Tatra Mt. Math. Publ. **16** (1999), 87–94

TATRA MOUNTAINS
Mathematical Publications

ON FUZZY SMOOTH FUNCTIONS IN MULTIDIMENSIONAL CASE

Martin Kalina

ABSTRACT. This paper is a continuation of [K2]. It begins by investigating partial fuzzy derivatives of fuzzy continuous, but crisp, multidimensional functions. Some basic properties of multidimensional fuzzy smooth functions are established.

In the one dimensional case several authors have made their contribution to fuzzy calculus and to the notion of fuzzy continuity — D. Dubois and H. Prade in [D-P1, D-P2, D-P3], M. Burgin and A. P. Šostak in [B-Š] and V. Janiš in [J1, J2, J3].

The notion of fuzzy smooth function, based on the fuzzy derivative of [K1] called ('nearness derivative', by V. Janiš in [J3]) was introduced in [K2]. This paper begins by studying fuzzy smooth functions in the multidimensional case. The functions we use always have their domain in $\mathbb{R}^n$ and range in $\mathbb{R}$.

1. Basic notions and notation.

Let us recall the notion of the fuzzy nearness relation from [K1].

Let $\mathscr{F}\colon \mathbb{R}\times\mathbb{R}\to[0,1]$. We say that $\mathscr{F}$ is a *relation of fuzzy nearness* iff the following holds:

(1) $x\,\mathscr{F}\,z=1 \Leftrightarrow z=x$,

(2) $x\,\mathscr{F}\,z = z\,\mathscr{F}\,x$,

(3) for each x,z,t if $x\le t\le z$ (or $z\le t\le x$) then $x\,\mathscr{F}\,t\ge x\,\mathscr{F}\,z$,

(4) for all x $\lim\limits_{z\to\pm\infty} x\,\mathscr{F}\,z=0$.

If moreover

(5) for all $\alpha\in\,]0,1]$ and all $x\in\mathbb{R}$ there exist unique $x_\alpha > x$ and $x_{-\alpha}<x$ such that $x\mathscr{F}x_\alpha = x\,\mathscr{F}\,x_{-\alpha}=\alpha$,

2000 Mathematics Subject Classification: 04A72, 26A24.
Keywords: fuzzy set, fuzzy smoothness, fuzzy continuity.
This paper has been supported by grant No. 1/4064/97

then $\mathscr{F}$ will be called *continuous fuzzy nearness relation.*

Property (4) of the above definition was added by V. Janiš in [J3].

Next, we define the fuzzy nearness relation in the n-dimensional case by the following:

Let $\mathscr{D}\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,1]$. We say that $\mathscr{D}$ is an *n-dimensional fuzzy nearness relation* iff the following holds:

(1) $(x_1, x_2, \dots, x_n)\,\mathscr{D}\,(y_1, y_2, \dots, y_n) = 1 \iff$
$(x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_n)$,

(2) $(x_1, x_2, \dots, x_n)\,\mathscr{D}\,(y_1, y_2, \dots, y_n) = (y_1, y_2, \dots, y_n)\,\mathscr{D}\,(x_1, x_2, \dots, x_n)$,

(3) $\forall t \in\,]0,1[\quad \forall (x_1, x_2, \dots, x_n),\ (y_1, y_2, \dots, y_n)$,

$$(x_1, x_2, \dots, x_n)\,\mathscr{D}\,(y_1, y_2, \dots, y_n) \le$$
$$(x_1, x_2, \dots, x_n)\,\mathscr{D}\,\big((y_1 - x_1)t + x_1, (y_2 - x_2)t + x_2, \dots, (y_n - x_n)t + x_n\big)\,,$$

(4) $\forall (k_1, k_2, \dots, k_n)$ such that $\sum\limits_{i=1}^{n} k_i^2 = 1$ and $\forall (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$,

$$\lim_{t\to\infty} (x_1, x_2, \dots, x_n)\,\mathscr{D}\,\big(t(x_1 \cdot k_1, x_2 \cdot k_2, \dots, x_n \cdot k_n) + (x_1, x_2, \dots, x_n)\big) = 0\,.$$

If moreover

(5) $\forall (k_1, k_2, \dots, k_n)$ such that $\sum\limits_{i=1}^{n} k_i^2 = 1$, $\forall \alpha \in\,]0,1[\,, \forall (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ there exist uniquely given $t_1 > 0$ and $t_2 < 0$ such that

$$(x_1, x_2, \dots, x_n)\,\mathscr{D}\,\big(t_1(x_1 \cdot k_1, x_2 \cdot k_2, \dots, x_n \cdot k_n) + (x_1, x_2, \dots, x_n)\big) =$$
$$(x_1, x_2, \dots, x_n)\,\mathscr{D}\,\big(t_2(x_1 \cdot k_1, x_2 \cdot k_2, \dots, x_n \cdot k_n) + (x_1, x_2, \dots, x_n)\big) = \alpha\,,$$

then the n-dimensional fuzzy nearness relation is said to be *continuous.*

Throughout this paper n will be a fixed natural number, $\mathscr{D}$ will denote a (not fixed) continuous n-dimensional fuzzy nearness relation (the term 'continuous' will be omitted, for brevity) and $\mathscr{Y}$ will denote a (not fixed) continuous fuzzy nearness relation (for one-dimensional case).

A (crisp) function $f\colon \mathbb{R}^n \to \mathbb{R}$ is said to be *fuzzy α-continuous (with respect to a given $\mathscr{Y}$) at a point $\mathcal{X} = (x_1, x_2, \dots, x_n)$* iff the following hold

$$\liminf_{\mathcal{Z}\to\mathcal{X}} f(\mathcal{Z})\ \ \mathscr{Y}\ \limsup_{\mathcal{Z}\to\mathcal{X}} f(\mathcal{Z}) \ge \alpha\,,$$
$$f(\mathcal{X})\ \ \mathscr{Y}\ \limsup_{\mathcal{Z}\to\mathcal{X}} f(z) \ge \alpha\,,$$
$$f(\mathcal{X})\ \ \mathscr{Y}\ \liminf_{\mathcal{Z}\to\mathcal{X}} f(\mathcal{Z}) \ge \alpha\,.$$

f is said to be *fuzzy α-continuous with respect to $\mathscr{Y}$ in the (n-dimensional) interval $I = I_1 \times I_2 \times \dots \times I_n$* iff it is fuzzy α-continuous with respect to $\mathscr{Y}$ at each point of I.

Lemma 1. *A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is continuous in an n-dimensional interval I iff for all $\alpha \in]0,1[$ it is fuzzy α-continuous in I with respect to every continuous fuzzy nearness relation $\mathscr{Y}$.*

Proof. This follows immediately from the above definition. □

Throughout this paper, $\mathcal{X}$ will always denote some point $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, x^i_α and $x^i_{-\alpha}$ the reals for which $x^i_{-\alpha} < x_i < x^i_\alpha$ and

$$(x_1, x_2, \dots, x^i_{-\alpha}, \dots, x_n) \ \mathscr{D} \ (x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, x^i_\alpha, \dots, x_n) \ \mathscr{D} \ (x_1, x_2, \dots, x_n) = \alpha\,.$$

Further, we set

$$\mathcal{X}^i_{-\alpha} = (x_1, x_2, \dots, x^i_{-\alpha}, \dots, x_n) \qquad \text{and} \qquad \mathcal{X}^i_\alpha = (x_1, x_2, \dots, x^i_\alpha, \dots, x_n)\,.$$

Put

$$f^{i\,\prime}_\alpha(\mathcal{X}) = \max\left(\frac{f(\mathcal{X}^i_\alpha) - f(\mathcal{X})}{x^i_\alpha - x_i}\,,\ \frac{f(\mathcal{X}^i_{-\alpha}) - f(\mathcal{X})}{x^i_{-\alpha} - x_i}\right)$$

and

$$f^{i\,\prime}_{-\alpha}(\mathcal{X}) = \min\left(\frac{f(\mathcal{X}^i_\alpha) - f(\mathcal{X})}{x^i_\alpha - x_i}\,,\ \frac{f(\mathcal{X}^i_{-\alpha}) - f(\mathcal{X})}{x^i_{-\alpha} - x_i}\right)\,.$$

The *partial nearness α-derivative of the function f for the ith variable* at the point $\mathcal{X}$ with respect to $\mathscr{D}$ is defined by the interval $\left[f^{i\,\prime}_{-\alpha}(\mathcal{X}),\ f^{i\,\prime}_\alpha(\mathcal{X})\right]$ and denoted by $\dfrac{\partial f}{\partial \mathscr{D}^i_\alpha}(\mathcal{X})$.

$\mathcal{K}$ will always denote an n-tuple $(k_1, k_2, \dots, k_n) \in \mathbb{R}^n$ satisfying $\sum\limits_{i=1}^{n} k_i^2 = 1$. Further, for any $\alpha \in]0,1[$, $t_{\mathcal{X},\mathcal{K}}(\alpha) > 0$ and $t_{\mathcal{X},\mathcal{K}}(-\alpha) < 0$ will denote reals such that

$$\begin{aligned} \mathcal{X} + t_{\mathcal{X},\mathcal{K}}(-\alpha) \cdot \mathcal{K} \ \mathscr{D} \ \mathcal{X} &= \alpha\,, \\ \mathcal{X} + t_{\mathcal{X},\mathcal{K}}(\alpha) \cdot \mathcal{K} \ \mathscr{D} \ \mathcal{X} &= \alpha\,, \end{aligned}$$

where '+' means the usual vector addition.

With partial nearness derivatives we may also define the partial nearness derivatives along lines. Denote by M and m the maximal and minimal value, respectively, from the values

$$\left\{\frac{f\big(\mathcal{X} + t_{\mathcal{X},\mathcal{K}}(-\alpha) \cdot \mathcal{K}\big) - f(\mathcal{X})}{t_{\mathcal{X},\mathcal{K}}(-\alpha)}\,,\ \frac{f\big(\mathcal{X} + t_{\mathcal{X},\mathcal{K}}(\alpha) \cdot \mathcal{K}\big) - f(\mathcal{X})}{t_{\mathcal{X},\mathcal{K}}(\alpha)}\right\}. \tag{A}$$

Then $[m, M]$ is said to be the *nearness α-derivative of f along the line corresponding to the vector $\mathcal{K}$*, at $\mathcal{X}$, and it will be denoted by $\dfrac{\partial f}{\partial \mathscr{D}_\alpha^{\mathcal{K}}}(\mathcal{X})$.

Now, we proceed to define fuzzy smoothness in the multidimensional case. Intuitively, an n-dimensional function is smooth if it has a tangent plane. So we require our functions to have a tangent plane in a 'fuzzy' sense. But first, yet more notation. Let $I_1 \subset \mathbb{R}$ and $I_2 \subset \mathbb{R}$ be arbitrary intervals and $\mathscr{Y}$ an arbitrary continuous fuzzy nearness relation. We denote $I_1 =_\alpha I_2$ iff

$$\max(I_1 \cup I_2) \;\mathscr{Y}\; \min(I_1 \cup I_2) \geq \alpha \,.$$

Let $[m_s, M_s]$ for $s \in \mathfrak{S}$ be a system of intervals. Then we put

$$\liminf_{s\in\mathfrak{S}} [m_s,\, M_s] = \left[\liminf_{s\in\mathfrak{S}} m_s,\, \liminf_{s\in\mathfrak{S}} M_s\right],$$

$$\limsup_{s\in\mathfrak{S}} [m_s,\, M_s] = \left[\limsup_{s\in\mathfrak{S}} m_s,\, \limsup_{s\in\mathfrak{S}} M_s\right].$$

2. Fuzzy smoothness.

Let $\alpha \in\,]0,1[$. A fuzzy α-continuous function f at a point $\mathcal{X}$ is said to be *α-smooth at the point $\mathcal{X}$* with respect to the pair $(\mathscr{D}, \mathscr{Y})$ iff for each $\mathcal{K} = (k_1, k_2, \dots, k_n)$ the following holds

$$\text{(1)} \qquad \frac{\partial f}{\partial \mathscr{D}_\alpha^{\mathcal{K}}}(\mathcal{X}) =_\alpha \sum_{i=1}^{n} k_i \frac{\partial f}{\partial \mathscr{D}_\alpha^{i}}(\mathcal{X}),$$

$$\text{(2)} \qquad \frac{\partial f}{\partial \mathscr{D}_\alpha^{\mathcal{K}}}(\mathcal{X}) =_\alpha \frac{\partial f}{\partial \mathscr{D}_\alpha^{\mathcal{K}}}(\mathcal{X}),$$

$$\text{(3)} \qquad \frac{\partial f}{\partial \mathscr{D}_\alpha^{\mathcal{K}}}(\mathcal{X}) =_\alpha \liminf_{\mathcal{Z}\to\mathcal{X}} \frac{\partial f}{\partial \mathscr{D}_\alpha^{\mathcal{K}}}(\mathcal{Z}),$$

$$\text{(4)} \qquad \frac{\partial f}{\partial \mathscr{D}_\alpha^{\mathcal{K}}}(\mathcal{X}) =_\alpha \limsup_{\mathcal{Z}\to\mathcal{X}} \frac{\partial f}{\partial \mathscr{D}_\alpha^{\mathcal{K}}}(\mathcal{Z}),$$

$$\text{(5)} \qquad \limsup_{\mathcal{Z}\to\mathcal{X}} \frac{\partial f}{\partial \mathscr{D}_\alpha^{\mathcal{K}}}(\mathcal{Z}) =_\alpha \liminf_{\mathcal{Z}\to\mathcal{X}} \frac{\partial f}{\partial \mathscr{D}_\alpha^{\mathcal{K}}}(\mathcal{Z}).$$

Roughly speaking, the first property is the existence of a 'fuzzy' tangent plane, the second one means that the interval we get as the α-derivative, is not too big, and the other properties are a kind of continuity of the α-derivatives along lines.

The function f is said to be *α-smooth on an n-dimensional interval I* with respect to the pair $(\mathscr{D}, \mathscr{Y})$, iff f is α-smooth at each point $\mathcal{X} \in I$ ($\alpha \in\,]0,1[$).

f is said to be *fuzzy smooth at a point $\mathcal{X}$* with respect to $(\mathscr{D}, \mathscr{Y})$, iff there exists α_0 such that for all $\alpha_0 < \alpha < 1$ f is α-smooth at $\mathcal{X}$.

f is said to be *fuzzy smooth on an n-dimensional interval I* with respect to $(\mathscr{D}, \mathscr{Y})$, iff there exists α_0 such that for all $\alpha_0 < \alpha < 1$, f is α-smooth on I.

Lemma 2. *If a function f is fuzzy smooth at a point $\mathcal{X}$ with respect to a pair $(\mathscr{D},\mathscr{Y})$ then the following holds for all n-tuples $\mathcal{K}$*

$$\limsup_{\alpha\to 1_-}\frac{f(\mathcal{X}+t_{\mathcal{X},\mathcal{K}}(-\alpha)\cdot\mathcal{K})-f(\mathcal{X})}{t_{\mathcal{X},\mathcal{K}}(-\alpha)}=\limsup_{\alpha\to 1_-}\frac{f(\mathcal{X}+t_{\mathcal{X},\mathcal{K}}(\alpha)\cdot\mathcal{K})-f(\mathcal{X})}{t_{\mathcal{X},\mathcal{K}}(\alpha)},$$

$$\liminf_{\alpha\to 1_-}\frac{f(\mathcal{X}+t_{\mathcal{X},\mathcal{K}}(-\alpha)\cdot\mathcal{K})-f(\mathcal{X})}{t_{\mathcal{X},\mathcal{K}}(-\alpha)}=\liminf_{\alpha\to 1_-}\frac{f(\mathcal{X}+t_{\mathcal{X},\mathcal{K}}(\alpha)\cdot\mathcal{K})-f(\mathcal{X})}{t_{\mathcal{X},\mathcal{K}}(\alpha)}.$$

P r o o f. This follows from the definition of fuzzy smoothness. □

Corollary. *If f is a fuzzy smooth function at a point $\mathcal{X}$ with respect to a pair $(\mathscr{D},\mathscr{Y})$ and if f has right- (or left-) hand side crisp partial derivative at $\mathcal{X}$ for the ith variable, then f has partial derivative at $\mathcal{X}$ for the ith variable and moreover*

$$\frac{\partial f}{\partial x_i}(\mathcal{X})=\lim_{\alpha\to 1_-}\frac{\partial f}{\partial\mathscr{D}^i_\alpha}(\mathcal{X}).$$

Comment. The above equality means two different things:

1. the intervals $\frac{\partial f}{\partial\mathscr{D}^i_\alpha}(\mathcal{X})$ tend to a single point as $\alpha\to 1_-$,
2. the limit point equals $\frac{\partial f}{\partial x_i}(\mathcal{X})$.

Lemma 3. *If f is a smooth function on a compact n-dimensional interval I then for each $\mathscr{D}$ there exists a continuous fuzzy nearness relation $\mathscr{Y}$ such that f is fuzzy smooth on I and, vice versa, for each $\mathscr{Y}$ there is a $\mathscr{D}$ such that f is fuzzy smooth on I.*

P r o o f. Let $\mathscr{D}$ be a given n-dimensional fuzzy nearness relation. Denote

$$F_\alpha(\mathcal{X})=\sup_{\mathcal{K}}\left|\frac{f(\mathcal{X}+t_{\mathcal{X},\mathcal{K}}(-\alpha)\cdot\mathcal{K})-f(\mathcal{X})}{t_{\mathcal{X},\mathcal{K}}(-\alpha)}-\frac{f(\mathcal{X}+t_{\mathcal{X},\mathcal{K}}(\alpha)\cdot\mathcal{K})-f(\mathcal{X})}{t_{\mathcal{X},\mathcal{K}}(\alpha)}\right|.$$

Then, since f is continuous in I and, because of the compactness of I, is also uniformly continuous, $\{F_\alpha\colon \alpha\in\,]0,1[\,\}$ is a system of functions with $\lim\limits_{\alpha\to 1_-}F_\alpha(\mathcal{X})=0$ for all $\mathcal{X}\in I$. Put $M_\alpha=\sup\{F_\alpha(x)\colon x\in I\}$. Then also $\lim\limits_{\alpha\to 1_-}M_\alpha=0$. Now, it is sufficient to take a decreasing continuous function $H\colon[0,1]\to\mathbb{R}$ such that $H(1)=0$ and for each $\beta\in\,]0,1[\ H(\beta)\geq\sup\{M_\alpha\colon\alpha\in[\beta,1[\,\}$. The fuzzy nearness relation $\mathscr{Y}$ can be defined by

$$x\,\mathscr{Y}\,y=\begin{cases}\alpha & \text{if } |x-y|=H(\alpha) \text{ for } \alpha\in\,]0,1]\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

We have just constructed $\mathscr{Y}$ for given $\mathscr{D}$. The construction of the n-dimensional fuzzy nearness relation $\mathscr{D}$ for given $\mathscr{Y}$ is left to the reader. □

EXAMPLE 1. Take

$$f(x_1, x_2) = \begin{cases} (x_1 + x_2) \cdot \sin\left(\frac{1}{x_1+x_2}\right) & \text{if } x_1 + x_2 > 0\,, \\ 0 & \text{if } x_1 + x_2 = 0\,, \\ -(x_1 + x_2) \cdot \sin\left(\frac{1}{x_1+x_2}\right) & \text{if } x_1 + x_2 < 0\,, \end{cases}$$

and let $\mathscr{Y}$ be an arbitrary continuous fuzzy nearness relation. Let $\mathscr{D}$ yield

$$\mathcal{X} \mathbin{\mathscr{D}} \mathcal{Z} = \begin{cases} 1 - \rho(\mathcal{X}, \mathcal{Z}) & \text{if } \rho(\mathcal{X}, \mathcal{Z}) \le 1\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where ρ is the usual Euclidean metric in $\mathbb{R}^n$. Then because of symmetry and continuity of f, the function is fuzzy smooth at 0. Since it is smooth at each point except 0, we may construct $\mathscr{Y}$ in such a way that for the intervals $I_{1,\alpha} = [-1, -\alpha] \times [-1, \alpha]$ and $I_{2,\alpha} = [\alpha, 1] \times [\alpha, 1]$ the function f is β-smooth for any $\beta \ge \hat{\alpha}$, where $\hat{\alpha}$ is a particular element of the interval $]\alpha, 1[$. Hence, f is fuzzy smooth at each point $(x_1, x_2) \in [-1, 1] \times [-1, 1]$.

We say that $f\colon \mathbb{R}^n \to \mathbb{R}$ is of *bounded variation in an n-dimensional interval* I, iff for all $\mathcal{X}, \mathcal{Z} \in I$ there is an M such that for all m and all $0 = t_0 < t_1 < t_2 < \dots t_m = 1$

$$\sum_{i=1}^{m} \left| f\big(t_{i-1}(\mathcal{Z} - \mathcal{X}) + \mathcal{X}\big) - f\big(t_i(\mathcal{Z} - \mathcal{X}) + \mathcal{X}\big) \right| \le M\,.$$

THEOREM 1. *If f is fuzzy smooth on a compact n-dimensional interval I with respect to the given pair $(\mathscr{D}, \mathscr{Y})$, then f is of bounded variation in I.*

P r o o f. Let f not be of bounded variation in I. Then there are points $\mathcal{X}, \mathcal{Z} \in I$ such that for all M there is an m and a sequence $0 = t_0 < t_1 < \dots < t_m$ such that

$$\sum_{i=1}^{m} \left| f\big(t_{i-1}(\mathcal{Z} - \mathcal{X}) + \mathcal{X}\big) - f\big(t_i(\mathcal{Z} - \mathcal{X}) + \mathcal{X}\big) \right| > M\,. \tag{B}$$

Since f is fuzzy smooth, and hence continuous on the compact n-dimensional interval I, it is bounded. Thus, we may find sequences $\{t_i\}_{i=1}^{\infty} \in\,]\,0, 1[$ and $\{\tilde{t}_i\}_{i=1}^{\infty} \in\,]\,0, 1[$ such that the points $S_i = t_i \cdot (\mathcal{Z} - \mathcal{X}) + \mathcal{X}$ are local maxima on the line given by the points $\mathcal{X}, \mathcal{Z}$ and $s_i = \tilde{t}_i \cdot (\mathcal{Z} - \mathcal{X}) + \mathcal{X}$ are local minima on that line. Because of (B), the slopes $\big(f(S_i) - f(s_i)\big)/(t_i - \tilde{t}_i)$, increase up to infinity, moreover, for all $\alpha \in]0, 1[$ we may find an i such that

$$S_i \mathbin{\mathscr{D}} s_i = \tilde{\alpha} \ge \alpha\,.$$

Hence, since f has a local maximum at S_i on our line for arbitrary $\alpha \in]0, 1[$ we may find i such that f is not α-smooth at S_i for any $(\mathscr{D}, \mathscr{Y})$, hence f is not α-smooth at that S_i. This contradicts the assumption that f is fuzzy smooth on I. □

Theorem 2. *There exist functions which are fuzzy smooth at each point of a compact n-dimensional interval I with respect to a particular pair $(\mathscr{D}, \mathscr{Y})$, but which are not fuzzy smooth on the interval I with respect to any pair $(\mathscr{D}, \mathscr{Y})$.*

Proof. The function f from Example 1 is fuzzy smooth at each point of the interval $[-1,1] \times [-1,1]$ with respect to the constructed pair $(\mathscr{D}, \mathscr{Y})$, but it is of unbounded variation, hence by Theorem 1 it cannot be fuzzy smooth with respect to any $(\mathscr{D}, \mathscr{Y})$ on the interval $[-1,1] \times [-1,1]$. □

Theorem 3. *There exist functions which are fuzzy smooth on an n-dimensional interval I with respect to a given pair $(\mathscr{D}, \mathscr{Y})$, but which are not smooth on I.*

Proof. Let us consider the following function

$$f(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 = 0\,, \\ \operatorname{sign} x_1 \sqrt{2|x_1| - x_1^2} & \text{if } x_1 \in \,]-1,0[\, \cup \,]0,1[\,, \\ -1 & \text{if } x < -1\,, \\ 1 & \text{if } x > 1\,. \end{cases}$$

Then f is not differentiable at $(0, x_2)$ for any $x_2 \in \mathbb{R}$ and hence it is not smooth on the interval $[-1,1] \times [-1,1]$. Define $\mathscr{D}$ by the following

$$\mathcal{X} \mathscr{D} \mathcal{Z} = \begin{cases} 1 - \rho(\mathcal{X}, \mathcal{Z}) & \text{if } \rho(\mathcal{X}, \mathcal{Z}) < 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Now, we are going to construct $\mathscr{Y}$ in such a way that f will be fuzzy smooth with respect to the couple $(\mathscr{D}, \mathscr{Y})$. For arbitrary $y \geq 0$ put

$$\alpha_y = \inf\left\{\alpha \in \,]0,1[\, : (\exists \mathcal{X} \in [-1,1] \times [-1,1])(\exists \mathcal{K})\left(y \in \frac{\partial f}{\partial \mathscr{D}_\alpha^{\mathcal{K}}}(\alpha)\right)\right\}.$$

Such an α_y exists for arbitrary $y \geq 0$. Further, using the notation from (A), for arbitrary $y \geq 0$ and $\alpha \in \left[\alpha_y, 1\right[$ put

$$R(y, \alpha) = \sup\left\{\frac{f\big(\mathcal{X} + t_{\mathcal{X},\mathcal{K}}(\pm\alpha) \cdot \mathcal{K}\big) - f(\mathcal{X})}{t_{\mathcal{X},\mathcal{K}}(\pm\alpha)}\,,\ y \in \frac{\partial f}{\partial \mathscr{D}_\alpha^{\mathcal{K}}}(\alpha)\right\}$$

and

$$L(y, \alpha) = \inf\left\{\frac{f\big(\mathcal{X} + t_{\mathcal{X},\mathcal{K}}(\pm\alpha) \cdot \mathcal{K}\big) - f(\mathcal{X})}{t_{\mathcal{X},\mathcal{K}}(\pm\alpha)}\,,\ y \in \frac{\partial f}{\partial \mathscr{D}_\alpha^{\mathcal{K}}}(\alpha)\right\}.$$

Since f is uniformly continuous in $[-1,1] \times [-1,1]$, such functions R and L do exist. To complete the construction, it is enough to take $\mathscr{Y}$ such that for any $y \geq 0$ and any $\alpha \in \left[\alpha_y, 1\right[$

$$L(y, \alpha) \ \mathscr{Y} \ R(y, \alpha) \geq \alpha\,.$$

□

REFERENCES

[B-Š] BURGIN, M.—ŠOSTAK, A.: *Fuzzification of the theory of continuous functions*, Fuzzy Sets and Systems **62** (1994), 71–81.

[D-P1] DUBOIS, D.—PRADE, H.: *Towards fuzzy differential calculus, part 1, Integration of fuzzy mappings*, Fuzzy Sets and Systems **8** (1982), 1–17.

[D-P2] DUBOIS, D.—PRADE, H.: *Towards fuzzy differential calculus, part 2: Integration on fuzzy intervals*, Fuzzy Sets and Systems **8** (1982), 105–116.

[D-P3] DUBOIS, D.—PRADE, H.: *Towards fuzzy differential calculus, part 3: Differentiation*, Fuzzy Sets and Systems **8** (1982), 225–233.

[J1] JANIŠ, V.: *Fixed points of fuzzy functions*, Tatra Mt. Math. Publ. **12** (1997), 13–19.

[J2] JANIŠ, V.: *Fuzzy uniformly continuous functions*, Tatra Mt. Math. Publ **14** (1998), 177–180.

[J3] JANIŠ, V.: *Nearness derivatives and fuzzy differentiability*, Fuzzy Sets and Systems (to appear).

[K1] KALINA, M.: *Derivatives of fuzzy functions and fuzzy derivatives*, Tatra Mt. Math. Publ. **12** (1997), 27–34.

[K2] KALINA, M.: *On fuzzy smooth functions*, Tatra Mt. Math. Publ **14** (1998), 153–159.

Received April 17, 1998

Department of Mathematics
Faculty of Civil Engineering
Slovak Technical University
Radlinského 11
SK–813 68 Bratislava
SLOVAKIA
E-mail: kalina@vox.svf.stuba.sk