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FUZZY CATEGORIES RELATED TO ALGEBRA AND TOPOLOGY

Alexander Šostak

ABSTRACT. The concept of a fuzzy category first introduced in [71], was studied in [75]. Roughly speaking, a fuzzy category is a category-like conglomerate in which "potential objects" and "potential morphisms" are respectively objects and morphisms only to a certain degree. In this paper, after briefly recalling basic concepts related to fuzzy categories, we discuss a series of examples of fuzzy categories connected with the categories of sets, groups, topological spaces, proximity spaces and metric spaces.

1. Introduction

Since the inception of fuzzy set theory (see [83], cf. also [65], [15]), the efforts of many researchers have been directed to develop fuzzy analogues of basic concepts of classical mathematics and to work out corresponding theories. Just to give *a few examples* of numerous investigations in this direction we shall mention here the following:

Topology. Topological structures on lattices of fuzzy sets were first considered in 1968 by C. L. Chang [8]; later Chang's ideas were developed in essentially different directions by J. A. Goguen [16], R. Lowen [48, 49], U. Höhle [21, 22, 23], B. Hutton [35], S. E. Rodabaugh [60, 62], Wang Guojun [76], A. Šostak [67, 70], Ying Mingsheng [78]–[82], U. Höhle and A. Šostak et al [30, 31].

Different approaches to fuzzy uniform structures were developed by B. Hutton [34], R. Lowen [50], S. E. Rodabaugh [61], D. Dzajanbajev and A. Šostak [10, 74], M. Burton, J. Gutierrez and M. A. de Prada [6, 18], et al. Various concepts of fuzzy proximity spaces were introduced and

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studied by A. K. Katsaras [36, 37], G. Artico and R. Moresco [1], J. Coulon and J. L. Coulon [9], S. Markin and A. Šostak [54], et al. Fuzzy syntopogeneous structures were considered in [38, 40, 74], etc.

(See also [73], [30], [31], [63] [45] and [5] for the surveys of different categories of topological type structures in the context of fuzzy sets.)

Algebra. Fuzzy groups were defined by A. Rosenfeld in 1971 [64]: this was probably the first attempt to consider an algebraic structure in connection with fuzzy sets. Rosenfelds's work was followed by many other papers in which algebraic structures were defined and studied in the context of fuzzy sets: see, e.g., A. Katsaras and D. B. Liu [39] (fuzzy subspaces of vector spaces), W. J. Liu [47], D. Malik and N. Mordeson [52] (fuzzy rings), [53] (fuzzy fields), V. Murali [56] (fuzzy subalgebras of universal algebras), [55] (fuzzy equivalence relations), Yuan Bo and Wo Wangming [77] (fuzzy ideals on distributive lattices) et al. Fuzzy topologies on algebraic structures were considered by D. H. Foster [13] and U. Höhle [24].

Measure theory. The concepts of a fuzzy σ-algebra and a fuzzy measurable function are due to E. P. Klement [41, 42], see also [44]. An alternative approach to the concept of a fuzzy measure and the corresponding theory was developed by U. Höhle [20]; see also E. P. Klement's papers [43], etc.

Of course, this list of directions of mathematics where certain fuzzy counterparts of classical mathematical concepts were defined and investigated could be continued and essentially extended. However, for our purposes it is important to emphasize that in all these works, as well as in the overwhelming majority of investigations in the field of "fuzzy mathematics" not mentioned here, the authors actually are working in usual, i.e., crisp categories. (It is not crucial here whether an author explicitly states the categorical framing of his research, or the categorical context reveals itself in the work only implicitly.) Even if the term "a fuzzy category" is used in a work as, e.g., in [58, Chapter 2.1], the question actually is about a category in the classical, crisp sense. "The fuzzy aspect" in all these investigations shows itself by a certain fuzzy framing of objects or/and of morphisms of the category: It can be a crisp structure on a lattice of fuzzy sets (as, e.g., in [8, 15, 48, 33, 35, 60, 62, 11, 12, 76], etc), or a fuzzy structure on some lattice (as, e.g., in [67, 70, 30, 78, 79], etc), or a certain structure (in particular, topological or algebraic) on a fuzzy set (as, e.g., in [64, 13, 47, 56, 52, 53]), or a fuzzy relation (in particular, a fuzzy equality) on a set (as, e.g., in [7, 55, 57, 26, 28, 29, 32], etc.), or fuzzy relations in the role of morphisms (see, e.g., [58, Chapter 2.1], [59]). In the sequel all these situations will be referred to as "categories of fuzzily structured sets".

As opposed to the categories of fuzzily structured sets, in the papers [71] and [72] the concept of a *fuzzy category* was introduced. Roughly speaking, a fuzzy category is a category-like conglomerate in which "potential objects" and "potential morphisms" are respectively objects and morphisms only to a certain degree (see Section 2 for the precise definition). Later, in a paper [75] we started systematically to develop the fundamentals of the theory of fuzzy categories; moreover [75] contains several examples of fuzzy categories.

The main purpose of this paper is to consider some concrete fuzzy categories which are mainly related to algebra and topology. These fuzzy categories are considered in Sections 3, 4, 5, 6 and 7. Some of these fuzzy categories were already introduced in [75], others appear here for the first time. In order to make the paper reasonably self-contained, before discussing concrete fuzzy categories, in Section 2 we expose those definitions and results from [75] which are used in the main text. In Section 1 we recall some definitions, facts and examples related to GL-monoids which will be needed both to introduce the general concept of a fuzzy category and to describe concrete examples of fuzzy categories.

1. Preliminaries

To introduce the concept of a fuzzy category in a sufficiently general setting and thus to come to natural non-trivial examples of (L-)fuzzy categories it is insufficient to restrict oneself to a purely lattice-theoretic context and therefore there appears the necessity to enrich the basic lattice L with an additional algebraic structure (or structures). Although different examples impose different demands on these structures, to make our exposition here more homogeneous, we shall assume that the underlying lattice L is enriched with a monoidal operation $*$ in such a way that the resulting triple $(L, \leq, *)$ is a GL-monoid. Although not the most general, this assumption is convenient both to state the necessary elements of the theory of fuzzy categories and to expose most of the examples of fuzzy categories known to us. Therefore, we shall briefly discuss here the concept of a GL-monoid and some related notions.

1.1 GL-monoids.

Let $(L, \leq)$ be a complete lattice, i.e., $(L, \leq)$ is a partially ordered set such that for every subset $A \subset L$ the join $\bigvee A$ and the meet $\bigwedge A$ are defined. In particular, $\bigvee L =: 1$ and $\bigwedge L =: 0$ are respectively the universal upper and the universal lower bounds in L (see, e.g., [2]). We assume that $0 \neq 1$, i.e., L has at least two elements.

A GL-monoid is a complete lattice enriched with a further binary operation $*$, i.e., a triple $(L, \leq, *)$ such that:

(1) $*$ is monotone, i.e., $\alpha \leq \beta$ implies $\alpha * \gamma \leq \beta * \gamma, \quad \forall \alpha, \beta, \gamma \in L$;

(2) $*$ is commutative, i.e., $\alpha * \beta = \beta * \alpha, \quad \forall \alpha, \beta \in L$;

(3) $(L, \leq, *)$ is integral, i.e., 1 acts as the unity: $\alpha * 1 = \alpha, \quad \forall \alpha \in L$;

(4) 0 acts as the zero element in $(L, \leq, *)$, i.e., $\alpha * 0 = 0, \quad \forall \alpha \in L$;

(5) $*$ is distributive over arbitrary joins, i.e., $\alpha * \left(\bigvee_i \beta_i\right) = \bigvee_i (\alpha * \beta_i)$, $\forall \alpha \in L$, $\forall \{\beta_i : i \in \mathcal{I}\} \subset L$;

(6) $(L, \leq, *)$ is divisible, i.e., $\alpha \leq \beta$ implies the existence of $\gamma \in L$ such that $\alpha = \beta * \gamma$

(see, e.g., [27], [28]; note that GL stands for *generalized logics*).

It is known that every GL-monoid is residuated, i.e., there exists a further binary operation "$\longmapsto$" (implication) on L satisfying the following condition:

$$\alpha * \beta \leq \gamma \Longleftrightarrow \alpha \leq (\beta \longmapsto \gamma), \qquad \forall \alpha, \beta, \gamma \in L.$$

Explicitly, the implication is given by

$$\alpha \longmapsto \beta = \bigvee \{\lambda \in L : \alpha * \lambda \leq \beta\}.$$

Below we list some properties of GL-monoids which will be useful in the sequel (see, e.g., [27], [28]):

(i) $\alpha \longmapsto \beta = 1 \Longleftrightarrow \alpha \leq \beta$;

(ii) $\alpha \longmapsto \left(\bigwedge_i \beta_i\right) = \bigwedge_i (\alpha \longmapsto \beta_i)$;

(iii) $\left(\bigvee_i \alpha_i\right) \longmapsto \beta = \bigwedge_i (\alpha_i \longmapsto \beta)$;

(v) $\alpha * \left(\bigwedge_i \beta_i\right) = \bigwedge_i (\alpha * \beta_i)$;

(vi) $(\alpha \longmapsto \gamma) * (\gamma \longmapsto \beta) \leq \alpha \longmapsto \beta$;

(vii) $\alpha \leq \beta \implies \gamma \longmapsto \alpha \leq \gamma \longmapsto \beta$.

1.2 Examples of GL-monoids.

1.2.1 Heyting algebras. Every complete infinitely distributive lattice $(L, \leq)$ viewed as a Heyting algebra (i.e., with $* = \wedge$) is obviously a GL-monoid.

1.2.2 MV-algebras. In the class of GL-monoids MV-algebra can be characterized as those GL-monoids, in which the double negation law holds, i.e., $(\alpha \longmapsto 0) \longmapsto 0 = \alpha$.

Notice that in an MV-algebra $\alpha \longmapsto 0 = 1$ iff $\alpha = 0$ and $\alpha \longmapsto 0 = 0$ iff $\alpha = 1$.

1.2.3 Continuous t-norms. Let $L = [0,1]$ be the closed unit interval endowed with the natural order relation $\leq$. According to K. Menger (see, e.g., the book by B. Schweizer and A. Sklar [66]), a *t-norm* on $[0,1]$ is a binary operation $T\colon [0,1] \times [0,1] \to [0,1]$ such that

(T1) $T(x,1) = x$, $T(x,0) = 0$ (boundary conditions);

(T2) $T(x,y) = T(y,x)$ (symmetry);

(T3) $T(x,y) \leq T(x',y')$ whenever $x \leq x'$ and $y \leq y'$ (isotonocity);

(T4) $T\big(x, T(y,z)\big) = T\big(T(x,y), z\big)$ (associativity).

It is known that a triple $([0,1], \leq, T)$ is a GL-monoid if and only if the t-norm T is continuous (see, e.g., [28, p. 72]).

Important examples of t-norms are:

$$T_m(x,y) = \min(x,y)\,;$$
$$T_p(x,y) = x \cdot y\,;$$
$$T_L(x,y) = \max(x+y-1, 0)\,.$$

(T_L is called the *Łukasiewicz conjunction or the Łukasiewicz t-norm*)

Moreover, it is known that every continuous t-norm is an ordinal sum of these t-norms.

In the sequel, $L = (L, \leq, *)$ *will always stand for a GL-monoid.*

2. L-fuzzy categories: basic concepts

Here we shall recall some definitions from [75] which will be needed in the sequel.

2.1 An L-fuzzy category.

An *L-fuzzy category* $\mathscr{C}$ consists of:

(1) A class $\mathcal{O}b(C)$ of potential objects.

(2) An L-fuzzy subclass ω of $\mathcal{O}b(C)$, $\omega\colon \mathcal{O}b(C) \longrightarrow L$.

(3) A class $\mathcal{M}(C) = \bigcup \big\{\mathcal{M}_C(X,Y)\colon X, Y \in \mathcal{O}b(C)\big\}$ of pairwise disjoint sets $\mathcal{M}_C(X,Y)$ for each pair of potential objects $X, Y \in \mathcal{O}b(C)$; the members of $\mathcal{M}_C(X,Y)$ are called potential morphisms from X to Y and the members of $\mathcal{M}(C)$ are called potential morphisms of the category $\mathscr{C}$.

(4) An L-fuzzy subclass μ of $\mathcal{M}(C)$, $\mu\colon \mathcal{M}(C) \longrightarrow L$, such that if $f \in \mathcal{M}_C(X,Y)$, then $\mu(f) \leq \omega(X) \wedge \omega(Y)$.

(5) A composition $\circ$ of morphisms is defined, i.e., for each triple $X, Y, Z \in \mathcal{O}b(C)$ of potential objects there exists a map

$$\circ\colon \mathcal{M}_C(X,Y) \times \mathcal{M}_C(Y,Z) \longrightarrow \mathcal{M}_C(X,Z) \qquad ((f,g) \to g \circ f),$$

such that the following axioms are satisfied:

(i) preservation of morphisms: $\mu(g \circ f) \geq \mu(g) * \mu(f)$;

(ii) associativity: if $f \in \mathcal{M}_C(X,Y)$, $g \in \mathcal{M}_C(Y,Z)$, and $h \in \mathcal{M}_C(Z,U)$, then $h \circ (g \circ f) = (h \circ g) \circ f$;

(iii) existence of identities: for each $X \in \mathcal{O}b(C)$ there exists an identity $e_X \in \mathcal{M}_C(X,X)$ such that $\mu(e_X) = \omega(X)$ and for all $X, Y, Z \in \mathcal{O}b(C)$, all $f \in \mathcal{M}_C(X,Y)$ and all $g \in \mathcal{M}_C(Z,X)$ it holds $f \circ e_X = f$ and $e_X \circ g = g$.

As in the classic situation, it is easy to see that the identity morphism is uniquely determined by X.

2.2 Remarks on terminology, notation and comments.

2.2.1. As in the classic Category Theory we shall usually write $f\colon X \to Y$ instead of $f \in \mathcal{M}_C(X,Y)$. Here X is called the *domain*, and Y the *codomain* of f.

2.2.2. Let $\mathcal{O}b_\alpha(C) = \{X \in \mathcal{O}b(C)\colon \omega(X) \geq \alpha\}$; $\mathcal{M}_\alpha(C) = \{f \in \mathcal{M}(C)\colon \mu(f) \geq \alpha\}$; $\mathcal{M}_\alpha^{\mathscr{C}}(X,Y) = \{f \in \mathcal{M}_C(X,Y)\colon \mu(f) \geq \alpha\}$. The elements of $\mathcal{O}b_\alpha(C)$ will be referred to as α-objects of the L-fuzzy category $\mathscr{C}$, while the elements of $\mathcal{M}_\alpha(C)$ will be called α-morphisms of the L-fuzzy category $\mathscr{C}$.

The elements of $\mathcal{O}b(C)$ will be referred to as potential objects or just as objects of the fuzzy category $\mathscr{C}$, respectively, the elements of $\mathcal{M}(C)$ will be called the potential morphisms or just morphisms of $\mathscr{C}$.

2.2.3. Given an L-fuzzy category $\mathscr{C} = \big(\mathcal{O}b(C), \omega, \mathcal{M}(C), \mu, \circ\big)$ and $X \in \mathcal{O}b(C)$, the intuitive meaning of the value $\omega(X)$ is the *degree* to which a potential object X of the L-fuzzy category $\mathscr{C}$ is indeed its object; similarly, for $f \in \mathcal{M}(C)$ the intuitive meaning of $\mu(f)$ is the degree to which a potential morphism f of $\mathscr{C}$ is indeed its morphism.

2.2.4. Let $\mathscr{C} = \big(\mathcal{O}b(C), \mathcal{M}(C), \circ\big)$ be a usual (i.e., crisp) category with $\mathcal{O}b(C)$ as the class of objects and $\mathcal{M}(C)$ as the class of morphisms, and $\circ$ as the composition law. Then $\mathscr{C}$ can be viewed also as an L-fuzzy category $\big(\mathcal{O}b(C), \omega, \mathcal{M}(C), \mu, \circ\big)$ where L-fuzzy classes $\omega\colon \mathcal{O}b(C) \to L$ and $\mu\colon \mathcal{M}(C) \to L$ are defined, respectively, by the equalities $\omega \equiv 1$, $\mu \equiv 1$.

Thus usual categories and L-fuzzy categories satisfying two additional conditions:

(cω) $\omega \equiv 1$;

(cμ) $\mu \equiv 1$,

are essentially the same thing.

2.2.5. Given an L-fuzzy category $\mathscr{C} = \big(\mathcal{O}b(C), \omega, \mathcal{M}(C), \mu, \circ\big)$ one can construct a crisp category $\mathscr{C}^{\perp} = \big(\mathcal{O}b(C), \mathcal{M}(C), \circ\big)$ by taking all potential objects and potential morphisms as objects and morphisms of $\mathscr{C}^{\perp}$, respectively, and leaving the composition law unchanged. The category $\mathscr{C}^{\perp}$ is called the (*crisp*) *bottom frame* or just the *bottom frame* of $\mathscr{C}$.

This construction can be generalized: let ι be an idempotent element of L and let $\mathscr{C}_\iota = \big(\mathcal{O}b_\iota(C), \mathcal{M}_\iota(C), \circ\big)$. Then $\mathscr{C}_\iota$ is a crisp category having $\mathcal{O}b_\iota(C)$ as objects and $\mathcal{M}_\iota(C)$ as morphisms. The composition law is the same as in $\mathscr{C}$. In particular, $\mathscr{C}_0 = \mathscr{C}^{\perp}$ and $\mathscr{C}_1 =: \mathscr{C}^{\top}$ is the (*crisp*) *top frame* of $\mathscr{C}$. (Of course, the case when $\mathscr{C}^{\top}$ is empty is not excluded.)

2.3 Fuzzy subcategories and quotient fuzzy categories.

Definition 2.3.1. Let $\mathscr{C} = \big(\mathcal{O}b(C), \omega, \mathcal{M}(C), \mu, \circ\big)$ be an L-fuzzy category. By an (L-fuzzy) subcategory of $\mathscr{C}$ we call an L-fuzzy category $\mathscr{C}' = \big(\mathcal{O}b(C), \omega', \mathcal{M}(C), \mu', \circ\big)$ where $\omega' \leq \omega$ and $\mu' \leq \mu$.

Thus an L-fuzzy category and its subcategory have the same classes of potential objects and potential morphisms and the same operation of composition. The only difference between a subcategory and the whole category is in L-fuzzy classes of objects and morphisms, i.e., in the degrees of belonging of potential objects and morphisms.

Remark 2.3.2. Let $\mathscr{C} = \big(\mathcal{O}b(C), \mathcal{M}(C), \circ\big)$ be a crisp category and $\mathscr{D} = \big(\mathcal{O}b(D), \mathcal{M}(D), \circ\big)$ be its (usual) subcategory. Then $\mathscr{D}$ can be identified with the fuzzy subcategory $\tilde{\mathscr{D}} = \big(\mathcal{O}b(C), \omega', \mathcal{M}(C), \mu', \circ\big)$ of $\big(\mathcal{O}b(C), \omega, \mathcal{M}(C), \mu, \circ\big)$, where $\omega \equiv 1$, $\mu \equiv 1$, such that $\omega'(X) = 1$ if $X \in \mathcal{O}b(D)$ and $\omega'(X) = 0$ otherwise; $\mu'(f) = 1$ if $f \in \mathcal{M}(D)$ and $\mu'(f) = 0$ otherwise. In particular, $\tilde{\mathscr{D}}^{\top} = \mathscr{D}$.

Definition 2.3.3. A subcategory $\mathscr{C}' = \big(\mathcal{O}b(C), \omega', \mathcal{M}(C), \mu', \circ\big)$ of an L-fuzzy category $\mathscr{C} = \big(\mathcal{O}b(C), \omega, \mathcal{M}(C), \mu, \circ\big)$ is called *full* if $\mu'(f) = \mu(f) \wedge \omega'(X) \wedge \omega'(Y)$ for every $f \in \mathcal{M}_C(X, Y)$.

Definition 2.3.4. An equivalence relation $\sim$ on the class of morphisms of an L-fuzzy category $\mathscr{C}$ is called a *congruence* on $\mathscr{C}$ provided that:

(1) each $\sim$-equivalence class is contained in $\mathcal{M}_C(X,Y)$ for some $X,Y \in \mathcal{O}b(C)$;

(2) $f \sim f'$ implies $\mu(f) = \mu(f')$;

(3) if $f \sim f'$ and $g \sim g'$ then $g \circ f \sim g' \circ f'$ whenever the composition is defined.

Definition 2.3.5. (**A quotient fuzzy category**)
Given a congruence relation $\sim$ on an L-fuzzy category $\mathscr{C}$, the quotient L-fuzzy category $\mathscr{C}/\sim$ can be defined in a natural way, patterned after the construction of the quotient category in classical situation (cf., e.g., [19, p.24]).

2.4 Dilution and concentration of fuzzy categories.

2.4.1. Let $\mathscr{C} = \big(\mathcal{O}b(C), \omega, \mathcal{M}(C), \mu, \circ\big)$ be a fuzzy category and, given a natural number n, let the mappings $\omega^n\colon \mathcal{O}b(C) \to L$, $\mu^n\colon \mathcal{M}(C) \to L$ be defined inductively as follows: $\omega^{(k+1)}(X) = \omega^k(X) * \omega(X)$; $\mu^{(k+1)}(f) = \mu^k(f) * \mu(f)$, $k \in \mathbb{N}$. Then $\mathscr{C}^n = \big(\mathcal{O}b(C), \omega^n, \mathcal{M}(C), \mu^n, \circ\big)$ is a fuzzy category; it is called the *nth power* of the original fuzzy category $\mathscr{C}$. In particular, the category $\mathscr{C}^2$ will be referred to as the *square* of the category $\mathscr{C}$. The operation $\mathscr{C} \to \mathscr{C}^n$ can be interpreted as the operation of *dilution* of the original category $\mathscr{C}$.

2.4.2. In case L has square roots (cf., e.g., [28]) we can define operation of *concentration* for fuzzy categories. Namely, given a fuzzy category $\mathscr{C} = \big(\mathcal{O}b(C), \omega, \mathcal{M}(C), \mu, \circ\big)$ let the fuzzy category $\mathscr{C}^{1/2}$ be defined by the equality $\big(\mathcal{O}b(C), \omega^{1/2}, \mathcal{M}(C), \mu^{1/2}, \circ\big)$. The fuzzy category $\mathscr{C}^{1/2}$ is referred to as the *square root* of the category $\mathscr{C}$ and the operation $\mathscr{C} \to \mathscr{C}^{1/2}$ can be interpreted as the operation of concentration of the original category $\mathscr{C}$.

Obviously, crisp categories are invariant under operations of dilution and concentration.

2.5 A general scheme for fuzzification of classical categories.

In this subsection we shall describe a general method convenient for fuzzifying classical categories. We shall repeatedly use this method in Sections 3 – 7 for constructing concrete fuzzy categories.

Let $\mathscr{C} = \big(\mathcal{O}b(C), \mathcal{M}(C), \circ\big)$ and $\mathscr{D} = \big(\mathcal{O}b(D), \mathcal{M}(D), \circ\big)$ be two classic categories and let $\Phi\colon \mathscr{C} \to \mathscr{D}$ be a functor. To define a new (again) classic category Cat we set $\mathcal{O}b(\mathrm{Cat}) = \mathcal{O}b(C)$. Further, given objects $\mathcal{X}, \mathcal{Y} \in \mathcal{O}b(\mathrm{Cat})$ we set $\mathcal{M}_{\mathrm{Cat}}(\mathcal{X}, \mathcal{Y}) = \mathcal{M}_D\big(\Phi(\mathcal{X}), \Phi(\mathcal{Y})\big)$. Thus the morphisms from $\mathcal{X}$ to $\mathcal{Y}$ in Cat are the same as the morphisms from $\Phi(\mathcal{X})$ to $\Phi(\mathcal{Y})$ in $\mathscr{D}$. The composition law in Cat is naturally induced by the composition law in $\mathscr{D}$.

To reflect in notation the way how Cat was obtained we write $\mathscr{C}_{\mathscr{D}\Phi}$ or just $\mathscr{C}_{\mathscr{D}}$ if it is clear what functor Φ is used. Obviously, in case when $\mathscr{C} = \mathscr{D}$ and $\Xi\colon \mathscr{C} \to \mathscr{C}$ is the identity functor, the resulting category $\mathscr{C}_{\mathscr{D}\Xi}$ is just the original category $\mathscr{C}$.

Now, defining in a certain way an L-subclass of objects $\omega\colon \mathcal{O}b(\mathscr{C}_{\mathscr{D}\Phi}) \to L$ and an L-subclass of morphisms $\mu\colon \mathcal{M}(\mathscr{C}_{\mathscr{D}\Phi}) \to L$ satisfying the axioms of the Definition 2.1 we come to a fuzzy category $\big(\mathcal{O}b(\mathscr{C}_{\mathscr{D}\Phi}), \omega, \mathcal{M}(\mathscr{C}_{\mathscr{D}\Phi}), \mu, \circ\big)$. In the sequel to denote the fuzzy category obtained in this way we shall just write $(\mathscr{C}_{\mathscr{D}\Phi}, \omega, \mu)$.

We shall often consider the situation when the L-subclass $\omega\colon \mathcal{O}b(\mathscr{C}_{\mathscr{D}\Phi}) \to L$ is defined by $\omega(\mathcal{X}) = 1$ for every $\mathcal{X} \in \mathcal{O}b(\mathscr{C}_{\mathscr{D}\Phi})$; in this case we write $\omega = \varepsilon$.

2.6 Functors between fuzzy categories.

Definition 2.6.1. Let $\mathscr{C} = \big(\mathcal{O}b(C), \omega_{\mathscr{C}}, \mathcal{M}(C), \mu_{\mathscr{C}}, \circ\big)$ and $\mathscr{D} = \big(\mathcal{O}b(D), \omega_{\mathscr{D}}, \mathcal{M}(D), \mu_{\mathscr{D}}, \circ\big)$ be fuzzy categories and let $F_1\colon \mathcal{O}b(C) \to \mathcal{O}b(D)$ and $F_2\colon \mathcal{M}(C) \to \mathcal{M}(D)$ be maps. Then the quadruple $F := (\mathscr{C}, \mathscr{D}, F_1, F_2)$ is called a functor provided the following properties are satisfied:

(1) $f \in \mathcal{M}_C(X, Y)$ implies $F_2(f) \in \mathcal{M}_D\big(F_1(X), F_1(Y)\big)$;

(2) F_2 preserves composition, i.e., $F_2(g \circ f) = F_2(g) \circ F_2(f)$ provided composition $g \circ f$ is defined;

(3) F preserves identities, i.e., $F_2(e_X) = e_{F_1(X)}$ for any $X \in \mathcal{O}b(C)$;

(4) $\mu_{\mathscr{C}}(f) \leq \mu_{\mathscr{D}}(F_2(f))$ for any $f \in \mathcal{M}(C)$.

(Notice that in [75] more general concepts of a γ-functor and a μ-functor were introduced. In particular, in terminology of [75] the functor defined here must be referred to as a 1-functor.)

In the sequel, instead of $F_1(A)$ and $F_2(f)$ we write $F(A)$ and $F(f)$, respectively. To denote a functor from $\mathscr{C}$ to $\mathscr{D}$ we generally write $F\colon \mathscr{C} \to \mathscr{D}$. Besides, in the sequel we shall usually write ω instead of $\omega_{\mathscr{C}}$ or $\omega_{\mathscr{D}}$ and μ instead of $\mu_{\mathscr{C}}$ or $\mu_{\mathscr{D}}$, since it is always clear from the context to which one of the categories the symbols ω and μ are related.

Remark 2.6.2. Let $\mathscr{C}$ and $\mathscr{D}$ be fuzzy categories and let $\mathscr{C}^{\perp}$ and $\mathscr{D}^{\perp}$ be the corresponding bottom frame categories (see 2.2.5). If $F\colon \mathscr{C} \to \mathscr{D}$ is a functor between fuzzy categories, then $F\colon \mathscr{C}^{\perp} \to \mathscr{D}^{\perp}$ is a functor in the classic sense.

3. Fuzzy categories of L-sets

Let L-SET be the category of L-(fuzzy) sets, i.e., its objects are pairs: $\mathcal{X} = (X, A)$ where X is a set and $A\colon X \to L$ is its L-(fuzzy) subset, and the morphisms $f \in \mathcal{M}(\mathcal{X}, \mathcal{Y})$, where $\mathcal{Y} = (Y, B)$, are mappings $f\colon X \to Y$ such that $B \circ f \geq A$ (see, e.g., [58], [15]).

Further, let L-SETREL and L-SETREL$'$ denote the categories of L-relations defined as follows. The objects of L-SETREL and L-SETREL$'$ are the same as in L-SET, i.e., pairs $\mathcal{X} = (X, A)$ where X is a set and $A\colon X \to L$ is its L-subset. Morphisms from $\mathcal{X}$ to $\mathcal{Y} = (Y, B)$ in this categories are L-relations $R\colon X \times Y \to L$ satisfying the inequality

$$R(x,y) \leq A(x) \wedge B(y)\,, \qquad \forall x \in X\,,\ y \in Y\,,$$

in the case of the category L-SETREL, and satisfying the inequality

$$\bigvee_{x \in X} \big[R(x,y) \wedge A(x)\big] \leq B(y)\,, \qquad \forall y \in Y\,,$$

in the case of the category L-SETREL$'$ (cf. [58]).

In this Section we discuss some approaches allowing to revisit these categories and to replace them by certain fuzzy categories.

3.1 Fuzzy categories of FL-SET $= (L\text{-}\mathrm{SET}, \omega, \mu_\omega)$ type.

Here we shall consider some fuzzy categories with the category L-SET as the bottom frame.

Given $\mathcal{X} = (X, A) \in \mathcal{O}b(L\text{-SET})$ let $\omega_1(\mathcal{X}) = \inf\limits_x A(x)$ and let $\omega_2(\mathcal{X}) = \inf\limits_x \big[A(x) \vee \big(A(x) \longmapsto 0\big)\big]$. Further, given a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ in L-SET, let $\mu_{\omega_i}(f) = \omega_i(\mathcal{X}) \wedge \omega_i(Y)$, $i = 1, 2$.

Thus we obtain fuzzy categories $F_i L$-SET $= (L\text{-SET},\ \omega_i,\ \mu_{\omega_i})$ $i = 1, 2$.[1]

Noticing that $\omega_2 \geq \omega_1$ we conclude that $F_1 L$-SET is a full fuzzy subcategory of $F_2 L$-SET. Moreover, in case L is a chain and $* = \wedge$, the both categories coincide, i.e., $F_2 L$-SET$=$ $F_1 L$-SET. Obviously, L-SET $= F_i L\text{-SET}^{\perp}$, $i = 1, 2$, i.e., L-SET is the bottom frame of the both categories. The top frame $F_1 L\text{-SET}^{\top}$ of the first one of the categories can be identified in an obvious way with the category SET of sets. On the other hand, in case when L is an MV-algebra, the top frame of the second category $F_2 L\text{-SET}^{\top}$ can be identified with the category SET^2 of pairs of sets (see, e.g., [17] for the definition of SET^2).

[1] Notice that in case $i = 1$ the monoidal operation $*$ does not have any impact on this fuzzy category, and hence actually we are not dealing with a GL-monoid $(L, \leq, \wedge, *)$, but with the underlying Heyting algebra $(L, \leq, \wedge, \wedge)$. Important consequences of this are caused by the fact of the idempotence of all $\alpha \in L$ with respect to $\wedge$.

In case when we restrict ourselves to consideration of those pairs $(X, A) \in \mathcal{O}b(L\text{-}\mathrm{SET})$ which have some additional structure, it is possible to suggest other natural ways of measuring the fuzziness of (X, A) and of defining corresponding fuzzy categories.

For example, let $L = [0,1]$ be the closed unit interval endowed with some t-norm (see 1.2.3), let X be a measurable space and let m be a measure on X such that $m(X) = 1$. To define a fuzzy category $[0,1]$-MSET, let $\mathcal{O}b([0,1]\text{-MSET})$ consist of all triples (X, m, A), where (X, m) is a measurable space such that $m(X) = 1$, and $A\colon X \to [0,1]$ is a measurable function. Further, let the morphisms $\mathcal{M}([0,1]\text{-MSET})$ be defined in the same way as in $([0,1]\text{-SET})$. Now, by setting $\omega_3(X, m, A) = \int\limits_X A\,dm$ and $\omega_4(X, m, A) = \int\limits_X 2 \cdot \left|A - \frac{1}{2}\right| dm$, we naturally arrive at fuzzy categories $F_3[0,1]$-MSET$=\big([0,1]\text{-MSET},\ \omega_3, \mu_{\omega_3}\big)$ and $F_4[0,1]$-MSET$=\big([0,1]\text{-MSET},\ \omega_4, \mu_{\omega_4}\big)$.

3.2 Fuzzy categories of FL- SET $= (L\text{- SET}, \omega, \mu)$ type.

In the previous subsection, starting from L-SET as the crisp bottom frame, we defined fuzzy categories by assigning to each object $\mathcal{X}$ and to each morphism $f \in \mathcal{M}(\mathcal{X}, \mathcal{Y})$ certain values $\omega(\mathcal{X}) \in L$ and $\mu(f) \in L$ which characterize the measure to which they are a real object and a real morphism, respectively, in the corresponding fuzzy category. In this subsection we also start from the category L-SET; however, leaving the class of potential objects unchanged, we extend here the class of potential morphisms. To describe this in a formal way it is convenient to apply the scheme exposed in 2.5.

Let $\mathscr{C} = L$-SET, $\mathscr{D}$ =SET and $\phi\colon \mathscr{C} \to \mathscr{D}$ be the functor assigning to each $\mathcal{X} = (X, A)$ the support set X and leaving morphisms unchanged. Then according to the scheme 2.5, we come to the category $L\text{-SET}_{\mathrm{SET}}$; its objects are the same as in L-SET, but its morphisms are all mappings between the corresponding support sets. Starting from this category as the crisp bottom frame, we define the fuzzy category $\mathcal{F}L$-SET by setting $\omega = \varepsilon$, i.e., $\omega(\mathcal{X}) = 1$ for every $\mathcal{X} \in \mathcal{O}b(L\text{-SET})$ and

$$\mu(f) = \bigwedge_{x \in X} \big(A(x) \longmapsto (B \circ f)(x)\big)$$

for every morphism f from $\mathcal{X} = (X, A)$ to $\mathcal{Y} = (Y, B)$ in the category $L\text{-SET}_{\mathrm{SET}}$.

To show that as the result we obtain a fuzzy category, we have to verify that $\mu(g \circ f) \geq \mu(g) * \mu(f)$ whenever $g\colon Y \to Z$, and $\mathcal{Z} = (Z, C) \in \mathcal{O}b(L\text{-SET}_{\mathrm{SET}})$ (see the diagram):

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "A"'] & Y \arrow[r, "g"] \arrow[d, "B"] & Z \arrow[dl, "C"] \\ & L & \end{tikzcd}$$

Diagram

Indeed,

$$\mu(g\circ f) = \bigwedge_{x\in X}\big(A \longmapsto C\circ(g\circ f)\big)(x) \geq \bigwedge_{x\in X}\Big[(A \longmapsto B\circ f)(x) * \big(B\circ f \longmapsto C\circ(g\circ f)\big)(x)\Big] \geq \Big[\bigwedge_{x\in X}(A \longmapsto B\circ f)(x)\Big] * \Big[\bigwedge_{x\in X}\big(B\circ f \longmapsto C\circ(g\circ f)\big)(x)\Big] = \mu(f) * \Big[\bigwedge_{x\in X}\Big(B\big(f(x)\big) \longmapsto (C\circ g)\big(f(x)\big)\Big)\Big] \geq \mu(f) * \Big[\bigwedge_{y\in Y}(B \longmapsto C\circ g)(y)\Big] = \mu(f) * \mu(g).$$

Since, obviously, $\mu(e_X) = 1$, we conclude that $\mathcal{F}L$-SET is indeed a fuzzy category. The inequality $A \leq B\circ f$ holds iff $A(x) \longmapsto B\circ f(x) = 1$ for each $x \in X$ i.e., iff $\mu(f) = 1$, and therefore L-SET is the top frame of $\mathcal{F}L$-SET.

(The category $\mathcal{F}L$-SET is especially visual in case when $L = [0,1]$ and $*$ is the Łukasiewicz conjunction, i.e., it is defined by the equality $\alpha * \beta = \max\{\alpha+\beta-1, 0\}$. In this case $\alpha \longmapsto \beta = \min\{1-\alpha+\beta, 1\}$ and hence the value $1-\mu(f) = \max\big\{\sup_{x\in X}\big(A(x)-(B\circ f)(x)\big), 0\big\}$ can be interpreted as the defectiveness degree of a mapping f from being a morphism in $[0,1]$-SET (cf., e.g., [70, Subsection 1.8])).

Taking in view fuzzy categories described in the previous subsection, we can modify the fuzzy category $\mathcal{F}L$-SET as follows. Let ω_i, $i = 1,2$, be defined as in 3.1 and let $\mu_i = \mu \wedge \mu_{\omega_i}$. As the result we come to a fuzzy category $(L\text{-SET}_{\text{SET}}, \omega_i, \mu_i)$ contained both in $(L\text{-SET}, \omega_i, \mu_{\omega_i})$ and in $(L\text{-SET}_{\text{SET}}, \varepsilon, \mu)$ as a full subcategory.

3.3 Fuzzy categories of $(L\text{-SETREL}, \omega, \mu_\omega)$ type.

Let $\mathcal{X} \in \mathcal{O}b(L\text{-SETREL}) = \mathcal{O}b(L\text{-SET})$ and let $\omega_i(\mathcal{X})$, $i = 1,2$, be defined as in 3.1. Further, given a morphism $R\colon \mathcal{X} \to \mathcal{Y}$ in L-SETREL, let $\mu_{\omega_i}(R) = \omega_i(\mathcal{X}) \wedge \omega_i(\mathcal{Y})$, $i = 1,2$. As the result we come to fuzzy categories $(L\text{-SETREL}, \omega_i, \mu_{\omega_i})$, $i = 1,2$. In a similar way, but starting with the category $L\text{-SETREL}'$ we come to fuzzy categories $(L\text{-SETREL}', \omega_i, \mu_{\omega_i})$, $i = 1,2$.

The "standard" functor $\Phi\colon L\text{-SET} \to L\text{-SETREL}$ defined by $\Phi(\mathcal{X}) = \mathcal{X}$ on objects and $\Phi(f) = R_f$, where

$$R_f(x,y) = A(x), \quad \text{if} \quad y = f(x) \qquad \text{and} \qquad R_f(x,y) = 0, \quad \text{if} \quad y \neq f(x),$$

on morphisms $f\colon (X,A) \to (Y,B)$ (cf., e.g., [58]), remains also a functor if considered in the "fuzzy situation", i.e., as the functor $\Phi\colon (L\text{-SET}, \omega_i, \mu_{\omega_i}) \to (L\text{-SETREL}, \omega_i, \mu_{\omega_i})$.

By setting $\Psi(\mathcal{X}) = \mathcal{X}$ on objects and $\Psi(R) = R$ on morphisms, we obtain the "standard" functor $\Psi\colon L$-SETREL$\to L$-SETREL$'$ (cf. [58]), which remains also a functor in the "fuzzy situation", i.e., as the functor $\Psi\colon (L\text{-SETREL}, \omega_i, \mu_{\omega_i}) \to (L\text{-SETREL}', \omega_i, \mu_{\omega_i})$, $i = 1, 2$.

3.4 Fuzzy categories of $(L\text{-}\mathrm{SETREL}_{L\text{-}\mathrm{REL}}, \varepsilon, \mu)$ type.

Let $\mathscr{C} = L$-SETREL be defined as above and let $\mathscr{D} = L$-REL be the category of L-relations. Its objects are sets, and morphisms from a set X to a set Y are L-relations $R\colon X \times Y \to L$ (recall that the composition of two relations $R\colon X \times Y \to L$ and $S\colon Y \times Z \to L$ is defined by $(R \circ S)(x, z) = \bigvee_{y \in Y} \big(R(x, y) \wedge S(y, z)\big)$). Further, let $\phi\colon \mathscr{C} \to \mathscr{D}$ be the functor assigning to $(X, A) \in \mathcal{O}b(L\text{-SETREL})$ its support set $X \in \mathcal{O}b\big(L\text{-REL}\big)$ and leaving morphisms unchanged. Further, let $\omega(\mathcal{X}) = 1$ for $\mathcal{X} = (X, A) \in \mathcal{O}b(L\text{-SETREL}_{L\text{-REL}})$ $\big(= \mathcal{O}b(L\text{-SETREL})\big)$ and set

$$\mu(R) = \bigwedge_{x \in X, y, \in Y} \big(R(x, y) \longmapsto A(x) \wedge B(y)\big)$$

for a morphism $R\colon (X, A) \longmapsto (Y, B)$ in $L\text{-}\mathrm{SETREL}_{L\text{-REL}}$ (i.e., $R\colon X \times Y \to L$ is an L-relation). An easy verification shows that $(L\text{-SETREL}_{L\text{-REL}}, \varepsilon, \mu)$ is indeed a fuzzy category.

Analogously, using the same scheme but starting from the category $\mathscr{C} = L\text{-}\mathrm{SETREL}'$ and setting

$$\mu'(R) = \bigwedge_{y \in Y} \left[\Big(\bigvee_{x \in X} R(x, y) \wedge A(x)\Big) \longmapsto B(y)\right]$$

we obtain a fuzzy category $(L\text{-}\mathrm{SETREL}'_{L\text{-REL}}, \varepsilon, \mu)$.

The functors Φ and Ψ introduced in the previous subsection, can be viewed upon also as functors

$$\Phi\colon (L\text{-SET}_{L\text{-SET}}, \varepsilon, \mu) \to (L\text{-SETREL}_{L\text{-REL}}, \varepsilon, \mu)$$

($(L\text{-SET}_{L\text{-SET}}, \varepsilon, \mu)$ is the fuzzy category introduced in 3.2) and

$$\Psi\colon (L\text{-SETREL}_{L\text{-REL}}, \varepsilon, \mu) \to (L\text{-SETREL}'_{L\text{-REL}}, \varepsilon, \mu')$$

respectively. To show this, it is sufficient to notice that $\mu(R_f) = \mu(f)$ for each mapping f and $\mu'(R) \leq \mu(R)$ for each L-relation R.

4. Fuzzy categories related to algebra

In the previous Section we considered different fuzzy categories whose objects are pairs (X, A), where X is a set and $A : X \to L$ is its L-subset. The subject of this Section are fuzzy categories whose objects are also pairs (X, A), however, now X will be supposed to be endowed with some algebraic structure, and this structure will make a certain impact on the conditions which the second component of the pair (X, A) must satisfy.

In order to make the exposition here more "homogeneous" and purposeful, we shall restrict ourselves to the case when X is a group. Thus, we shall proceed from the category GR of groups and propose different fuzzification schemes for GR. Patterned after these schemes, a reader can develop similar fuzzifications for other categories of algebra, such as the category RING of rings, the category MOD of modules, the category MON of monoids, the category VECT of vector spaces, etc.

4.1 Fuzzy categories of $(L\text{-}\mathrm{GR}, \omega, \mu)$ type.

Let L-GR be the category of L-groups (see, e.g., [64] or [58, Chapter 2.3]). Thus, objects of L-GR are L-groups, i.e., pairs (X, G) where $(X, \cdot)$ is a group and $G\colon X \to L$ is its L-subset closed under operation of product (i.e., $G(x \cdot y) \geq G(x) * G(y)$, $\forall x, y \in X$) and closed under the operation of taking the inverses (i.e., $G(x^{-1}) \geq G(x)$, $\forall x \in X$). Given two L-groups $\mathcal{X} = (X, G)$ and $\mathcal{Y} = (Y, H)$, a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is a group homomorphism $f\colon X \to Y$ such that $G \leq H \circ f$.

Now we can define fuzzy subclasses $\omega_i\colon \mathcal{O}b(L\text{-}\mathrm{GR}) \to L$ and $\mu_{\omega_i}\colon \mathcal{M}(L\text{-}\mathrm{GR}) \to L$ by the same formulae as in 3.1 and obtain fuzzy categories $(L\text{-}\mathrm{GR}, \omega_i, \mu_{\omega_i})$, $i = 1, 2$.

4.2 Fuzzy categories of $(L\text{-}\mathrm{GR}_{\mathrm{GR}}, \omega, \mu)$ type.

Let L-$\mathrm{GR}_{\mathrm{GR}}$ be the category defined according to 2.5, i.e., its objects are the same as in L-GR, and its morphisms $f\colon \mathcal{X} \to \mathcal{Y}(\mathcal{X} = (X, A),\ \mathcal{Y} = (Y, B) \in \mathcal{O}b(L\text{-}\mathrm{GR}))$ are arbitrary homomorphisms $f\colon X \to Y$ of the corresponding support groups. By setting $\omega(\mathcal{X}) = 1$ and

$$\mu(f) = \bigwedge_{x \in X} \big(A(x) \longmapsto (B \circ f)(x)\big)$$

we obtain a fuzzy category $(L\text{-}\mathrm{GR}_{\mathrm{GR}},\ \varepsilon, \mu)$. One can modify this category by introducing some "measure" for the class of objects, too. For example, defining ω_i, $i = 1, 2$, in the same way as in 4.1 and setting $\bar{\mu}_i = \mu \wedge \mu_{\omega_i}$, we come to fuzzy categories $(L\text{-}\mathrm{GR}_{\mathrm{GR}},\ \omega_i, \bar{\mu}_i)$, $i = 1, 2$.

4.3 Fuzzy categories of $(L\text{-}\widetilde{\mathrm{GR}}, \omega, \mu)$ type.

In the fuzzy categories considered in the previous subsections, the class of potential objects was $\mathcal{O}b(L\text{-}\mathrm{GR})$ and ω was "estimating" the degree to which

$\mathcal{X} \in \mathcal{O}b(L\text{-GR})$ belongs to the given fuzzy category. Thus, ω in some sense "narrowed" the class of objects if compared with the original category L-GR. On the other hand, here we start with "widening" of the class of potential objects from L-GR and then consider a fuzzy subclass ω of this larger class $\mathcal{O}b(L\text{-}\widetilde{\mathrm{GR}})$ of potential objects. Here are the details:

Let $L\text{-}\widetilde{\mathrm{GR}}$ be the category whose objects are pairs (X, A), where X is a group and $A\colon X \to L$ is its L-*subset* (i.e., no validity of axioms from 4.1 on A is assumed) and whose morphisms are arbitrary homomorphisms of the corresponding support groups. By setting

$$\omega(X,A) = \left[\bigwedge_{x,x' \in X} (A(x) * A(x') \longmapsto A(x \cdot x')) \right] \wedge \left[\bigwedge_{x \in X} (A(x^{-1}) \longmapsto A(x)) \right],$$

we define an L-fuzzy class of objects. Now there are several ways how to complete this construction and to obtain a fuzzy category. In particular, we may set $\mu_\omega(f) = \omega(\mathcal{X}) \wedge \omega(\mathcal{Y})$ for a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ and thus come to the fuzzy category $(L\text{-}\widetilde{\mathrm{GR}},\ \omega, \mu_\omega)$. Another possibility is to set $\bar{\mu}(f) = \mu_\omega(f) \wedge \mu(f)$ where μ is defined as in 4.2. Thus we arrive at a fuzzy category $(L\text{-}\widetilde{\mathrm{GR}},\ \omega, \bar{\mu})$. Notice that its top frame $(L\text{-}\widetilde{\mathrm{GR}},\ \omega, \mu_\omega)^\top$ is the category L-GR.

By assigning to each $(X, A) \in \mathcal{O}b(L\text{-GR})$ the pair $(X, A) \in \mathcal{O}b(L\text{-SET})$, i.e., "forgetting" the algebraic structure of X and leaving morphisms unchanged, we define a forgetful functor $\Phi\colon (L\text{-}\widetilde{\mathrm{GR}}, \omega, \mu_\omega) \to (L\text{-}SET_{\mathrm{SET}}, \varepsilon, \mu)$ where $(L\text{-}\mathrm{SET}_{\mathrm{SET}}, \varepsilon, \mu)$ is the fuzzy category defined in 3.2.

5. Fuzzy categories related to topology

5.1 Fuzzy categories of FL-TOP $= (L\text{-}\mathrm{TOP}_{\mathrm{SET}}, \varepsilon, \mu)$ type.

Let L-TOP be the category of Chang-Goguen L-topological spaces [8, 16]. Recall that the objects of L-TOP are L-topological spaces, i.e., pairs (X, τ) where X is a set and τ is a family of its L-fuzzy subsets (i.e., $\tau \subset L^X$) such that (1) $0, 1 \in \tau$; (2) $U, V \in \tau \Longrightarrow U \wedge V \in \tau$ and (3) $U_i \in \tau,\ \forall i \in \mathcal{I} \Longrightarrow \bigvee_i U_i \in \tau$.

The morphisms $f\colon (X, \tau_X) \to (Y, \tau_Y)$ in L-TOP are mappings $f\colon X \to Y$ such that $V \in \tau_Y \implies f^{-1}(V) \in \tau_X$.

Originating from the category L-TOP, we shall define a fuzzy category FL-TOP as follows. Let $L\text{-TOP}_{\mathrm{SET}}$ be the category defined according to 2.5, i.e., its objects are L-topological spaces and its morphisms are all mappings between the corresponding support sets. We set $\omega = \varepsilon$.

The degree to which a mapping $f\colon X \to Y$ is a morphism in the fuzzy category FL-TOP is defined by the equality:

$$\mu(f) = \bigwedge_{V\in\tau_Y} \left[\wedge_{x\in X}\big(f^{-1}(V) \longmapsto \operatorname{Int} f^{-1}(V)\big)(x)\right],$$

where the interior $\operatorname{Int} A$ of an L-fuzzy set $A \in L^X$ in (X,τ) is defined as $\operatorname{Int} A = \bigvee\{U\colon U\in\tau,\, U \le A\}$.

Since, obviously, $\mu(e_X) = 1$ for the identical morphism $e_X\colon (X,\tau)\to(X,\tau)$, to ground that $FL\text{-TOP} = (L\text{-TOP}_{\text{SET}},\ \varepsilon,\mu)$ thus obtained is indeed a fuzzy category we have to show only that $\mu(g\circ f) \ge \mu(g) * \mu(f)$ for any mappings $f\colon (X,\tau_X)\to(Y,\tau_Y)$, $g\colon (Y,\tau_Y)\to(Z,\tau_Z)$. However, this follows from the following chain of (in)equalities (see (vi), (vii) in Section 1.1):

$$\begin{aligned}
\mu(g\circ f) &= \inf_{W\in\tau_Z}\ \inf_{x\in X} \Big((g\circ f)^{-1}(W) \longmapsto \operatorname{Int}\big((g\circ f)^{-1}(W)\big)\Big)(x)\\
&= \inf_{W\in\tau_Z}\ \inf_{x\in X} \Big(f^{-1}\big(g^{-1}(W)\big) \longmapsto \operatorname{Int} f^{-1}\big(g^{-1}(W)\big)\Big)(x)\\
&\ge \inf_{W\in\tau_Z}\ \inf_{x\in X} \Big[\Big(f^{-1}\big(g^{-1}(W)\big) \longmapsto f^{-1}\big(\operatorname{Int}\big(g^{-1}(W)\big)\big)\Big)(x)\\
&\qquad * \Big(f^{-1}(\operatorname{Int}(g^{-1}(W))) \longmapsto \operatorname{Int}\big(f^{-1}\big(g^{-1}(W)\big)\big)\Big)(x)\Big]\\
&\ge \Big[\inf_{W\in\tau_Z}\ \inf_{x\in X} \Big(g^{-1}(W) \longmapsto \operatorname{Int}\big(g^{-1}(W)\big)\Big)\big(f(x)\big)\Big]\\
&\qquad * \Big[\inf_{W\in\tau_Z}\ \inf_{x\in X} \Big(f^{-1}\big(\operatorname{Int}\big(g^{-1}(W)\big)\big) \longmapsto \operatorname{Int}\big(f^{-1}\big(g^{-1}(W)\big)\big)\Big)(x)\Big]\\
&\ge \Big[\inf_{W\in\tau_Z}\ \inf_{y\in Y} \Big(g^{-1}(W) \longmapsto \operatorname{Int}\big(g^{-1}(W)\big)\Big)(y)\Big]\\
&\qquad * \Big[\inf_{W\in\tau_Z}\ \inf_{x\in X} \Big(f^{-1}\big(\operatorname{Int}\big(g^{-1}(W)\big)\big) \longmapsto \operatorname{Int}\big(f^{-1}\big(g^{-1}(W)\big)\big)\Big)(x)\Big]\\
&\ge \mu(g) * \Big[\inf_{V\in\tau_Y}\ \inf_{x\in X} \Big(f^{-1}(V) \longmapsto \operatorname{Int}\big(f^{-1}(V)\big)\Big)(x)\Big] = \mu(g)*\mu(f).
\end{aligned}$$

Remarks.

(a) The scheme of fuzzification applied here becomes especially lucid when $L = [0,1]$ and $*$ is the Łukasiewicz conjunction. Then $\alpha \longmapsto \beta = \min(1-\alpha+\beta, 1)$ and hence $\mu(f)$ shows how far f deviates from being continuous: $\mu(f) = 1 - d(f)$ where the value $d(f) = \sup_{V\in\tau_Y}\ \sup_{x\in X} \big(f^{-1}(V) - \operatorname{Int}\big(f^{-1}(V)\big)\big)(x)$ can be called *the defect of continuity* of the mapping $f\colon (X,\tau_X)\to(Y,\tau_Y)$ of the corresponding Chang-Goguen [0,1]-topological spaces. In particular, if (X,τ_X), (Y,τ_Y) are ordinary topological spaces, then $d(f) = 0$ if f is continuous and $d(f) = 1$ otherwise (see [70]).

(b) Let L be a continuous lattice [14] and let $\omega_L\colon$ TOP $\to L$-TOP be the embedding omega functor first introduced by R. Lowen [48] and later extended by T. Kubiak to the case of an arbitrary continuous lattice see [46]). Then the fuzzy category $F\omega_L(\mathrm{TOP})$ obtained by "restriction" of the above fuzzification scheme to the subcategory $\omega_L(\mathrm{TOP})$ of the category L-TOP can be viewed upon as a fuzzification of the category **TOP** of topological spaces.

5.2 Fuzzy categories of FL-FTOP $= (L$-FTOP$_{\mathrm{SET}}, \varepsilon, \mu)$ type.

Following [67, 70], by an L*-fuzzy topological space* we call a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is an L-fuzzy topology on it, i.e., a mapping $\mathcal{T}\colon L^X \to L$ endowed with the following properties:

(1) $\mathcal{T}(0) = \mathcal{T}(1) = 1$;

(2) $\mathcal{T}(U \wedge V) \geq \mathcal{T}(U) \wedge \mathcal{T}(V)$, $\forall\, U, V \in L^X$;

(3) $\mathcal{T}\Big(\bigvee_{i\in\mathcal{I}} U_i\Big) \geq \bigwedge_{i\in\mathcal{I}} \mathcal{T}(U_i)$, $\forall \{U_i\colon i \in \mathcal{I}\} \subset L^X$.

A mapping $f\colon X \to Y$ where $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ are L-fuzzy topological spaces is called *continuous* if $\mathcal{T}_X\big(f^{-1}(V)\big) \geq \mathcal{T}_Y(V)$ for each $V \in L^Y$. Let L-FTOP stand for the category of L-fuzzy topological spaces and continuous mappings between them.

To define a fuzzification FL-FTOP of the category L-FTOP, we first consider the category L-FTOP$_{\mathrm{SET}}$ (cf., 2.5) whose objects are the same as in L-FTOP and whose morphisms are all mappings between the corresponding support sets. Further, let $\omega = \varepsilon$ characterize the class of objects and let

$$\mu(f) = \bigwedge_{V\in L^Y} \Big(\mathcal{T}_Y(V) \longmapsto \mathcal{T}_X\big(f^{-1}(V)\big)\Big)$$

for $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y) \in \mathcal{O}b(L$-FTOP$)$ and a mapping $f\colon X \to Y$.

Thus we come to a fuzzy category FL-FTOP $=(L$-FTOP$_{\mathrm{SET}}, \varepsilon, \mu)$. Indeed, if $e_X\colon (X, \tau_X) \to (X, \tau_X)$ is the identity mapping, then obviously, $\mu(e_X) = 1$. Therefore we have to check only that $\mu(g \circ f) \geq \mu(g) * \mu(f)$ for the composition $g \circ f$ of mappings $f\colon (X, \tau_X) \to (Y, \tau_Y)$ and $g\colon (Y, \tau_Y) \to (Z, \tau_Z)$:

$$\begin{aligned}\mu(g\circ f) &= \bigwedge_{W\in L^Z}\Big(\mathcal{T}_Z(W)\longmapsto\mathcal{T}_X\big(f^{-1}(g^{-1}(W))\big)\Big)\\
&\geq \bigwedge_{W\in L^Z}\Big[\Big(\mathcal{T}_Z(W)\longmapsto\mathcal{T}_Y\big(g^{-1}(W)\big)\Big)\\
&\qquad *\Big(\mathcal{T}_Y\big(g^{-1}(W)\big)\longmapsto\mathcal{T}_X\big(f^{-1}(g^{-1}(W))\big)\Big)\Big]\\
&\geq \Big[\bigwedge_{W\in L^Z}\Big(\mathcal{T}_Z(W)\longmapsto\mathcal{T}_Y\big(g^{-1}(W)\big)\Big)\Big]\\
&\qquad *\Big[\bigwedge_{V\in L^Y}\Big(\mathcal{T}_Y(V)\longmapsto\mathcal{T}_X\big(f^{-1}(V)\big)\Big)\Big] = \mu(g)*\mu(f).\end{aligned}$$

5.3 Fuzzy categories of FL- FTOP $= (L\text{-}\widetilde{\mathrm{FTOP}}_{\mathrm{SET}},\omega,\mu)$ type.

Let L-$\widetilde{\mathrm{FTOP}}$ be the category whose objects are pairs $(X,\mathcal{T})$ where X is a set and $\mathcal{T}\colon X\to L$ is a mapping (no axioms on $\mathcal{T}$ are assumed) and whose morphisms from $(X,\mathcal{T}_X)$ to $(Y,\mathcal{T}_Y)$ are "continuous like" mappings $f\colon X\to Y$ such that $\mathcal{T}_X\big(f^{-1}(V)\big)\geq\mathcal{T}_Y(V)$ for every $V\in L^Y$. By setting

$$\begin{aligned}\omega(X,\mathcal{T}) &= \big(1\longmapsto\mathcal{T}(0)\big)\bigwedge\big(1\longmapsto\mathcal{T}(1)\big)\\
&\bigwedge\Big[\bigwedge_{U,V\in L^X}\big(\mathcal{T}(U)\wedge\mathcal{T}(V)\longmapsto\mathcal{T}(U\wedge V)\big)\Big]\\
&\bigwedge\Big[\bigwedge_{\{U_i:\ i\in\mathcal{I}\}}\Big(\big(\wedge_{i\in\mathcal{I}}\mathcal{T}(U_i)\big)\longmapsto\mathcal{T}\big(\vee_{i\in\mathcal{I}}U_i\big)\Big)\Big]\end{aligned}$$

and

$$\mu_\omega(f)=\omega(X,\mathcal{T}_X)\wedge\omega(Y,\mathcal{T}_Y)$$

we come to a fuzzy category $(L\text{-}\widetilde{\mathrm{FTOP}},\omega,\mu_\omega)$.

Taking as the bottom frame category the category L-$\widetilde{\mathrm{FTOP}}_{\mathrm{SET}}$ (cf 2.5) instead of the category L-$\widetilde{\mathrm{FTOP}}$ and setting

$$\mu(f)=\omega(X,\mathcal{T}_X)\wedge\omega(Y,\mathcal{T}_Y)\bigwedge\Big[\bigwedge_{V\in L^Y}\Big(\mathcal{T}_Y(V)\longmapsto\mathcal{T}_X\big(f^{-1}(V)\big)\Big)\Big]$$

we obtain the fuzzy category FL-FTOP $= (L\text{-}\widetilde{\mathrm{FTOP}}_{\mathrm{SET}},\ \omega,\mu)$.

6. Fuzzy categories of proximity spaces

Extending somewhat the definition of a fuzzy proximity introduced in [54], by an L*-fuzzy proximity* on a set X we shall call a mapping $\delta\colon L^X \times L^X \to L$ such that for all $A, B, C, A', B' \in L^X$:

(0P) $A \le A',\ B \le B' \Longrightarrow \delta(A,B) \le \delta(A',B')$;

(1P) $\delta(0,1) = 0$;

(2P) $\delta(A,B) \le \delta(B,A)$;

(3P) $\delta(A,\ B \vee C) \le \delta(A,B) \vee \delta(A,C)$;

(4P) $\delta(A,B) \ge\ A * B$;

(5P) $$\delta(A,B) \ge \Big[\bigwedge_{E \in L^X} \Big(\delta(A,E) \vee \delta\big(B,(E \longmapsto 0)\big)\Big)\Big] \bigwedge \Big[\bigwedge_{E \in L^X} \Big(\delta(B,E) \vee \delta\big(A,(E \longmapsto 0)\big)\Big)\Big].$$

(Notice that axioms (0P) and (3P) together are equivalent to the following stronger version of the axiom (3P): (3'P) $\delta(A, B \vee C) = \delta(A,B) \vee \delta(A,C)$.)

A pair (X,δ) is called an L*-fuzzy proximity space*. A mapping $f\colon (X,\delta_X) \to (Y,\delta_Y)$ is called *proximally continuous*, or *a proximity mapping*, if

$$\delta_X(A,B) \le \delta_Y\big(f(A), f(B)\big) \qquad \text{for all} \quad A, B \in L^X.$$

(In [54] we restricted ourselves to the case when $L = [0,1]$, and $*$ was either a Łukasiewicz conjunction or $* = \wedge$.)

Let L-FPROX denote the category of L-fuzzy proximity spaces and proximity mappings.

Remark. In [36] and [37] A. K. Katsaras proposed two alternative definitions of fuzzy proximities which can be viewed as special cases of our L-fuzzy proximities. Contrary to our L-fuzzy proximities, Katsaras fuzzy proximities are essentially two-valued, i.e., they must satisfy the following additional property:

(KP) $\delta\big([0,1]^X \times [0,1]^X\big) \subset \{0,1\}$.

Moreover, to obtain Katsaras fuzzy proximities introduced in [37] we must take $L = [0,1]$ endowed with Łukasiewicz conjunction. The corresponding category will be denoted K_1-Prox. To characterize Katsaras fuzzy proximities from [36], we must modify somewhat the above general definition of an L-fuzzy proximity. Namely, we must take $* = \wedge$ in the axiom (4P), but define residuation which appear in the axiom (5P) by means of the Łukasiewicz conjunction. The corresponding category will be denoted K_2-Prox.

In this section we shall discuss some ways in which the category L-FPROX (and, as a by-product, the categories of Katsaras fuzzy proximity spaces) can be fuzzified.

6.1 Fuzzy categories of $(L\text{-}\mathrm{FPROX}_{\mathrm{SET}}, \varepsilon, \mu)$ type: fuzzification of morphisms.

Let L-$\mathrm{FPROX}_{\mathrm{SET}}$ be a category obtained from L-FPROX according to the scheme 2.5 and let $\omega = \varepsilon$. Further, given a mapping $f\colon (X, \delta_X) \to (Y, \delta_Y)$, let

$$\mu(f) = \bigwedge_{A,B \in L^X} \Big(\delta_X(A, B) \longmapsto \delta_Y\big(f(A), f(B)\big)\Big).$$

To check that $(L\text{-}\mathrm{FPROX}_{\mathrm{SET}}, \varepsilon, \mu)$[2] thus obtained is indeed a fuzzy category we have to show that $\mu(g \circ f) \geq \mu(g) * \mu(f)$.

Let $f\colon (X, \delta_X) \to (Y, \delta_Y)$, $g\colon (Y, \delta_Y) \to (Z, \delta_Z)$. Then, applying (vi) from Section 1.2, we get:

$$\begin{aligned}
\mu(g \circ f) = &\bigwedge_{A,B \in L^X} \Big[\delta_X(A, B) \longmapsto \delta_Z\big((g \circ f)(A)\,,\,(g \circ f)(B)\big)\Big] \\
\geq &\bigwedge_{A,B \in L^X} \big(\delta_X(A, B)\big) \longmapsto \delta_Y\big(f(A), f(B)\big) \\
* &\bigwedge_{A,B \in L^X} \Big(\delta_Y\big(f(A), f(B)\big) \longmapsto \delta_Z\big(g\big(f(A)\big)\,,\, g\big(f(B)\big)\big)\Big) \\
\geq &\ \mu(f) * \bigwedge_{C,D \in L^Y} \Big(\delta_Y(C, D) \longmapsto \delta_Z\big(g(C), g(D)\big)\Big) = \mu(f) * \mu(g)\,.
\end{aligned}$$

The full fuzzy subcategory FKPROX of $F\mathbf{I}$-PROX, whose objects satisfy the property (KP), can be considered as a fuzzification of the category of Katsaras fuzzy proximity spaces.

6.2 Fuzzy categories of $(L\text{-}\widetilde{\mathrm{FPROX}}, \omega, \mu)$ type.

Let now L-$\widetilde{\mathrm{FPROX}}$ be the category of "proximity-like" spaces, whose objects are pairs (X, δ), where X is a set and $\delta\colon L^X \times L^X \to L$ is a mapping (no axioms are assumed) and whose morphisms $f\colon (X, \delta_X) \to (Y, \delta_Y)$ are mappings satisfying the inequality

$$\delta_X(A, B) \leq \delta_Y\big(f(A), f(B)\big) \qquad \text{for all} \quad A, B \in L^X\,.$$

By replacing the inequality $\leq$ in an axiom (iP), i$= 0, 1, 2, 3, 4, 5$ in the definition of an L-fuzzy proximity by implication sign $\longmapsto$, one naturally arrives at a fuzzy predicate $\omega_i\colon L^{L^X \times L^X} \longmapsto L$, which can be interpreted as the fuzzy version of

[2] Note that such estimation of a "potential morphism" in the category L-$\mathrm{FPROX}_{\mathrm{SET}}$ in case $L = [0, 1]$ is endowed with the Łukasiewicz conjunction, which appeared earlier in our paper [54].

the corresponding crisp axiom (iP).[3] Namely, given a mapping $\delta\colon L^X \times L^X \to L$ let the value $\omega_i(\delta) \in L$, $i = 0,1,2,3,4,5$, be defined as follows:

$\omega_0(\delta) = \bigwedge\{\delta(A,B) \mapsto \delta(A',B')\colon A \le A',\, B \le B'\}$ (predicate of monotonicity);

$\omega_1(\delta) = \delta(0,1) \mapsto 0$ (the predicate of universality);

$\omega_2(\delta) = \bigwedge\limits_{A,B\in L^X} \Big(\delta(A,B) \mapsto \delta(B,A)\Big)$ (the predicate of symmetricity);

$\omega_3(\delta) = \bigwedge\limits_{A,B,C\in L^X} \Big(\delta(A,\, B\vee C) \mapsto \delta(A,B) \vee \delta(A,C)\Big)$ (the predicate of union);

$\omega_4(\delta) = \bigwedge\limits_{A,B\in L^X} \Big(A * B \mapsto \delta(A,B)\Big)$ (the predicate of coincidence);

$$\omega_5(\delta) = \bigwedge_{A,B\in L^X} \Big[\bigwedge_{E\in L^X} \Big(\delta(A,E) \vee \delta\big(B,(E \longmapsto 0)\big)\Big) \mapsto \delta(A,B)\Big]$$
$$\bigwedge \Big[\bigwedge_{E\in L^X} \Big(\delta(B,E) \vee \delta\big(A,(E \longmapsto 0)\big)\Big) \longmapsto \delta(A,B)\Big]$$

Obviously, δ satisfies the condition (iP) if and only if $\omega_i(\delta) = 1$, $i = 0,1,2,3,4,5$.

Starting from the category L-$\widetilde{\text{PROX}}$ as the bottom crisp frame and basing on the fuzzy predicates ω_i, $i = 0,1,2,3,4,5$, we can define fuzzy categories (FL-PROX $= L$-$\widetilde{\text{PROX}}$, ω, μ_ω), where $\omega(X,\delta) = \bigwedge\limits_i \omega_i(\delta)$ for $(X,\delta) \in \mathcal{O}b(L\text{-}\widetilde{\text{PROX}})$ and $\mu_\omega(f) = \omega(X,\delta_X) \wedge \omega(Y,\delta_Y)$ for a mapping $f\colon (X,\delta_X) \to (Y,\delta_Y)$.

In a similar way but starting from the category L-$\widetilde{\text{FPROX}}_{\text{SET}}$ (cf 2.5) instead of the category L-$\widetilde{\text{FPROX}}$ and setting

$$\bar\mu(f) = \mu_\omega(f) \wedge \mu(f)\,,$$

where $\mu(f)$ is defined as in 6.1 we come to a fuzzy category (L-$\widetilde{\text{FPROX}}_{\text{SET}}$, $\omega, \bar\mu$).

7. Fuzzification of some categories of metric spaces

Let BMetr be the category of bounded metric spaces (X,ρ), such that $\rho \le 1$, and continuous mappings, and let NEBMetr be its subcategory determined by

[3] Note that here our ideas have something in common with the ideology of the semantic analysis used by Ying Mingsheng in his theory of fuzzifying topologies, see [78]–[82].

all non-expanding mappings. Further, let $\widetilde{\mathrm{BMetr}}$ be the category whose objects are "metric-like" spaces, i.e., pairs (X,ρ) where X is a set and $\rho\colon X\times X\to[0,1]$ is a mapping (no axioms are assumed), and whose morphisms from (X,ρ_X) to (Y,ρ_Y) are mappings $f\colon X\to Y$ such that for every $x\in X$ and for every $\varepsilon>0$ there exists $\delta>0$, such that $\rho_Y\big(f(x),f(x')\big)<\varepsilon$ whenever $\rho(x,x')<\delta$.

In this section, the closed unit interval $[0,1]$ *will be always in the role of* L.

7.1 Fuzzy categories of $(\mathrm{NEBMetr}_{\mathrm{SET}},\varepsilon,\mu)$ type.

Starting from the category NEBMetr of bounded metric spaces (recall that we assume $\rho\le 1$) and non-expanding mappings, and applying the construction described in 2.5, we arrive at the category $\mathrm{NEBMetr}_{\mathrm{SET}}$.

We set $\omega(X,\rho)=1$ for every $X\in\mathcal{O}b(\mathrm{NEBMetr})$ and

$$\mu(f)=\bigwedge_{x,x'\in X}\big(\rho_Y\big(f(x),f(x')\big)\longmapsto\rho_X(x,x')\big)$$

for every mapping $f\colon(X,\rho_X)\to(Y,\rho_Y)$. An easy verification shows that μ thus defined satisfies the axioms of 2.1 and thus we arrive at a fuzzy category $(\mathrm{NEBMetr}_{\mathrm{SET}},\ \varepsilon,\mu)$. This fuzzy category can be interpreted as a certain *fuzzification* of the category NEBMetr of bounded metric spaces and their non-expanding mappings. (Note that in case when $*$ is the Łukasiewicz t-norm on the interval $[0,1]$, this category earlier appeared in [3].)

A similar construction, but when SET is replaced by the category TOP, leads us to the fuzzy category $(\mathrm{NEBMetr}_{\mathrm{TOP}},\ \varepsilon,\mu)$ in which only *continuous* mappings are taken into account.

7.2 Fuzzy categories of $(\widetilde{\mathrm{BMetr}},\omega,\mu)$ type.

Let $(X,\rho)\in\mathcal{O}b(\widetilde{\mathrm{BMetr}})$, then by setting

$$\omega(X,\rho)=\bigwedge_{x,x\in X}\big(\rho(x,x)\longmapsto 0\big)\wedge\bigwedge_{x,y\in X}\big(\rho(x,y)\longmapsto\rho(y,x)\big)$$
$$\wedge\bigwedge_{x,y,z\in X}\big(\rho(x,z)\longmapsto\big(\rho(x,y)+\rho(y,z)\wedge 1\big),$$

we define a fuzzy subclass ω of $\mathcal{O}b(\widetilde{\mathrm{BMetr}})$; the intuitive meaning of ω is the evaluation "how far" is this metric-like function ρ from being a "real" (pseudo)metric. Now, by setting $\mu_\omega(f)=\omega\big((X,\rho_X)\big)\wedge\omega\big((Y,\rho_Y)\big)$ for a morphism $f\colon(X,\rho_X)\to(Y,\rho_Y)$, we obtain a fuzzy category $(\widetilde{\mathrm{BMetr}},\ \omega,\mu_\omega)$.

Further, let $\widetilde{\mathrm{BMetr}}_{\mathrm{SET}}$ be the category obtained from $\widetilde{\mathrm{BMetr}}$ by means of construction 2.5 and let $\bar\mu(f)=\mu_\omega\wedge\mu$, where μ_ω is defined as above, and μ is defined by the same formula as in Subsection 7.1. As the result we obtain the category $(\widetilde{\mathrm{BMetr}}_{\mathrm{SET}}, \omega, \bar{\mu})$. Replacing in the above construction the category SET by the category TOP, we come to the fuzzy category $(\widetilde{\mathrm{BMetr}}_{\mathrm{TOP}}, \omega, \bar{\mu})$.

Conclusion

By a series of examples we have exposed here some ideas how categories of the classical mathematics can be fuzzified, and outlined several fuzzification methods. We assume that this work can be continued in several directions. First, one can develop other examples of fuzzy categories based either on the ideas presented in this work or on some other schemes suitable for fuzzification of classical categories. Second, it would be interesting to make a deeper investigation of (some of) these categories and to develop the corresponding theories. In particular, we think that for the fuzzy categories related to metric spaces which were discussed in Section 7 it would be tempting to work out elements of a theory, which in a reasonable way could be interpreted as a fuzzy analogue of the classical theory of metric spaces. Finally, the interrelations between different fuzzified categories must be investigated. This concerns first of all, in our opinion, the fuzzy categories of topological nature, i.e., those ones which were discussed in Sections 5, 6 and 7.

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Department of Mathematics
University of Latvia
LV–1586 Riga
LATVIA
E-mail: sostaks@com.latnet.lv