Tatra Mt. Math. Publ. **16** (1999), 11–19

TATRA MOUNTAINS
Mathematical Publications

LATTICES GENERATED BY BINARY FUZZY RELATIONS

Radim Bělohlávek

ABSTRACT. We present a construction which yields a complete lattice from an arbitrary binary fuzzy relation, generalizing Birkhoff's result from the early age of lattice theory. We show possible applications in the modeling of vague concepts and in conceptual analysis.

1. Introduction

As early as in the first edition of the programmatic book on lattice theory [3], G. Birkhoff showed the following result: For a binary relation $I \subseteq G \times M$ define the mappings ${}^{\uparrow}\colon 2^G \mapsto 2^M$, ${}^{\downarrow}\colon 2^M \mapsto 2^G$ by

$$A^{\uparrow} = \{m \in M \colon \langle g, m\rangle \in I \text{ for each } g \in A\}, \tag{1}$$
$$B^{\downarrow} = \{g \in G \colon \langle g, m\rangle \in I \text{ for each } m \in B\}, \tag{2}$$

for each $A \subseteq G$, $B \subseteq M$. Then the mappings ${}^{\uparrow}$ and ${}^{\downarrow}$ form a Galois connection between the complete lattices $\langle 2^G, \subseteq\rangle$ and $\langle 2^M, \subseteq\rangle$ and the composite mappings ${}^{\uparrow\downarrow}\colon 2^G \mapsto 2^G$, ${}^{\downarrow\uparrow}\colon 2^M \mapsto 2^M$ are closure operators (see later). Birkhoff then demonstrates on several examples that such a case arises quite often in mathematics.

The aim of our paper is to show that the Birkhoff's construction may be generalized: starting from a binary **L**-fuzzy relation and considering the complete residuated lattice as the structure of truth values we get generalizations of the main theorems of the classical case. In Section 2 we introduce necessary notions. Section 3 contains the results. In Section 4 we propose application to conceptual modeling (cf. [12]).

2000 Mathematics Subject Classification: 03B52, 04A72, 06A15, 94D05.
Keywords: fuzzy relation, complete lattice, residuated lattice, Galois connection, conceptual modeling.

Supported by grant No. 029/98 of the IGS University of Ostrava and by grant No. 201/96/0985 of the GAČR.

2. Preliminaries

Let $\mathbf{U} = \langle U, \subseteq \rangle$ and $\mathbf{V} = \langle V, \subseteq \rangle$ be ordered sets. The mappings $^{\uparrow}$ and $^{\downarrow}$, $^{\uparrow}\colon U \mapsto V$, $^{\downarrow}\colon V \mapsto U$, form a Galois connection between $\mathbf{U}$ and $\mathbf{V}$ whenever the conditions

$$\text{if} \quad A_1 \subseteq A_2 \quad \text{then} \quad A_1^{\uparrow} \supseteq A_2^{\uparrow}, \tag{3}$$

$$\text{if} \quad B_1 \subseteq B_2 \quad \text{then} \quad B_1^{\downarrow} \supseteq B_2^{\downarrow}, \tag{4}$$

$$A \subseteq A^{\uparrow\downarrow} \quad \text{and} \quad B \subseteq B^{\downarrow\uparrow}, \tag{5}$$

are satisfied for every $A, A_1, A_2 \in U$, $B, B_1, B_2 \in V$.

Recall that an $\mathbf{L}$-fuzzy set [4] in a universe X is any mapping $A\colon X \mapsto L$ where L is the support of a suitable structure $\mathbf{L}$ of truth values. If $X = G \times M$ then A is called a binary $\mathbf{L}$-fuzzy relation between G and M. The crucial question is that of the choice of the structure $\mathbf{L}$ of truth values. From the point of view of fuzzy logic in narrow sense [5, 6, 7, 9, 10] it seems to be a natural requirement for $\mathbf{L}$ to be a residuated lattice [5, 6, 9, 11]. Several classes of special residuated lattices have been axiomatized (i.e., there is a complete logical calculus with the semantics defined over the respective class of residuated lattices) [6, 7, 9, 10]. By a residuated lattice we mean an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$ such that

$\langle L, \wedge, \vee, 0, 1 \rangle$ is a lattice with the least element 0 and the greatest element 1 (let $\leq$ be the lattice order),

$\langle L, \otimes, 1 \rangle$ is a commutative semigroup with the unit 1, i.e., $\otimes$ is associative, commutative, and the identity $x \otimes 1 = x$ holds,

$\otimes$ and $\rightarrow$ satisfy the adjunction property, i.e.,

$$x \leq y \rightarrow z \qquad \text{iff} \qquad x \otimes y \leq z$$

holds for each $x, y, z \in L$.

In each residuated lattice, $x \leq y$ implies $x \otimes z \leq y \otimes z$ (isotonicity), and $x \leq y$ implies $z \rightarrow x \leq z \rightarrow y$ (isotonicity in the second argument) and $x \rightarrow z \geq y \rightarrow z$ (antitonicity in the first argument). The operation $\otimes$ is thus a t-norm (see, e.g., [6]), $\rightarrow$ is called a residuum. A residuated lattice $\mathbf{L}$ is complete if the lattice $\langle L, \wedge, \vee, 0, 1 \rangle$ is complete [3]. The most studied and applied set of truth values is the real interval $[0, 1]$. The most important are the Łukasiewicz, Gödel, and product algebras (see [6] for their role and the definitions).

Another important structure of truth values is given by $L = \{a_0, a_1, \dots, a_n\}$, $0 = a_0 < \cdots < a_n = 1$, with the t-norm $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and the corresponding residuum $a_k \rightarrow a_l = a_{\min(n-k+l,n)}$. Note that each of the preceding residuated lattices is complete. It may be easily checked that there is just one residuated lattice with the support $2 = \{0, 1\}$, we denote it by $\mathbf{2}$. $\mathbf{2}$ is the algebra of truth values of classical logic. The class of all $\mathbf{L}$-fuzzy sets in X will be denoted by L^X. If $\mathbf{L}$ is complete then the relation $\subseteq$ (defined by $A \subseteq B$ iff $A(x) \leq B(x)$ for each $x \in X$) makes L^X into a complete lattice.

3. Lattice generated by fuzzy relation

Suppose $\mathbf{L}$ is a complete residuated lattice and I is a binary $\mathbf{L}$-fuzzy relation between G and M, i.e., $I \in L^{G\times M}$. We first introduce operators ${}^{\uparrow}\colon L^G \mapsto L^M$ and ${}^{\downarrow}\colon L^M \mapsto L^G$ generalizing the classical case. The definition (1) of ${}^{\uparrow}$ may be reformulated as follows

$$m \in A^{\uparrow} \qquad \text{iff} \qquad \text{for all } g \in G\colon \text{ if } g \in A \text{ then } \langle g, m\rangle \in I\,.$$

Rewriting into the language of complete residuated lattices (the chosen algebras of fuzzy logic) yields

$$A^{\uparrow}(m) = \bigwedge_{g\in G} A(g) \to I(g,m) \qquad \text{for } m \in M\,. \tag{6}$$

Similarly, for ${}^{\downarrow}$ we put

$$B^{\downarrow}(g) = \bigwedge_{m\in M} B(m) \to I(g,m) \qquad \text{for } g \in G\,. \tag{7}$$

Thus (6), (7) are equivalent to (1), (2) for $\mathbf{L} = \mathbf{2}$. However, (6), (7) may be used for any complete residuated lattice $\mathbf{L}$.

Lemma 1. *Let $\mathbf{L}$ be a complete residuated lattice, let $I \in L^{G\times M}$ be a $\mathbf{L}$-fuzzy relation between G and M. Then the operators ${}^{\uparrow}$ and ${}^{\downarrow}$ given by* (6) *and* (7) *form a Galois connection between $\langle L^G, \subseteq\rangle$ and $\langle L^M, \subseteq\rangle$.*

P r o o f . Prove (3). If $A_1 \subseteq A_2$ then $A_1(g) \leq A_2(g)$ for each $g \in G$. From the antitonicity of $\to$ in the first argument we conclude $A_1(g) \to I(g,m) \geq A_2(g) \to I(g,m)$ for each $g \in G$, $m \in M$. Hence

$$\begin{aligned} A_1^{\uparrow}(m) &= \bigwedge_{g\in G} \big(A_1(g) \to I(g,m)\big) \\ &\geq \bigwedge_{g\in G} \big(A_2(g) \to I(g,m)\big) = A_2^{\uparrow}(m)\,, \end{aligned}$$

i.e., $A_1^{\uparrow} \supseteq A_2^{\uparrow}$. The condition (4) may be proved analogously.

Prove (5). Let $A \underset{\sim}{\subset} G$, $g \in G$. Prove first

$$A(g) \leq \bigwedge_{h\in G} \big(A(h) \to I(h,m)\big) \to I(g,m) \tag{8}$$

for each $m \in M$. By the adjunction property, (8) holds iff

$$A(g) \otimes \bigwedge_{h \in G} \big(A(h) \to I(h,m)\big) \le I(g,m)$$

which is due to commutativity of $\otimes$ and the adjunction property equivalent to

$$\bigwedge_{h \in G} \big(A(h) \to I(h,m)\big) \le A(g) \to I(g,m)$$

which holds evidently. From (8) and from the properties of infimum it follows

$$A(g) \le \bigwedge_{m \in M} \Big(\bigwedge_{h \in G} \big(A(h) \to I(h,m)\big) \to I(g,m) \Big) = A^{\uparrow\downarrow}(g)\,,$$

i.e., $A \subseteq A^{\uparrow\downarrow}$. The proof of the second part of (5) is symmetric. We have proved the assertion. $\square$

The following assertion follows from the properties of Galois connections, see, e.g., [3].

Lemma 2. *Under the conditions of Lemma 1, it holds*

$$A^{\uparrow} = A^{\uparrow\downarrow\uparrow} \qquad \text{and} \qquad B^{\downarrow} = B^{\downarrow\uparrow\downarrow}\,. \tag{9}$$

Moreover, the composite mappings $^{\uparrow\downarrow}$, $^{\downarrow\uparrow}$ *are closure mappings* [3] *on* $\langle L^G, \subseteq \rangle$, $\langle L^M, \subseteq \rangle$, *respectively, i.e., the conditions* (5)

$$A^{\uparrow\downarrow} = A^{\uparrow\downarrow\uparrow\downarrow}, \qquad B^{\downarrow\uparrow} = B^{\downarrow\uparrow\downarrow\uparrow}\,, \tag{10}$$

$$A_1 \subseteq A_2 \text{ implies } A_1^{\uparrow\downarrow} \subseteq A_2^{\uparrow\downarrow}, \qquad B_1 \subseteq B_2 \text{ implies } B_1^{\downarrow\uparrow} \subseteq B_2^{\downarrow\uparrow} \tag{11}$$

hold.

Note that closure mappings in L^X for $\mathbf{L}$ being a complete lattice have been studied (under the name fuzzy closure operators) in [2].

Denote further $\mathcal{L}(G,M,I) = \{\langle A,B \rangle \in L^G \times L^M \colon A^{\uparrow} = B,\ B^{\downarrow} = A\}$ and put for $\langle A_1, B_1 \rangle, \langle A_2, B_2 \rangle \in \mathcal{L}(G,M,I)$

$$\langle A_1, B_1 \rangle \le \langle A_2, B_2 \rangle \quad \text{iff} \quad A_1 \subseteq A_2 \quad \text{iff} \quad B_1 \supseteq B_2\,.$$

One may easily verify that the definition of $\le$ is correct.

Theorem. $\mathcal{L}(G,M,I)$ *is under* $\le$ *a complete lattice where*

$$\bigwedge_{j \in J} \langle A_j, B_j \rangle = \left\langle \bigcap_{j \in J} A_j\,, \Big(\bigcap_{j \in J} A_j\Big)^{\uparrow} \right\rangle, \tag{12}$$

$$\bigvee_{j \in J} \langle A_j, B_j \rangle = \left\langle \bigcap_{j \in J} (B_j)^{\downarrow}\,, \bigcap_{j \in J} B_j \right\rangle. \tag{13}$$

Hence, $^{\uparrow}$ and $^{\downarrow}$ are dual isomorphisms between the lattices of $^{\uparrow\downarrow}$-closed **L**-*fuzzy sets in G and $^{\downarrow\uparrow}$-closed* **L**-*fuzzy sets in M.*

P r o o f. It follows from the previous statements and definitions that

$$\bigcap_{j\in J} A_j \subseteq \Big(\bigcap_{j\in J} A_j\Big)^{\uparrow\downarrow}$$

and

$$\Big(\bigcap_{j\in J} A_j\Big)^{\uparrow\downarrow} \subseteq A_k^{\uparrow\downarrow} = A_k \qquad \text{for all} \quad k \in J\,,$$

i.e., $\Big(\bigcap\limits_{j\in J} A_j\Big)^{\uparrow\downarrow} \subseteq \bigcap\limits_{j\in J} A_j$. Hence $\Big(\bigcap\limits_{j\in J} A_j\Big)^{\uparrow\downarrow} = \bigcap\limits_{j\in J} A_j$ and $\Big\langle \bigcap\limits_{j\in J} A_j, \Big(\bigcap\limits_{j\in J} A_j\Big)^{\uparrow} \Big\rangle \in \mathcal{L}(G,M,I)$. If $\langle A,B\rangle \leq \langle A_j,B_j\rangle$ for all $j\in J$ then $A\subseteq A_j$ for all $j\in J$, and $A \subseteq \bigcap\limits_{j\in J} A_j$. We conclude $\langle A,B\rangle \leq \Big\langle \bigcap\limits_{j\in J} A_j, \Big(\bigcap\limits_{j\in J} A_j\Big)^{\uparrow} \Big\rangle$, proving (12). (13) can be proved symmetrically. $\square$

4. Applications in conceptual analysis

Starting from the B i r k h o f f's result, R. W i l l e begun to develop the theory of concept lattices as a part of his program concerning the restructuring of lattice theory [12]. The theory of concept lattices resulted into the nowadays well-developed theory with applications in conceptual modeling and conceptual data analysis. In the following we show that our generalization seems to be an interesting one also from the point of view of conceptual modeling and conceptual data analysis. We first introduce necessary notions which are direct generalizations of the notions of the classical case introduced in [12]. We obtain the classical notions setting $\mathbf{L} = \mathbf{2}$.

By the school of Port-Royal logicians, a concept is given by its extent (the collection of objects which are determined by the concept) and its intent (the collection of attributes characterizing the concept). Thus, e.g., the extent of the concept DOG is the collection of all dogs (we leave aside the question of time and possible worlds) and its intent is the collection of all properties of dogs (e.g., "to bark", "to be a mammal" etc.). Surely, not all concepts have sharp extents and intents. Most of the empirical concepts are vague (consider, e.g., the concept BIG DOG — bigger dogs-objects belong more to the extent of this concept than smaller dogs). In the framework of fuzzy set theory it therefore seems to be natural to model both the extent and the intent by fuzzy sets. Call therefore the triple $\langle G,M,I\rangle$, where $I\in L^{G\times M}$ a (**L**-fuzzy) context [12]. G will be called the set of objects, M will be called the set of attributes, I models the relation "to have an attribute" between objects and attributes. A ($\mathbf{L}$-fuzzy) concept is then a pair $\langle A, B \rangle \in L^G \times L^M$ such that $A^{\uparrow} = B$, $B^{\downarrow} = A$. A and B represent the extent and the intent of the concept $\langle A, B \rangle$. The condition $A^{\uparrow} = B$, $B^{\downarrow} = A$ says: B is the (fuzzy) set of all attributes common to all of the objects of A (cf. [6]) and A is the (fuzzy) set of all objects which share all of the attributes of B (cf. [7]). The relation $\leq$ then models the hierarchy of concepts ("to be a subconcept"). The fundamental properties of this hierarchy as described by Theorem seem to be very natural and confirmed by our intuition. For further developments in the theory of fuzzy concepts see [1].

Consider the following example which is a fuzzified version of the example presented in [12]. For $\mathbf{L}$ we put $L = \{0, \frac{1}{2}, 1\}$ with the structure described in Section 2. The fuzzy context is given by the Table 1. The set G contains nine elements (Mercury, ... , Pluto), the set M contains four attributes ("size small", ... , "near to Sun").

TABLE 1. Fuzzy context given by planets and their attributes.

	size		from Sun	
	small (ss)	large (sl)	far (df)	near (dn)
Mercury(Me)	1	0	0	1
Venus (V)	1	0	0	1
Earth (E)	1	0	0	1
Mars (Ma)	1	0	$\frac{1}{2}$	1
Jupiter (J)	0	1	1	$\frac{1}{2}$
Saturn	0	1	1	$\frac{1}{2}$
Uranus (U)	$\frac{1}{2}$	$\frac{1}{2}$	1	0
Neptune (N)	$\frac{1}{2}$	$\frac{1}{2}$	1	0
Pluto (P)	1	0	1	0

The corresponding lattice of fuzzy concepts is depicted in Fig. 1.

To get a deeper insight, the elements (i.e., fuzzy concepts) of the lattice are identified in Tab. 2.

A few remarks to this example. First, note that the concepts which were found depend on the fuzzy context. The fuzzy context is given by a subjective judgement (e.g., to what degree we consider "Mars is far from Sun" to be true). The subjective judgement influences therefore the resulting structure of concepts. Second, there are apparently natural concepts in the concept lattice (e.g., 14 ("small planet near to Sun")), as well as concepts which "were found" (e.g., 26 ("a planet far from Sun which is at least partially large")). Concept no. 1 is an

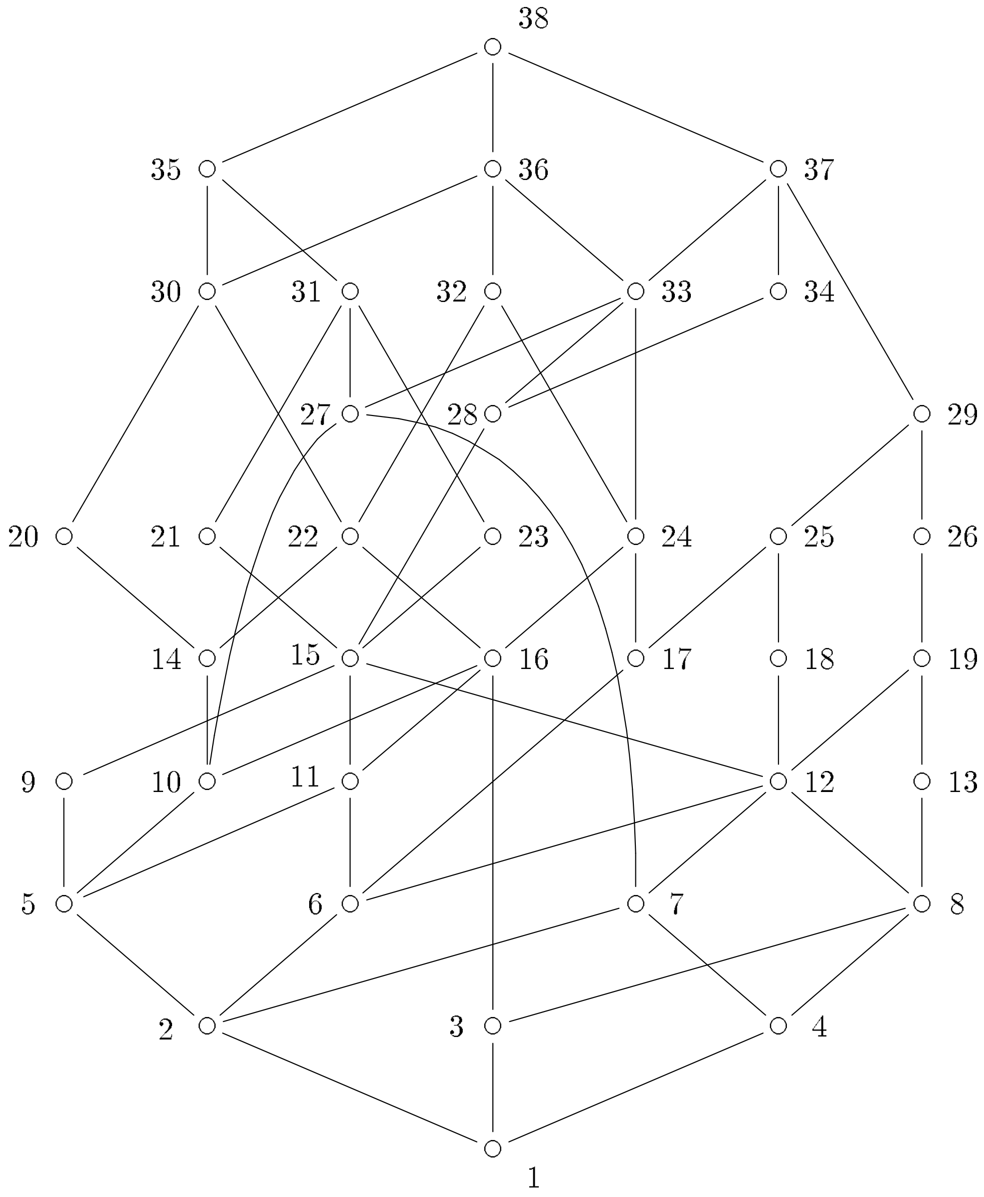

Figure 1. Lattice of fuzzy concepts

example of an empirically empty concept. Concepts which do not contain any element in the degree 1 in their extents (e.g., 1, 2, 3) could be called partially (empirically) empty. There is nothing wrong about these concepts just as is nothing wrong about empirically empty concepts.

Table 2. Fuzzy concepts of the context of Tab. 1.

no.	extent									intent			
	Me	V	E	Ma	J	S	U	N	P	ss	sl	df	dn
1.	0	0	0	0	0	0	0	0	0	1	1	1	1
2.	0	0	0	$\frac{1}{2}$	0	0	0	0	0	1	$\frac{1}{2}$	1	1
3.	0	0	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	1	$\frac{1}{2}$
4.	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	1	1	1
5.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	1
6.	0	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{1}{2}$
7.	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1
8.	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	1	$\frac{1}{2}$
9.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
10.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	0	0	0	1	0	$\frac{1}{2}$	1
11.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
12.	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$
13.	0	0	0	0	1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	1	1	$\frac{1}{2}$
14.	1	1	1	1	0	0	0	0	0	1	0	0	1
15.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
16.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$
17.	0	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	1	0
18.	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0
19.	0	0	0	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$
20.	1	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	0	0	1
21.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
22.	1	1	1	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$
23.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
24.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	$\frac{1}{2}$	0
25.	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$	0	1	0
26.	0	0	0	$\frac{1}{2}$	1	1	1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	0
27.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1
28.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
29.	0	0	0	$\frac{1}{2}$	1	1	1	1	1	0	0	1	0
30.	1	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$
31.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$
32.	1	1	1	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	0	0
33.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0
34.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0
35.	1	1	1	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$
36.	1	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$	0	0	0
37.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	1	1	1	0	0	$\frac{1}{2}$	0
38.	1	1	1	1	1	1	1	1	1	0	0	0	0

REFERENCES

[1] BĚLOHLÁVEK, R.: *Network processing indeterminacy.* Doctoral thesis, xviii+198 pp., Technical University of Ostrava, (1998).

[2] BIACINO, L.—GERLA, G.: *An extension principle for closure operators*, J. Math. Anal. Appl. (to appear).

[3] BIRKHOFF, G.: *Lattice Theory*, 1-st edition, AMS Coll. Publ., Vol. 25, Providence, R. I., 1940.

[4] GOGUEN, J. A.: *L-fuzzy sets*, J. Math. Anal. Appl. **18** (1967), 145–174.

[5] GOGUEN, J. A.: *The logic of inexact concepts*, Synthese **19** (1968–69), 325–373.

[6] HÁJEK, P.: *Metamathematics of Fuzzy Logic.* A book to appear by Kluwer.

[7] HÖHLE, U.: *Commutative residuated monoids*, in: Non-classical logics and their applications to fuzzy subsets (U. Höhle, E. P. Klement, eds.), Kluwer, Dordrecht, 1995.

[8] *Arbeiten der Forschungsgruppe Begriffsanalyse bis 1989, Arbeiten der Forschungsgruppe Begriffsanalyse ab 1990*, Lists of citations, TH Darmstadt, 1994.

[9] NOVÁK, V.: *Fuzzy Sets and Their Applications*, Adam-Hilger, Bristol, 1989.

[10] PAVELKA, J.: *On fuzzy logic I, II, III*, Z. Math. Logik Grundlag. Math. **25** (1979), 45–52, 119–139, 447–464.

[11] WARD, M.—DILWORTH, R. P.: *Residuated lattices*, Trans. Amer. Math. Soc. **45** (1939), 335–354.

[12] WILLE, R.: *Restructuring lattice theory: an approach based on hierarchies of concepts*, in: Ordered Sets, Reidel, Dordrecht-Boston, 1982, pp. 445–470.

Received March 31, 1998

Department of Computer Science
Technical University of Ostrava
tř. 17. listopadu
CZ–708 33 Ostrava
CZECH REPUBLIC

Institute for Res. and Appl.
of Fuzzy Modeling
University of Ostrava
Bráfova 7
CZ–701 03 Ostrava
CZECH REPUBLIC
E-mail: belohlav@osu.cz