Tatra Mt. Math. Publ. **16** (1999), 143–149

TATRA MOUNTAINS
Mathematical Publications

ON THE PRODUCT MV ALGEBRAS

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ABSTRACT. In an MV algebra M a binary operation (product) is introduced axiomatically. Then the product of M-values measures is constructed. The result is applied to the construction of the joint observable of two observables with values in M. Finally so-called divisible MV algebras are examined.

1. Introduction

A prototype of an MV algebra is a set $\mathscr{F}$ of fuzzy sets $f\colon \Omega \to [0,1]$ closed under the following binary operations $\oplus, \odot$ defined by

$$f \oplus g = \min(f+g, 1)\,,$$
$$f \odot g = \max(f+g-1, 0)\,,$$

and a binary operation $*$ defined by

$$f^* = 1 - f\,.$$

(A motivating example for $\oplus$ is the composition $f \oplus g$ of two grew pictures, $f\colon \Omega \to [0,1]$, $g\colon \Omega \to [0,1]$.) Evidently $1_\Omega = f \oplus f^* \in \mathscr{F}$ and $O_\Omega = 1_\Omega^* \in \mathscr{F}$.

Generally an MV algebra is an algebraic system $(M, \oplus, \odot, *, 1, 0)$, where M is a set, $\oplus, \odot$ are binary operations, $*$ is a unary operation 1 and 0 are fixed elements, satisfying the following properties:

(i) $a \oplus b = b \oplus a$,

(ii) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$,

(iii) $a \oplus 0 = a$,

(iv) $a \oplus 1 = 1$,

(v) $(a^*)^* = a$,

(vi) $0^* = 1$,

2000 Mathematics Subject Classification: 03G20, 06F20, 28B10.

Keywords: MV algebra, product of measures, joint observable.

This paper has been supported by grant VEGA 2/6087/99.

(vii) $a \oplus a^* = 1$,

(viii) $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$,

(ix) $a \odot b = (a^* \oplus b^*)^*$.

Of course, by the Mundici theorem ([3]), any MV algebra can be represented by a commutative l-group. Recall that a commutative l-group is an algebraic system $(G, +, \leq)$, where $(G, +)$ is a commutative group, $(G, \leq)$ is a partially ordered set being a lattice and $a \leq b$ implies $a + c \leq b + c$. Let $(G, +, \leq)$ be a commutative l-group, 0 be the neutral element of $(G, +)$ and $u \in G$, $u > 0$. On the interval $[0, u)$ define the following operations:

$$a \oplus b = (a + b) \wedge u\,,$$
$$a \odot b = (a + b - u) \vee 0\,,$$
$$a^* = u - a\,.$$

Then $\mathscr{M}G = \big([0, u], \oplus, \odot, *, u, 0\big)$ becomes an MV algebra. The Mundici theorem states that to any MV algebra there exists a commutative l-group G with a strong unit u (i.e., to any $a \in G$ there exists $n \in \mathbb{N}$ such that $a \leq nu$) such that

$$M \simeq \mathscr{M}G\,.$$

The first aim of the paper is a product measure theorem for M-valued measures. Therefore we introduce the notion of the product MV algebra.

Definition 1. A *product MV algebra* is an algebraic system $(M, \oplus, \odot, \cdot, *, u, 0)$, where $(M, \oplus, \odot, *, u, 0)$ is an MV algebra and $\cdot$ is a binary operation satisfying the following conditions:

(i) $u \cdot u = u$.

(ii) The operation $\cdot$ is commutative and associative.

(iii) If $a + b \leq u$, then $c \cdot (a + b) = c \cdot a + c \cdot b$ for any $c \in M$.

(iv) If $a_n \searrow 0$, $b_n \searrow 0$, then $a_n \cdot b_n \searrow 0$.

In Section 2 we shall prove the existence of the product of M-valued measures. This result is used for the construction of the joint observable of two observables in Section 3. Finally, in Section 4, so-called weakly divisible MV algebras are considered as a special case of the product MV algebra.

2. Product of measures

The notion of an observable is a special case of a measure. Therefore we first prove the product measure theorem and then the joint observable existence theorem as its special case.

Definition 2. Let M be an MV algebra. By a *measure with values in M* we shall understand any mapping $\mu\colon \mathscr{R} \to M$ defined on a ring $\mathscr{R}$ of subsets of a set X satisfying the following conditions:

(i) $\mu(X) = 1$.

(ii) If $A, B \in \mathscr{R}$, $A \cap B = \emptyset$, then $\mu(A \cup B) = \mu(A) + \mu(B)$.

(iii) If $A_n \in \mathscr{R}$ $(n = 1, 2, \dots)$, $A \in \mathscr{R}$, $A_n \nearrow A$, then $\mu(A_n) \nearrow \mu(A)$.

Definition 3. Let μ, ν, resp. be M-valued measures defined on measurable spaces $(X, \mathscr{S})$, $(Y, \mathscr{T})$, resp. Then by the *product measure* of μ and ν (if it exists) we understand a mapping $\mu \times \nu\colon \sigma(\mathscr{D}) \to M$ (where $\mathscr{D} = \{A \times B\colon\ A \in \mathscr{S},\ B \in \mathscr{T}\}$ such that

$$\mu \times \nu(C \times D) = \mu(C) \cdot \nu(D)$$

for any $C \in \mathscr{S}$, $D \in \mathscr{T}$.

Definition 4. A σ-complete MV algebra M is called weakly σ-distributive, if for any bounded double sequence $(a_{ij})_{i,j}$ such that $a_{ij} \searrow 0$ $(j \to \infty$, $i = 1, 2, \dots)$ it is

$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \bigvee_{i=1}^{\infty} a_{i\varphi(i)} = 0 .$$

Theorem 1. *Let $\mu, \nu\colon \mathscr{B}(\mathbb{R}) \to M$ be measures defined on the σ-algebra of Borel subsets of $\mathbb{R}$. Let M be weakly σ-distributive. Then there exists the product measure $\mu \times \nu$.*

Proof. Define $\mathscr{P} = \big\{[a,b) \times [c,d)\colon a < b,\ c < d\big\}$ and $\lambda\colon \mathscr{P} \to M$ by

$$\lambda(A \times B) = \mu(A) \cdot \nu(B) .$$

It is not difficult to see that λ is additive on $\mathscr{P}$. Therefore it is possible to extend λ to the ring $s(\mathscr{P})$ generated by $\mathscr{P}$:

$$\lambda\Big(\bigcup_{i=1}^{n} A_i \times B_i\Big) = \sum_{i=1}^{n} \lambda(A_i \times B_i) .$$

Again, λ is additive.

Denote by $\mathscr{C}$ the family of all compact sets in $\mathbb{R}^2$. The mapping $\lambda\colon s(\mathscr{P}) \to M$ is compact in the following sense: To any $D \in s(\mathscr{P})$ there are $C_i \in \mathscr{C}$ and $A_i \in s(\mathscr{P})$ such that $A_i \subset C_i \subset D$ and

$$\lambda(D \setminus A_i) \searrow 0 \qquad (j \to \infty) .$$

Indeed, if

$$D = \bigcup_{j=1}^{n} [a_j, b_j) \times [c_j, d_j),$$

then it is sufficient to put

$$C_i = \bigcup_{j=1}^{n} \left[a_j, b_j - \tfrac{1}{i}\right] \times \left[c_j, d_j - \tfrac{1}{j}\right],$$

$$A_i = \bigcup_{j=1}^{n} \left[a_j, b_j - \tfrac{2}{j}\right) \times \left[c_j, d_j - \tfrac{2}{j}\right),$$

and to use (iv) of Definition 1.

Now by the Alexandrov theorem ([4], Theorem 9.5.3), λ is upper continuous in $\emptyset$. Therefore $\lambda\colon s(\mathscr{P}) \to M$ is a measure.

By the measure extension theorem ([4], Theorem 3.4.2) there exists exactly one measure $\mu \times \nu\colon \sigma\big(s(\mathscr{P})\big) \to M$ extending λ. Of course, $\sigma\big(s(\mathscr{P})\big) = \mathscr{B}(\mathbb{R}^2)$ and

$$\mu \times \nu(C \times D) = \lambda(C \times D) = \mu(C) \cdot \nu(D)$$

for any $C \in \mathscr{B}(\mathbb{R})$, $D \in \mathscr{B}(\mathbb{R})$. □

3. Joint observable

Definition 5. An *observable* is a mapping $x\colon \mathscr{B}(\mathbb{R}) \to M$ satisfying the following conditions:

(i) $x(\mathbb{R}) = 1$.

(ii) If $A \cap B = \emptyset$, then $x(A \cup B) = x(A) + x(B)$.

(iii) If $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.

Definition 6. Let $x, y\colon \mathscr{B}(\mathbb{R}) \to M$ be two observables. The *joint observable* of the observables x, y is a mapping $h\colon \mathscr{B}(\mathbb{R}^2) \to M$ satisfying the following conditions:

(i) $h(\mathbb{R}^2) = 1$.

(ii) If $A \cap B = \emptyset$, then $h(A \cup B) = h(A) + h(B)$.

(iii) If $A_n \nearrow A$, then $h(A_n) \nearrow h(A)$.

(iv) $h(C \times D) = x(C) \cdot y(D)$ for any $C, D \in \mathscr{B}(\mathbb{R})$.

THEOREM 2. *Let M be a σ-complete weakly σ-distributive MV algebra. Then to any observables x, y there exists their joint observable.*

Proof. Evidently $x\colon \mathscr{B}(\mathbb{R}) \to M$, $y\colon \mathscr{B}(\mathbb{R}) \to M$ are measures. By Theorem 1 there exists their product $x \times y\colon \mathscr{B}(\mathbb{R}^2) \to M$. Put $h = x \times y$. Then

$$h(\mathbb{R}^2) = x(\mathbb{R}) \cdot y(\mathbb{R}) = u \cdot u = u\,,$$

h is additive and continuous and

$$h(C \times D) = x(C) \cdot y(D)\,.$$

□

Let us return to our basic example $\mathscr{F}$ of an MV algebra, $\mathscr{F} \subset [0,1]^{\Omega}$.

THEOREM 3. *Let $\mathscr{F}$ be the family of all fuzzy sets $f\colon \Omega \to [0,1]$ measurable with respect to a given σ-algebra of subsets of Ω. Then $\mathscr{F}$ is σ-complete weakly σ-distributive MV algebra.*

Proof. Let $f_{ij} \in \mathscr{F}$, $f_{ij} \searrow 0$ $(j \to \infty)$. Fix i and put $g_j = f_{ij}$, $g(\omega) = \lim\limits_{j\to\infty} g_j(\omega)$. Since $g \in \mathscr{F}$ and $g \le g_j$ $(j = 1, 2, \dots)$, we obtain $g \le 0$, hence $f_{ij}(\omega) = g_j(\omega) \searrow 0$ $(j \to \infty)$ for any $\omega \in \Omega$. Since $g_j(\omega) \in \mathbb{R}$, to any $\omega \in \Omega$, any $\varepsilon > 0$ and any $i \in \mathbb{N}$ there exists $\Psi(i, \omega)$ such that

$$f_{i\Psi(i,\omega)}(\omega) < \varepsilon\,.$$

Fix ω and define $\varphi\colon \mathbb{N} \to \mathbb{N}$ by the formula $\varphi(i) = \Psi(i, \omega)$. Then to any ω and any i there exists $\varphi\colon \mathbb{N} \to \mathbb{N}$ such that

$$\left(\bigvee_{i=1}^{\infty} f_{i\varphi(i)}(\omega)\right) = \bigvee_{i=1}^{\infty} \left(f_{i\varphi(i)}(\omega)\right) \le \varepsilon\,.$$

Therefore

$$\bigwedge_{\varphi\colon \mathbb{N}\to\mathbb{N}} \bigvee_{i=1}^{\infty} f_{i\varphi(i)}(\omega) \le \varepsilon$$

for any $\varepsilon > 0$, hence

$$\bigwedge_{\varphi\colon \mathbb{N}\to\mathbb{N}} \bigvee_{i=1}^{\infty} f_{i\varphi(i)}(\omega) = 0$$

for any $\omega \in \Omega$. Define $g_{\varphi}\colon \Omega \to \mathbb{R}$ by

$$g_{\varphi}(\omega) = \bigvee_{i=1}^{\infty} f_{i\varphi(i)}(\omega)\,.$$

We have proved

$$\bigwedge_{\varphi} g_{\varphi}(\omega) = 0$$

for any $\omega \in \Omega$. Evidently $g_{\varphi} \geq 0$ for any φ. Let $h \in \mathscr{F}$, $h \leq g_{\varphi}$ for any φ. Then $h(\omega) \leq g_{\varphi}(\omega)$ for any φ. Therefore $h(\omega) \leq \bigwedge_{\varphi} g_{\varphi}(\omega) = 0$. Since $h(\omega) \leq 0$ for any $\omega \in \Omega$, we obtain $h \leq 0$, hence

$$0 = \bigwedge_{\varphi} g_{\varphi} = \bigwedge_{\varphi} \bigvee_{i=1}^{\infty} f_{i\varphi(i)} .$$

We have proved that $\mathscr{F}$ is weakly σ-distributive. □

4. Weakly divisible MV algebras

Definition 7. An MV algebra $(M, \oplus, \odot, *, u, 0)$ is called *weakly divisible*, if to any $n \in \mathbb{N}$ there exists $a \in M$ such that $na = u$.

Theorem 4. *Any σ-complete weakly divisible MV algebra is isomorphic to an MV algebra $\mathscr{F}$ of all functions $f \colon \Omega \to [0,1]$ continuous on a compact Hausdorff topological space Ω.*

Proof. [1], Theorem 3.9. □

Theorem 5. *Any σ-complete weakly divisible MV algebra is a product MV algebra.*

Proof. Let $F \colon M \to \mathscr{F}$ be an isomorphism stated in Theorem 4. Define

$$a \cdot b = F^{-1}\big(F(a) \cdot F(b)\big) ,$$

where $F(a) \cdot F(b)$ is the usual product of real functions. Evidently the binary operation $\cdot$ on M is commutative and associative, and

$$\begin{aligned} u \cdot u &= F^{-1}\big(F(u) \cdot F(u)\big) = F^{-1}(1_{\Omega} \cdot 1_{\Omega}) \\ &= F^{-1}(1_{\Omega}) = u . \end{aligned}$$

If $a + b \leq u$, then $F(a) + F(b) \leq F(u) = 1_{\Omega}$, hence

$$F(c) \cdot F(a+b) = F(c) \cdot F(a) + F(c) \cdot F(b) \leq 1_{\Omega} ,$$

and

$$\begin{aligned} c \cdot (a+b) &= F^{-1}\big(F(c) \cdot F(a+b)\big) \\ &= F^{-1}\big(F(c) \cdot F(a)\big) + F^{-1}\big(F(c) \cdot F(b)\big) \\ &= c \cdot a + c \cdot b . \end{aligned}$$

Let $a_n \searrow 0$, $b_n \searrow 0$. Then $F(a_n) \searrow 0_{\Omega}$, $F(b_n) \searrow 0_{\Omega}$, hence

$$a_n \cdot b_n = F^{-1}\big(F(a_n) \cdot F(b_n)\big) \searrow F^{-1}(0_{\Omega}) = 0 .$$

□

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Received April 3, 1998

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