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ON NOWHERE DENSENESS OF CERTAIN FUZZY CONTROLLERS CONTAINING PRERESTRICTED NUMBER OF RULES

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ABSTRACT. Although in the literature various results can be found claiming that fuzzy controllers are universal approximators, recently it was shown [3, 4] that for Sugeno controllers this property does not hold any longer if the number of the rules is bounded. Namely, it was shown that the set of Sugeno controllers with bounded number of rules is nowhere dense in the space of continuous functions. In this paper a generalization of the referred result is presented for a wider class of fuzzy controllers. This class contains Takagi–Sugeno-, and Takagi–Sugeno–Kang type fuzzy controllers as well.

1. Introduction

Considering the success of fuzzy controllers with nonlinear plants, several authors raised the question, what is the reason of this versatility of fuzzy controllers. In the beginning of the 90's an answer was given from mathematical point of view [1, 2, 5, 9]: fuzzy control systems are universal approximators in the sense that it is possible to construct such rule based fuzzy controllers that approximate any (nonlinear) continuous functions with arbitrary accuracy, i.e., the fuzzy control system can implement any necessary control function.

On the other hand, the practical limitation of these theorems was investigated in [3, 4] by Moser for Sugeno controllers. It turned out that under the assumption of bounded number of rules, Sugeno controllers are nowhere dense in the L^p space (space of continuous function equipped with the norm $\|\cdot\|_p$), which means that the set of Sugeno controllers lies "almost" discretely in the L^p space.

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This paper is a generalization of the results for Sugeno controllers, more precisely speaking, we consider fuzzy controllers of a very general form originated from Sugeno and Takagi–Sugeno controllers. This paper is strongly based on the results and proofs of M o s e r 's paper, hence, we restrict ourselves to point out the necessary modifications to adjust the proofs to the more general case.

The main idea of this paper is to show that the technique used in the essential proof of [3, 4] is suitable also for a wider class of fuzzy controllers. This class contains fuzzy controllers with consequents that can be expressed as tensor product of univariate functions. For convenience, we call them T-controllers, where T stands for the abbreviation of "tensor" (product consequent).

2. Preliminaries

Definition 1. Let us consider a fuzzy rule base with rules of the structure

$$R_{i,j}\colon \quad \text{If} \quad x \text{ is } A_i \quad \text{and} \quad y \text{ is } B_j \quad \text{then} \quad \sum_{k=1}^{m_{i,j}} f_{k,(i,j)}(x)\, g_{k,(i,j)}(y)\,, \tag{1}$$

where the number of the rules is bounded by n_1 in the first and by n_2 in the second variable. The functions $f_{k,(i,j)}(x)$ and $g_{k,(i,j)}(y)$ are arbitrary univariate functions, furthermore, there is an overall upper bound m such that for all indices, $m_{i,j} \leq m$ $(1 \leq i \leq n_1,\ 1 \leq j \leq n_2)$ holds. For brevity, when context makes clear that the denotation concerns the rule $R_{i,j}$, we write f_k and g_k instead of $f_{k,(i,j)}(x)$ and $g_{k,(i,j)}(x)$, respectively. We shall refer to the fuzzy controller containing rules of form (1) as *T-controller.*

Let us denote the set of fuzzy controllers containing rules of the form (1) by

$$\mathbf{T}_{(n_1,n_2),m}(X \times Y)\,, \tag{2}$$

where the pair (n_1, n_2) is an upper bound for the number of rules in the input spaces X and Y, and m is the upper bound for the number of functions summed in the consequent part of the rules. For $p \in [1, \infty]$ we introduce the set

$$\mathbf{T}^{(p)}_{(n_1,n_2),m}(X \times Y)\,, \tag{3}$$

which is a subset of (2) and $L^p(X \times Y)$ as well. Therefore, the set (3) is equipped with the L^p-norm $\|\cdot\|_p$. The elements of sets (3) and (2) are called T-controllers of order $\big((n_1, n_2), m\big)$.

Note that the set of Sugeno controllers is a subset of T-controllers where indices $m_{i,j}$ equal to 1, $f_1 = c_{i,j}$ and $g_1 = 1$ for all i and j. Further, let us consider the so-called Takagi–Sugeno (TS-) controllers on domain $(X \times Y)$ [8], which are fuzzy systems with linear consequents, i.e., with rules like

$$\text{If} \quad x \text{ is } A_i \quad \text{and} \quad y \text{ is } B_j \qquad \text{then} \quad a_{ij} + b_{ij}x + c_{ij}y \,. \tag{4}$$

The set of TS-controllers is also a subset of $\mathbf{T}_{(n_1,n_2),m}(X \times Y)$, because with indices $m_{i,j} = 3$ and with $f_1 = a_{i,j}$ $f_2 = b_{i,j}x$, $f_3 = g_1 = g_2 = 1$, and $g_3 = c_{i,j}y$ in (1) for all $1 \le i \le n_1$ and $1 \le j \le n_2$ we regain the rule of the form (4).

Furthermore, a generalization of Takagi–Sugeno type controllers, the set formed by the Takagi–Sugeno–Kang type (TSK-) controllers [7] is a subset of the set of T-controllers. The TSK-controller models the controlled system by describing its nature by means of fuzzy rules with bounded polynomial consequent as

$$\text{If} \quad x \text{ is } A_i \quad \text{and} \quad y \text{ is } B_j \qquad \text{then} \quad P_m^{ij}(x,y)\,, \tag{5}$$

where $P_m^{ij}(x,y)$ is a polynomial of maximum order m having the form

$$P_m^{ij}(x,y) = \sum_{p=0}^{m}\sum_{q=0}^{m} a_{pq}^{ij} x^p y^q \,. \tag{6}$$

If indices $m_{i,j} = (m+1)^2$, and functions $f_{pq} = a_{pq}^{ij} x^p$ and $g_{pq} = y^q$ $(1 \le p,$ $q \le m)$, then (1) and (5) are equivalent.

The input-output function of T-controller $t \in \mathbf{T}_{(n_1,n_2),m}(\Omega)$, $t : \Omega \to \mathbb{R}$ is given by

$$t(x,y) = \frac{\sum\limits_{i=1}^{n_1}\sum\limits_{j=1}^{n_2} \tilde{\mu}_i(x)\tilde{\nu}_j(y) \sum\limits_{k=1}^{m_{i,j}} f_{k,(i,j)}(x)\, g_{k,(i,j)}(y)}{\sum\limits_{i=1}^{n_1}\sum\limits_{j=1}^{n_2} \tilde{\mu}_i(x)\tilde{\nu}_j(y)}\,, \tag{7}$$

where the families of membership functions provide a complete cover of the input space Ω. Analogously as for Sugeno controllers in [3, 4], we can get

$$t(x,y) = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \tilde{\mu}_i(x)\tilde{\nu}_j(y) \sum_{k=1}^{m_{i,j}} f_{k,(i,j)}(x)\, g_{k,(i,j)}(y)$$

from (7), requiring the families of membership functions to be fuzzy partitions in Ruspini sense [6]. From now, we consider Ω to be $[0,1]^2$.

3. The results on Sugeno controllers [3, 4]

Moser has proven the nowhere denseness of Sugeno controllers in two steps. First, it was shown (see Lemma 1) that a special function cannot be approximated arbitrarily well by Sugeno controllers with bounded number of rules. Then it was presented in Theorem 1 that for every Sugeno controller and for arbitrary $\varepsilon > 0$ there exists a function ω in the ε-environment of the Sugeno controller which cannot be approximated with arbitrary accuracy by Sugeno controllers. This immediately implies that the set of Sugeno controllers with prerestricted number of rules is nowhere dense in the space of continuous functions. The construction of ω in Theorem 1 is based on the special function of the lemma.

Lemma 1. (Moser [4]). *Let $\omega : [0,1]^2 \to \mathbb{R}$ be a function of the form $\omega(x,y) = \alpha$ if $y \geq x$ and $\omega(x,y) \neq \alpha$ else, where $\alpha \in \mathbb{R}$. Then for each $p \in [1,\infty]$ and $n_1, n_2 \in \mathbb{N}$ there holds*

$$\inf\left\{\|\omega - s\|_p : s \in \mathbf{Sug}^{(p)}_{(n_1,n_2)}\right\} > 0\,. \tag{8}$$

Here $\mathbf{Sug}^{(p)}_{(n_1,n_2)}$ denotes the set of Sugeno controllers containing maximum n_1 terms in the first, and n_2 terms in the second variable, and which is equipped with the L^p-norm.

The proof proceeds indirectly. First, it is pointed out, under the assumption that the function ω can be approximated arbitrarily well, that ω has to be the limit of a sequence of Sugeno controllers at least for almost all points. Then it is shown that this limit function is again a Sugeno controller, and hence a tensor product. Finally, due to the special form of the function ω contradiction is derived.

Theorem 1. (Moser [4]). *To each $p \in [1,\infty]$, $\varepsilon > 0$ and*

$$s \in \mathbf{Sug}^{(p)}_{(n_1,n_2)}([0,1]^2)\,,$$

$n_1, n_2 \in \mathbb{N}$, there is a continuous function

$$\omega \in L^p([0,1]^2) \setminus \mathrm{cl}\left[\mathbf{Sug}^{(p)}_{(n_1,n_2)}([0,1]^2)\right]$$

fulfilling $\|\omega - s\|_p < \varepsilon$, where $\mathrm{cl}\big[\mathbf{Sug}^{(p)}_{(n_1,n_2)}(\Omega)\big]$ denotes the closure of the subset $\mathbf{Sug}^{(p)}_{(n_1,n_2)}(\Omega)$ of $L^p(\Omega)$ with respect to the topology induced by the L^p-norm $\|\cdot\|_p$.

4. Main result

Our result is a modification of the essential lemma in [3, 4]. Its main idea is that T-controllers can also be expressed as bounded tensor products of univariate functions. First, we prove that functions of a special form cannot be approximated with arbitrary accuracy by T-controllers, regardless of what L^p-norm is chosen. Then the method of Theorem 1 can be applied for showing the nowhere denseness of the set $T^{(p)}_{(n_1,n_2),m}([0,1]^2)$ in the set of continuous functions.

Lemma 2. *Let $\omega : [0,1] \to \mathbb{R}$ be a function of the form $\omega(x,y) = \alpha$ if $y \geq x$, $\omega(x,y) \neq \alpha$ else, where $\alpha \in \mathbb{R}$. Then for each $p \in [1,\infty]$ and $n_1, n_2 \in \mathbb{N}$ there holds*

$$\inf\left\{ \|\omega - t\|_p \colon t \in \mathbf{T}^{(p)}_{(n_1,n_2),m} \right\} > 0 \,. \tag{9}$$

P r o o f . The proof follows the train of thought of Lemma 1. First, we suppose that opposing the assumption (9) there exist a sequence $\hat{t}_n$, $n \in \mathbb{N}$, of T-controllers, $t \in \mathbf{T}^{(p)}_{(n_1,n_2),m}$, which ensures norm-convergence to the function ω. Analogously as in Lemma 1, norm-convergence implies pointwise convergence almost everywhere. So there exists a sequence t_n of T-controllers such that

$$\lim_n t_n = \omega - \alpha \,,$$

where

$$t_n(x,y) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \mu_i^{(n)}(x)\, \nu_j^{(n)}(y) \sum_{k=1}^{m_{i,j}} f^{(n)}_{k,(i,j)}(x)\, g^{(n)}_{k,(i,j)}(y) \,.$$

Let Ω' be the set where pointwise convergence holds.

We also apply diagonal construction due to Cantor on the basis of Ω^*, which is countable subset of Ω' lying dense in $[0,1]^2$ as well.

The modification in this step is that, instead of convergence of the real numbers in the consequent, we require that for each pair $(x_r, y_r) \in \Omega^*$ the sums

$$\sum_{k=1}^{m_{i,j}} f^{(r,n)}_{k,(i,j)}(x_k)\, g^{(r,n)}_{k,(i,j)}(y_k) \tag{10}$$

converge to a real number as n tends to infinity. Similarly as in Lemma 1, by the upper index (r,n) we denote a subsequence of the sequence denoted by upper index (n), furthermore, we demand the sequence with upper index $(r+1,n)$ to be a subsequence of the sequence with upper index (r,n), (see Lemma 1). Finally, from the above we may suppose that sequences $\big(\mu_i^{(n,n)}\big)_n$, $\big(\nu_j^{(n,n)}\big)_n$, and similarly $\big(f^{(n,n)}_{k,(i,j)}\big)_n$ and $\big(g^{(n,n)}_{k,(i,j)}\big)_n$ are convergent for all indices $1 \leq i \leq n_1$, $1 \leq j \leq n_2$, and $1 \leq k \leq m_{i,j}$ where $m_{i,j} \leq m$.

It follows that the input-output functions of T-controller $t_{n,n}$, converges to the input-output function of T-controller

$$\tilde{t}(x,y) = \sum_{i,j} \tilde{\mu}_i(x)\, \tilde{\nu}_j(y) \sum_k \tilde{f}_{p,(i,j)}(x)\, \tilde{g}_{q,(i,j)}(y) \,, \tag{11}$$

where

$$\lim_n \mu_i^{(n,n)} = \tilde{\mu}_i \,, \quad \lim_n \nu_j^{(n,n)} = \tilde{\nu}_j \,, \quad \lim_n f^{(n,n)}_{k,(i,j)} = \tilde{f}_{k,(i,j)} \,, \quad \lim_n g^{(n,n)}_{k,(i,j)} = \tilde{g}_{k,(i,j)} \,.$$

We aim at showing that (11) is in fact a tensor product of univariate functions with bounded order N, in order to introduce induction similarly as in the referred paper [3].

It is obvious that (11) can be expressed as

$$\tilde{t}(x,y)=\sum_{i}^{N}\phi_i(x)\psi_i(y)\sum_{k}^{m}\tilde{f}_{k,i}(x)\,\tilde{g}_{k,i}(y)\,, \tag{12}$$

where $N=n_1\cdot n_2$. (11) can be rewritten as

$$\begin{aligned}\tilde{t}(x,y)=&\sum_{i}^{N}\phi_i(x)\,\tilde{f}_{1,i}(x)\,\psi_i(y)\,\tilde{g}_{1,i}(y)\\&+\sum_{i}^{N}\phi_i(x)\,\tilde{f}_{2,i}(x)\,\psi_i(y)\,\tilde{g}_{2,i}(y)+\dots\\&\dots+\sum_{i}^{N}\phi_i(x)\,\tilde{f}_{m,i}(x)\,\psi_i(y)\,\tilde{g}_{m,i}(y)\\=&\sum_{i=1}^{m\cdot N}\tilde{\phi}_i(x)\,\tilde{\psi}_i(y)\,,\end{aligned} \tag{13}$$

where

$$\tilde{\phi}_j(x)=\begin{cases}\phi_i(x)\,\tilde{f}_{1,i}(x) & \text{for } j=1,\dots,N,\\ \phi_i(x)\,\tilde{f}_{2,i}(x) & \text{for } j=N+1,\dots,2N,\\ \quad\vdots & \\ \phi_i(x)\,\tilde{f}_{m,i}(x) & \text{for } j=(m-1)N+1,\dots,mN.\end{cases}\qquad (1\le i\le N)\,. \tag{14}$$

Analogously,

$$\tilde{\psi}_j(y)=\begin{cases}\psi_i(y)\,\tilde{g}_{1,i}(y) & \text{for } j=1,\dots,N,\\ \psi_i(y)\,\tilde{g}_{2,i}(y) & \text{for } j=N+1,\dots,2N,\\ \quad\vdots & \\ \psi_i(y)\,\tilde{g}_{m,i}(y) & \text{for } j=(m-1)N+1,\dots,m\cdot N.\end{cases}\qquad (1\le i\le N)\,. \tag{15}$$

Observe that all functions $\tilde{\phi}_i(x)$ and $\tilde{\psi}_i(y)$ $(1\le i\le m\cdot N)$ are univariate functions depending only on x and y, respectively. Hence, the formula (13) shows that the limit of input-output function of T-controllers, $\tilde{t}(x,y)$, can be expressed as tensor product of univariate functions with bounded order $m\cdot N$.

Now, we can again refer to Lemma 1. The final contradiction can be achieved analogously, by means of induction on the basis of tensor product, thus, it is omitted. □

Example 1. In the case of TS-controllers (see (4)), (14) changes as

$$\tilde{\phi}_j(x) = \begin{cases} \phi_i(x)\tilde{a}_i & \text{for } j = 1, \dots, N, \\ \phi_i(x)\tilde{b}_i x & \text{for } j = N+1, \dots, 2N, \\ \phi_i(x) & \text{for } j = 2N+1, \dots, 3N, \end{cases} \qquad (1 \le i \le N),$$

where $\tilde{a}_i$ and $\tilde{b}_i$ are the parameters of linear consequents of the limit-TS-controller. (15) changes analogously.

Example 2. In the case of TSK-controllers (see (5) and (6)), (14) changes as

$$\tilde{\phi}_j(x) = \begin{cases} \phi_i(x)\tilde{a}^i_{00} & \text{for } j = 1, \dots, N, \\ \phi_i(x)\tilde{a}^i_{01} & \text{for } j = N{+}1, \dots, 2N, \\ \quad\vdots & \\ \phi_i(x)\tilde{a}^i_{0m} & \text{for } j = (m \cdot N){+}1, \dots, (m{+}1) \cdot N, \\ \phi_i(x)\tilde{a}^i_{10}x & \text{for } j = \big((m{+}1) \cdot N\big){+}1, \dots, (m{+}2) \cdot N, \\ \quad\vdots & \\ \phi_i(x)\tilde{a}^i_{1m}x & \text{for } j = \big((2m{+}1) \cdot N\big){+}1, \dots, 2(m{+}1) \cdot N, \\ \quad\vdots & \\ \quad\vdots & \\ \phi_i(x)\tilde{a}^i_{m0}x^m & \text{for } j = \big((m(m{+}1)) \cdot N\big){+}1, \dots, \big(m(m+1){+}1\big) \cdot N, \\ \quad\vdots & \\ \phi_i(x)\tilde{a}^i_{mm}x^m & \text{for } j = \big(((m{+}1)^2 - 1) \cdot N\big){+}1, \dots, (m{+}1)^2 \cdot N, \end{cases}$$

$$(1 \le i \le N),$$

where coefficients $\tilde{a}^i_{pq}$ $(1 \le i \le N,\ 0 \le p, q \le m)$ are parameters of polynomial consequents of the limit-TSK-controller. (15) changes analogously.

The proof of the next theorem is identical with the one of Theorem 1, as it exploits only the result of the Lemma 2, which is analogous to Lemma 1 and does not use particular properties of Sugeno or T-controllers.

Theorem 2. *To each $p \in [1, \infty]$, $\varepsilon > 0$ and $t \in \mathbf{T}^{(p)}_{(n_1,n_2),m}$, $n_1, n_2, m \in \mathbb{N}$, there is a continuous function*

$$\omega \in L^p(\Omega) \setminus \mathrm{cl}\big(\mathbf{T}^{(p)}_{(n_1,n_2),m}\big)$$

fulfilling $\|\omega - t\|_p < \varepsilon$.

As Theorem 2 guarantees that to each $\varepsilon > 0$ and $t \in \mathbf{T}^{(p)}_{(n_1,n_2),m}$ there is a function $\omega \in L^p(\Omega) \setminus \mathrm{cl}\big(\mathbf{T}^{(p)}_{(n_1,n_2),m}\big)$ with $\|\omega - t\|_p$, we immediately obtain that $\mathbf{T}^{(p)}_{(n_1,n_2),m}$ is nowhere dense in $L^p(\Omega)$.

As we mentioned previously, Takagi–Sugeno and Takagi–Sugeno–Kang controllers are special cases of T-controllers, hence, all results in this section are suitable for them as well.

5. Conclusion

Theorem 2 points out that if we restrict the number of rules, the corresponding set of T-controllers is nowhere dense in the space of continuous functions. This contributes to the opinion that not the great variety of design parameters but rather the number of rules used in the controller is the reason of the universal approximation property of various fuzzy controllers. Comparing this result with theorems stating the universal approximation property of various classes of fuzzy controllers, this paper should reveal the practical limitation of fuzzy controllers for function approximation.

The initial result of Moser [3, 4] cannot be generalized by this techniques for further fuzzy controllers, e.g., for Mamdani type controllers. Nevertheless, we think that such practical restriction for the rule base, as its boundedness, may lead to similar result for Mamdani controllers, as well. However, it requires different proving method.

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