cycle-condition. Then*

$$\operatorname{per}(A) = \operatorname{lcm}\,\big\{\operatorname{per}(\mathscr{K})\colon \mathscr{K} \in \mathrm{SCC}^*(\mathscr{G})\big\}\,.$$

P r o o f . By Theorem 3.2, the set of all common multiples of the set $\big\{\operatorname{per}(\mathscr{K})\colon \mathscr{K} \in \mathrm{SCC}^*(\mathscr{G})\big\}$ is equal to the set of all multiples of $\operatorname{per}(A)$. □

4. Algorithm for verifying the zero weight cycles condition for a given matrix

In this section we present an $O(n^3)$ algorithm for verifying the condition that the weight of any cycle in the digraph of a given matrix equals zero.

DEFINITION 4.1. For $A \in \mathscr{W}(n,n)$, $\Delta(A)$ denotes the formal matrix power sum

$$\Delta(A) \;=\; A \oplus A^{(2)} \oplus A^{(3)} \oplus \cdots = (\delta_{ij}\colon\; i,j \in n)\,,$$

called the *weak transitive closure* of A.

DEFINITION 4.2. For $A \in \mathscr{W}(n,n)$, A is called *p-regular* if there exists some natural p such that

$$\Delta(A) = A \oplus A^{(2)} \oplus A^{(3)} \oplus \cdots \oplus A^{(p)}\,.$$

THEOREM 4.1. *The cycle-condition can be verified in $O(n^3)$ time.*

Proof. For verifying the zero-weight cycles condition of a given matrix we will show that the maximum of all weights of all cycles equals to zero as well as the minimum of all weights of all cycles.

For a given matrix $A \in \mathscr{W}(n,n)$, we can obtain in $O(n^3)$ time, by Floyd-Warshall algorithm [9], the matrix $A^{\{n+1\}}$ in which the element $a_{ij}^{\{n+1\}}$ is equal to the maximal weight of all paths from i to j with lengths at most n. If the equation $a_{ii}^{\{n+1\}} \le 0$ holds true for any $i \in n$, then the weight of each cycle, of arbitrary length, is non-positive, too. This implies that

$$A^{\{n+1\}} = \Delta(A).$$

In $\Delta(A)$, the element δ_{ij} represents, for any $i, j \in n$, the maximal weight of all paths from i to j, regardless of their length. Therefore, for any $i \in n$, the diagonal element δ_{ii} represents the maximal weight of all cycles containing the node i.

Now, let us take, instead of a given matrix A, the matrix

$$-A' = (-a_{ij})$$

(in which we change the signs of all finite elements in A, while the signs of the infinite elements remain unchanged, i.e., A' does not denote the transposed matrix of A). We can find, in the same way as above, the weak transitive closure of $-A'$. For any $i \in n$ the diagonal element represents the maximal weight of all cycles in $\mathscr{G}(-A')$ containing node i, what is the minus value of the minimal weight of all cycles in $\mathscr{G}(A)$ containing node i, at the same time. The weight of all cycles is equal to zero if and only if both parts of the algorithm give the positive answer, i.e., both maximum and minimum of the cycle weights are equal to zero. $\square$

5. Algorithm for computing the matrix period

In this section we present an $O(n^3)$ algorithm to compute the matrix period for a given matrix A using Theorem 3.2.

THEOREM 5.1. *Let $A \in \mathscr{W}(n,n)$ fulfil the cycle-condition. Then* $\operatorname{per}(A)$ *can be computed in $O(n^3)$ time.*

Proof. The algorithm consists of three parts. First, we shall find the strongly connected components of the $\mathscr{G}(A)$. As mentioned in previous section, by Floyd-Warshall algorithm we can compute the weak transitive closure

$$\Delta(A) = (\delta_{ij}:\ i, j \in n)$$

for a given matrix $A \in \mathscr{W}(n,n)$ in $O(n^3)$ time. For any pair $i, j \in n$, there are two cases: either δ_{ij} is finite, then there exists a path from i to j in the digraph $\mathscr{G}$, or δ_{ij} equals to ε, in the opposite case. Therefore, the nodes i and j are contained in the same non-trivial strongly connected component if and only if both (δ_{ij}) and (δ_{ji}) are finite.

Second, we shall compute the period of each non-trivial strongly connected component. These periods can be found in $O(n^3)$ time by the Balcer-Veinott algorithm [1] described also in [3].

Finally, the period $\operatorname{per}(A)$ can be computed in $O(n \log n)$ time as the least common multiple of the periods of all non-trivial strongly connected components in digraph $\mathscr{G}$ by using the Euclid algorithm for computing $\gcd\{a,b\}$ for $a, b \in \mathbb{N}$ and the formula

$$\operatorname{lcm}\{a,b\} = \frac{ab}{\gcd\{a,b\}}\,.$$

□

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