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BASIC PROPERTIES OF IMPLICATORS IN A RESIDUAL FRAMEWORK

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ABSTRACT. A general class of binary operators on the unit interval is considered. Typical members of this class are conjunctors. With each of these operators, three other binary operators are associated, of which, in some cases, one can be considered as a conjunctor and the two others as implicators. The most important of these operators is the residual operator. In this paper, these associated operators are studied in detail, identifying necessary and sufficient conditions to be imposed on a given operator to ensure a certain property of one of its associated operators. In particular, properties as the neutrality principle, contrapositivity and the exchange principle are investigated.

1. Introduction

In this paper, a general class of binary operators on the unit interval is considered, containing operators with increasing, left-continuous second partial maps and satisfying an additional boundary condition. With each of these operators, three other binary operators are associated. Starting from a conjunctor with some additional properties, for instance, one of them can also be considered as a conjunctor, while the two others can be considered as implicators. The idea behind this paper stems from [3, 4] and can be stated as follows: what conditions should a given operator fulfill in order to guarantee a desired behaviour of its residual operator? This investigation is carried out on an *individual* basis, i.e., for each property, such as the neutrality principle, contrapositivity and the exchange principle, separately. Combinations of these results then lead to unusual characterizations of t-norms, for instance.

Our paper is organized as follows. In Section 2, we briefly recall the usual definitions of negators, conjunctors and implicators. In Section 3, we associate three binary operators with a given binary operator and relate these constructions to Fodor's closure theorems. In Section 4, we characterize the operators yielding a

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residual implicator; surprisingly, not all corner conditions of a conjunctor have to be imposed. In the three subsequent sections, similar characterizations are established for the neutrality principle, the contrapositivity and the exchange principle.

2. Logical operators

In this section, we list the basic definitions of negators, strict negators and strong negators generalizing the Boolean negation, of conjunctors, t-seminorms and t-norms generalizing the Boolean conjunction and of implicators generalizing the Boolean implication.

Definition 2.1.

(i) A *negator* $\mathcal{N}$ is a decreasing unary operator on $[0,1]$ that satisfies $\mathcal{N}(0)=1$ and $\mathcal{N}(1)=0$.

(ii) A negator $\mathcal{N}$ is called *strict* if it is a permutation.

(iii) A negator $\mathcal{N}$ is called *strong* if it is involutive, i.e., $(\forall x \in [0,1])$ $\big(\mathcal{N}(\mathcal{N}(x))=x\big)$.

Note that for any strict negator $\mathcal{N}$, its inverse $\mathcal{N}^{-1}$ is also a strict negator, and both are continuous.

Definition 2.2.

(i) A *conjunctor* $\mathcal{C}$ is an increasing binary operator on $[0,1]$ that satisfies $\mathcal{C}(0,1)=\mathcal{C}(1,0)=0$ and $\mathcal{C}(1,1)=1$.

(ii) A *t-seminorm* $\mathcal{C}$ is a conjunctor that satisfies $(\forall x \in [0,1])\big(\mathcal{C}(1,x) = \mathcal{C}(x,1)=x\big)$.

(iii) A *t-norm* $\mathcal{T}$ is a t-seminorm that is commutative and associative.

Definition 2.3. An *implicator* $\mathcal{I}$ is a hybrid monotonous binary operator on $[0,1]$ (i.e., with decreasing first and increasing second partial maps) that satisfies $\mathcal{I}(0,0)=\mathcal{I}(1,1)=1$ and $\mathcal{I}(1,0)=0$.

For any implicator $\mathcal{I}$, the partial map $\mathcal{I}(\,\cdot\,,0)$ is a negator, denoted $\mathcal{N}_{\mathcal{I}}$ and called the negator induced by $\mathcal{I}$. Notice that from here on, by a binary operator $\mathcal{O}$ we always mean a binary operator on $[0,1]$.

3. Fodor's closure theorems

In this section, we associate three binary operators with a given binary operator and relate them to Fodor's closure theorems.

Definition 3.1. Consider a binary operator $\mathcal{O}$ and a negator $\mathcal{N}$. Then the binary operators $R[\mathcal{O}]$, $S_{\mathcal{N}}[\mathcal{O}]$ and $C_{\mathcal{N}}[\mathcal{O}]$ are defined by, for any $(a,b) \in [0,1]^2$:

$$\begin{aligned} R[\mathcal{O}](a,b) &= \sup\{x \in [0,1] \colon \mathcal{O}(a,x) \le b\}\,, \\ S_{\mathcal{N}}[\mathcal{O}](a,b) &= \mathcal{N}\big(\mathcal{O}(a,\mathcal{N}(b))\big)\,, \\ C_{\mathcal{N}}[\mathcal{O}](a,b) &= S_{\mathcal{N}}\big[R[\mathcal{O}]\big](a,b)\,. \end{aligned}$$

The operator $R[\mathcal{O}]$ is usually called the *residual operator* of $\mathcal{O}$, a term which only makes sense for operators $\mathcal{O}$ belonging to the class $\mathscr{F}$ described below. One easily verifies that for a strict negator $\mathcal{N}$ it always holds that $S_{\mathcal{N}^{-1}}\big[S_{\mathcal{N}}[\mathcal{O}]\big] = \mathcal{O}$; obviously, it also holds that $\mathcal{O}_1 = \mathcal{O}_2 \iff S_{\mathcal{N}}[\mathcal{O}_1] = S_{\mathcal{N}}[\mathcal{O}_2]$.

In this paper, three classes of binary operators will prove to be of extreme importance. The first one is based on a property known as the *residuation condition*, while the two others depend on the choice of a negator $\mathcal{N}$.

(i) $\mathscr{F}$ is the class of binary operators $\mathcal{O}$ for which it holds that

$$\big(\forall(a,b,c) \in [0,1]^3\big)\Big(\mathcal{O}(a,b) \le c \iff b \le R[\mathcal{O}](a,c)\Big)\,;$$

(ii) $\mathscr{G}_{\mathcal{N}}$ is the class of binary operators $\mathcal{O}$ for which it holds that

$$\big(\forall(a,b,c) \in [0,1]^3\big)\Big(\mathcal{O}(a,b) \le c \iff \mathcal{O}\big(a,\mathcal{N}(c)\big) \le \mathcal{N}(b)\Big)\,;$$

(iii) $\mathscr{H}_{\mathcal{N}}$ is the class of binary operators $\mathcal{O}$ for which it holds that

$$\big(\forall(a,b,c) \in [0,1]^3\big)\Big(\mathcal{O}(a,b) \le c \iff \mathcal{O}\big(\mathcal{N}(c),b\big) \le \mathcal{N}(a)\Big)\,.$$

The classes $\mathscr{F}$ and $\mathscr{G}_{\mathcal{N}}$ appear in the work of Fodor [10] and Fodor [11], respectively, and the class $\mathscr{H}_{\mathcal{N}}$ appears in the work of Demirli [3]. The following important theorem gives us insight in the extent of the class $\mathscr{F}$. It follows from the theory of Galois-connections (see, e.g., [13]).

Theorem 3.1. *Consider a binary operator $\mathcal{O}$. The following statements are equivalent:*

(i) *$\mathcal{O}$ belongs to the class $\mathscr{F}$;*

(ii) *the second partial maps of $\mathcal{O}$ are increasing, left-continuous and*

$$\big(\forall a \in [0,1]\big)\big(\mathcal{O}(a,0) = 0\big)\,;$$

(iii) *the second partial maps of $R[\mathcal{O}]$ are increasing, right-continuous and*

$$\big(\forall a \in [0,1]\big)\big(R[\mathcal{O}](a,1) = 1\big)\,.$$

This theorem implies that any operator $\mathcal{O} \in \mathscr{F}$ satisfies two corner conditions of a conjunctor: $\mathcal{O}(0,0) = \mathcal{O}(1,0) = 0$. The corresponding residual operator $R[\mathcal{O}]$ satisfies two corner conditions of an implicator: $R[\mathcal{O}](0,1) = R[\mathcal{O}](1,1) = 1$.

The importance of the classes $\mathscr{F}$ and $\mathscr{G}_{\mathcal{N}}$ can be estimated from the following closure theorems.

THEOREM 3.2. [10] *Consider a binary operator $\mathcal{O}$ and a strong negator $\mathcal{N}$, then the following equivalence holds:*

$$\mathcal{O} \in \mathscr{F} \iff R\big[C_{\mathcal{N}}[\mathcal{O}]\big] = S_{\mathcal{N}}[\mathcal{O}]\,.$$

This theorem can be reformulated in the following equivalent way.

COROLLARY 3.1. *Consider a binary operator $\mathcal{O}$ and a strong negator $\mathcal{N}$, then the following equivalence holds:*

$$\mathcal{O} \in \mathscr{F} \iff C_{\mathcal{N}}\big[C_{\mathcal{N}}[\mathcal{O}]\big] = \mathcal{O}\,.$$

PROPOSITION 3.1. *Consider a binary operator $\mathcal{O}$ and a strict negator $\mathcal{N}$, then the following equivalence holds:*

$$\mathcal{O} \in \mathscr{F} \iff C_{\mathcal{N}}[\mathcal{O}] \in \mathscr{F}\,.$$

Proof. We will, for instance, give the proof from left to right. Consider $\mathcal{O} \in \mathscr{F}$. Let $(a,b,c) \in [0,1]^3$. One easily verifies the following chain of equivalences:

$$\begin{aligned} C_{\mathcal{N}}[\mathcal{O}](a,b) \le c &\iff \mathcal{N}\big(R[\mathcal{O}]\big(a,\mathcal{N}(b)\big)\big) \le c \\ &\iff R[\mathcal{O}]\big(a,\mathcal{N}(b)\big) \ge \mathcal{N}^{-1}(c) \\ &\iff \mathcal{O}\big(a,\mathcal{N}^{-1}(c)\big) \le \mathcal{N}(b)\,. \end{aligned}$$

Similarly, taking into account the decreasingness and continuity of $\mathcal{N}$, it follows that

$$\begin{aligned} b \le R\big[C_{\mathcal{N}}[\mathcal{O}]\big](a,c) &\iff b \le \sup\big\{x : \mathcal{N}\big(R[\mathcal{O}]\big(a,\mathcal{N}(x)\big) \le c\big\} \\ &\iff b \le \sup\big\{x : R[\mathcal{O}]\big(a,\mathcal{N}(x)\big) \ge \mathcal{N}^{-1}(c)\big\} \\ &\iff b \le \sup\big\{\mathcal{N}^{-1}(x) : R[\mathcal{O}](a,x) \ge \mathcal{N}^{-1}(c)\big\} \\ &\iff b \le \mathcal{N}^{-1}\big(\inf\big\{x : R[\mathcal{O}](a,x) \ge \mathcal{N}^{-1}(c)\big\}\big) \\ &\iff \mathcal{N}(b) \ge \inf\big\{x : R[\mathcal{O}](a,x) \ge \mathcal{N}^{-1}(c)\big\}\,. \end{aligned}$$

Since $\mathcal{O} \in \mathscr{F}$, Theorem 3.1 states that $R[\mathcal{O}](a,\cdot\,)$ is increasing and right-continuous, and, hence, we can continue

$$\begin{aligned} b \le R\big[C_{\mathcal{N}}[\mathcal{O}]\big](a,c) &\iff \mathcal{N}(b) \in \big\{x : R[\mathcal{O}](a,x) \ge \mathcal{N}^{-1}(c)\big\} \\ &\iff R[\mathcal{O}]\big(a,\mathcal{N}(b)\big) \ge \mathcal{N}^{-1}(c) \\ &\iff \mathcal{O}\big(a,\mathcal{N}^{-1}(c)\big) \le \mathcal{N}(b)\,. \end{aligned}$$

Combining the above, we conclude that

$$C_{\mathcal{N}}[\mathcal{O}](a,b) \le c \iff b \le R\big[C_{\mathcal{N}}[\mathcal{O}]\big](a,c)\,,$$

for any $(a,b,c) \in [0,1]^3$, i.e., $C_{\mathcal{N}}[\mathcal{O}] \in \mathscr{F}$. The converse part is analogous. □

Theorem 3.3. [11] *Consider a binary operator $\mathcal{O}$ and a strong negator $\mathcal{N}$, then the following equivalence holds:*

$$\mathcal{O} \in \mathscr{G}_{\mathcal{N}} \iff R[\mathcal{O}] = S_{\mathcal{N}}[\mathcal{O}] .$$

Corollary 3.2. *Consider a binary operator $\mathcal{O}$ and a strong negator $\mathcal{N}$, then the following equivalence holds:*

$$\mathcal{O} \in \mathscr{G}_{\mathcal{N}} \iff C_{\mathcal{N}}[\mathcal{O}] = \mathcal{O} .$$

Corollary 3.1 and Theorem 3.3 imply that for any strong negator $\mathcal{N}$, the set $\mathscr{G}_{\mathcal{N}}$ is included in the set $\mathscr{F}$.

Proposition 3.2. *Consider a binary operator $\mathcal{O}$ and a negator $\mathcal{N}$. If $\mathcal{O}$ is commutative, then the following equivalence holds:*

$$\mathcal{O} \in \mathscr{G}_{\mathcal{N}} \iff \mathcal{O} \in \mathscr{H}_{\mathcal{N}} .$$

Proof. We will, for instance, give the proof from left to right. Consider $\mathcal{O} \in \mathscr{G}_{\mathcal{N}}$. Let $(a,b,c) \in [0,1]^3$, then the commutativity of $\mathcal{O}$ and the fact that $\mathcal{O} \in \mathscr{G}_{\mathcal{N}}$ imply the following chain of equivalences, for any $(a,b,c) \in [0,1]^3$:

$$\mathcal{O}(a,b) \le c \iff \mathcal{O}(b,a) \le c \iff \mathcal{O}\big(b,\mathcal{N}(c)\big) \le \mathcal{N}(a) \iff \mathcal{O}\big(\mathcal{N}(c),b\big) \le \mathcal{N}(a) .$$

Hence, $\mathcal{O} \in \mathscr{H}_{\mathcal{N}}$. □

It was initially Dubois and Prade [6] who proved the closure property (Theorem 3.2) to emphasize that the process of generating implicators from conjunctors (see Section 4) is closed. Fodor's results (Theorems 3.2 and 3.3) extend this by identifying the 'most general classes of operators' for which the closure theorems hold.

4. Residual implicators

It is well known that for a t-norm $\mathcal{T}$, for instance, the operator $R[\mathcal{T}]$ is an implicator. Next, we want to determine, within the class $\mathscr{F}$, the necessary and sufficient conditions for an operator $\mathcal{O}$ to have an implicator as residual operator $R[\mathcal{O}]$. This will guarantee at the same time that $C_{\mathcal{N}}[\mathcal{O}]$ is a conjunctor for any strong negator $\mathcal{N}$.

Theorem 4.1. *Consider $\mathcal{O} \in \mathscr{F}$. Then the following equivalences hold:*

(i) *the first partial maps of $\mathcal{O}$ are increasing if and only if the first partial maps of $R[\mathcal{O}]$ are decreasing;*

(ii) $\mathcal{O}(0,1) = 0 \iff R[\mathcal{O}](0,0) = 1$;

(iii) $\big(\forall b \in]0,1]\big)\big(\mathcal{O}(1,b) > 0\big) \iff R[\mathcal{O}](1,0) = 0$.

Proof.

(i) We first prove the implication from left to right. Consider $(a,a') \in [0,1]^2$ such that $a \leq a'$, and $b \in [0,1]$, then $\{x\colon \mathcal{O}(a',x) \leq b\} \subseteq \{x\colon \mathcal{O}(a,x) \leq b\}$, and hence $R[\mathcal{O}](a,b) \geq R[\mathcal{O}](a',b)$.

Next, we show converse implication. Consider $(a,a') \in [0,1]^2$ such that $a \leq a'$, and $b \in [0,1]$, then we have to show that $\mathcal{O}(a,b) \leq \mathcal{O}(a',b)$. Since $\mathcal{O} \in \mathscr{F}$, the inequality $\mathcal{O}(a',b) \leq \mathcal{O}(a',b)$ implies that $b \leq R[\mathcal{O}]\big(a', \mathcal{O}(a',b)\big)$. Since $R[\mathcal{O}]$ is an implicator, its first partial maps are decreasing, and $b \leq R[\mathcal{O}]\big(a, \mathcal{O}(a',b)\big)$. Again using the fact that $\mathcal{O} \in \mathscr{F}$, it follows that $\mathcal{O}(a,b) \leq \mathcal{O}(a',b)$.

(ii) Assume that $\mathcal{O}(0,1) = 0$. By definition, we have $R[\mathcal{O}](0,0) = \sup\{x\colon \mathcal{O}(0,x) = 0\}$. Due to Theorem 3.1, the second partial maps of $\mathcal{O}$ are increasing, and hence $\mathcal{O}(0,1) = 0$ implies that $(\forall\, x \in [0,1])\big(\mathcal{O}(0,x) = 0\big)$. Therefore $R[\mathcal{O}](0,0) = 1$.

Next, assume that $R[\mathcal{O}](0,0) = 1$. Since the second partial maps of $\mathcal{O}$ are increasing, it follows that certainly $(\forall\, x \in]0,1])\big(\mathcal{O}(0,x) = 0\big)$. Now suppose that $\mathcal{O}(0,1) > 0$, then $\mathcal{O}(0,\,\cdot\,)$ is not left-continuous in 1, which contradicts Theorem 3.1.

(iii) If $(\forall\, x \in]0,1])\big(\mathcal{O}(1,x) > 0\big)$, it follows that $R[\mathcal{O}](1,0) = \sup\{x\colon \mathcal{O}(1,x) = 0\} = 0$. Conversely, if $R[\mathcal{O}](1,0) = \sup\{x\colon \mathcal{O}(1,x) = 0\} = 0$, then it certainly holds that $(\forall\, x \in]0,1])\big(\mathcal{O}(1,x) > 0\big)$.

□

Corollary 4.1. *Consider $\mathcal{O} \in \mathscr{F}$. Then $R[\mathcal{O}]$ is an implicator if and only if*

(i) *the first partial maps of $\mathcal{O}$ are increasing;*

(ii) $\mathcal{O}(0,1) = 0$;

(iii) $(\forall\, b \in]0,1])\big(\mathcal{O}(1,b) > 0\big)$.

Notice that in the foregoing corollary, the operator $\mathcal{O}$ need not be a conjunctor, since the condition $\mathcal{O}(1,1) = 1$ is not required. Also, the condition $\mathcal{O}(0,1) = 0$ implies that $(\forall\, b \in [0,1])\big(\mathcal{O}(0,b) = 0\big)$. On the other hand, considering a conjunctor $\mathcal{O} \in \mathscr{F}$ is not sufficient to obtain an implicator, since the condition $(\forall\, b \in]0,1])\big(\mathcal{O}(1,b) > 0\big)$ is not necessarily satisfied.

Theorem 4.2. *Consider $\mathcal{O} \in \mathscr{F}$ and a strict negator $\mathcal{N}$. Then the following equivalences hold:*

(i) *the first partial maps of $C_{\mathcal{N}}[\mathcal{O}]$ are increasing if and only if the first partial maps of $\mathcal{O}$ are increasing;*

(ii) $C_{\mathcal{N}}[\mathcal{O}](0,1) = 0 \iff \mathcal{O}(0,1) = 0$;

(iii) $(\forall b \in]0,1])\big(C_{\mathcal{N}}[\mathcal{O}](1,b) > 0\big) \iff \mathcal{O}(1,1) = 1$.

Proof. The first two equivalences are immediate consequences of Theorem 4.1. Let us show the third one. We will first give the proof from left to right. Assume that $(\forall b \in]0,1])\big(C_{\mathcal{N}}[\mathcal{O}](1,b) > 0\big)$. Suppose that $\mathcal{O}(1,1) = c < 1$. Due to the continuity of $\mathcal{N}$, there exists $b \in]0,1]$ such that $c = \mathcal{N}(b)$. It then follows that $C_{\mathcal{N}}[\mathcal{O}](1,b) = \mathcal{N}\big(R[\mathcal{O}](1,\mathcal{N}(b))\big) = \mathcal{N}\big(R[\mathcal{O}](1,c)\big)$. Since $\mathcal{O}(1,1) = c$, it follows that $R[\mathcal{O}](1,c) = 1$ and hence $C_{\mathcal{N}}[\mathcal{O}](1,b) = 0$, a contradiction.

Next, assume that $\mathcal{O}(1,1) = 1$. Let $b \in [0,1]$ such that $C_{\mathcal{N}}[\mathcal{O}](1,b) = 0$. It then holds that $\sup\{x \colon \mathcal{O}(1,x) \leq \mathcal{N}(b)\} = 1$. Since the second partial maps of $\mathcal{O}$ are increasing and left-continuous, it then follows that $\mathcal{O}(1,1) = 1 \leq \mathcal{N}(b)$, and hence $b = 0$. □

Corollary 4.2. *Consider $\mathcal{O} \in \mathscr{F}$ and a strict negator $\mathcal{N}$. Then $R\big[C_{\mathcal{N}}[\mathcal{O}]\big]$ is an implicator if and only if $\mathcal{O}$ is a conjunctor.*

Corollary 4.3. *Consider $\mathcal{O} \in \mathscr{F}$ and a strong negator $\mathcal{N}$. Then $C_{\mathcal{N}}[\mathcal{O}]$ is a conjunctor if and only if $R[\mathcal{O}]$ is an implicator.*

In view of Proposition 3.1 and Theorem 3.2, Corollary 4.2 is only a special case of the following, easily proven more general theorem, also valid outside the current framework.

Theorem 4.3. *Consider a binary operator $\mathcal{O}$ and a strict negator $\mathcal{N}$. Then $S_{\mathcal{N}}[\mathcal{O}]$ is an implicator if and only if $\mathcal{O}$ is a conjunctor.*

Example 4.1. Consider a t-norm $\mathcal{T} \in \mathscr{F}$. Let $\alpha \in]0,1[$ and define the binary operator $\mathcal{O}$ by $\mathcal{O}(a,b) = \alpha\mathcal{T}(a,b)$. According to Corollary 4.1, the residual operator $R[\mathcal{O}]$, given by

$$R[\mathcal{O}](a,b) = R[\mathcal{T}]\big(a, \tfrac{b}{\alpha}\big)\,,$$

is an implicator. Since $\mathcal{O}(1,1) = \alpha < 1$, the binary operator $\mathcal{O}$ is not a conjunctor. Theorem 4.3 then implies that, for any strict negator $\mathcal{N}$, the binary operator $S_{\mathcal{N}}[\mathcal{O}]$ is not an implicator.

5. Neutrality principle

In classical propositional logic, the tautology "(true $\Rightarrow q$) $\iff q$" is known as the *neutrality principle*. It is extended to the multivalued framework in the following way. Notice that, formally, we introduce the property for any binary operator $\mathcal{O}$.

Definition 5.1. A binary operator $\mathcal{O}$ satisfies the *neutrality principle* if

$$\big(\forall\, b \in [0,1]\big)\big(\mathcal{O}(1,b) = b\big)\,.$$

Definition 5.2. [1] An implicator $\mathcal{I}$ is called a *border implicator* if it satisfies the neutrality principle.

Theorem 5.1. *Consider $\mathcal{O} \in \mathscr{F}$, then $R[\mathcal{O}]$ satisfies the neutrality principle if and only if $\mathcal{O}$ satisfies the neutrality principle.*

Proof. The proof from right to left being trivial, we give the proof from left to right. Consider $b \in [0,1]$. Since $R[\mathcal{O}](1,b) = b$ and $\mathcal{O} \in \mathscr{F}$, it follows that $\mathcal{O}(1,b) \leq b$. Now suppose that $\mathcal{O}(1,b) < b$. Let $\mathcal{O}(1,b) = c$, then $R[\mathcal{O}](1,c) = \sup\big\{x \colon \mathcal{O}(1,x) \leq c\big\} \geq b > c$, a contradiction. Hence, $\mathcal{O}(1,b) = b$. $\square$

Theorem 5.1 and Corollary 4.1 then lead to the following characterization of residual border implicators.

Corollary 5.1. *Consider $\mathcal{O} \in \mathscr{F}$. Then $R[\mathcal{O}]$ is a border implicator if and only if $\mathcal{O}$ is a conjunctor that satisfies the neutrality principle.*

Theorem 5.2. *Consider a binary operator $\mathcal{O}$ and a strong negator $\mathcal{N}$. Then $S_{\mathcal{N}}[\mathcal{O}]$ satisfies the neutrality principle if and only if $\mathcal{O}$ satisfies the neutrality principle.*

Proof. Let $b \in [0,1]$, then, due to the involutivity of $\mathcal{N}$, the following chain of equivalences holds:

$$S_{\mathcal{N}}[\mathcal{O}](1,b) = b \iff \mathcal{N}\big(\mathcal{O}\big(1,\mathcal{N}(b)\big)\big) = b \iff \mathcal{O}\big(1,\mathcal{N}(b)\big) = \mathcal{N}(b)\,.$$

If $\mathcal{O}$ satisfies the neutrality principle, then $S_{\mathcal{N}}[\mathcal{O}]$ clearly also does. Since $\mathcal{N}$ is continuous, for any $b' \in [0,1]$ there exists $b \in [0,1]$ such that $\mathcal{N}(b) = b'$. It then follows that $S_{\mathcal{N}}[\mathcal{O}]$ satisfies the neutrality principle if and only if $\mathcal{O}$ does. $\square$

Proposition 3.1 and Theorems 3.2, 5.1 and 5.2 then lead to the following corollary.

Corollary 5.2. *Consider $\mathcal{O} \in \mathscr{F}$ and a strong negator $\mathcal{N}$. Then $C_{\mathcal{N}}[\mathcal{O}]$ satisfies the neutrality principle if and only if $\mathcal{O}$ satisfies the neutrality principle.*

Corollaries 4.2, 5.1 and 5.2 allow the following formulations.

Corollary 5.3. *Consider $\mathcal{O} \in \mathscr{F}$ and a strong negator $\mathcal{N}$. Then $R\big[C_{\mathcal{N}}[\mathcal{O}]\big]$ is a border implicator if and only if $\mathcal{O}$ is a conjunctor that satisfies the neutrality principle.*

Corollary 5.4. *Consider $\mathcal{O} \in \mathscr{F}$ and a strong negator $\mathcal{N}$. Then $R[\mathcal{O}]$ is a border implicator if and only if $R\big[C_{\mathcal{N}}[\mathcal{O}]\big]$ is a border implicator.*

6. Contrapositivity

A very important property of the classical implication is the contrapositivity, expressed in the tautology "$(p \Rightarrow q) \iff (\neg q \Rightarrow \neg p)$". It has been studied extensively for implicators, too (see, e.g., [12]). Again, we formally introduce the property for any binary operator $\mathcal{O}$.

Definition 6.1. A binary operator $\mathcal{O}$ is called *contrapositive w.r.t. a negator* $\mathcal{N}$ if

$$\left(\forall (a,b) \in [0,1]^2\right)\Big(\mathcal{O}(a,b) = \mathcal{O}\big(\mathcal{N}(b), \mathcal{N}(a)\big)\Big).$$

Theorem 6.1. *Consider $\mathcal{O} \in \mathscr{F}$ and a negator $\mathcal{N}$. Then $R[\mathcal{O}]$ is contrapositive w.r.t. $\mathcal{N}$ if and only if $\mathcal{O} \in \mathscr{H}_{\mathcal{N}}$.*

Proof. Since $\mathcal{O} \in \mathscr{F}$, the second partial maps of $\mathcal{O}$ are increasing and left-continuous. This implies that, for any $(a,b) \in [0,1]^2$:

$$\{x \colon \mathcal{O}(a,x) \le b\} = \big[0, R[\mathcal{O}](a,b)\big],$$
$$\big\{x \colon \mathcal{O}\big(\mathcal{N}(b),x\big) \le \mathcal{N}(a)\big\} = \big[0, R[\mathcal{O}]\big(\mathcal{N}(b), \mathcal{N}(a)\big)\big].$$

The theorem now follows immediately. □

In fact, this theorem was our main motivation behind the introduction of the class $\mathscr{H}_{\mathcal{N}}$. The following proposition combines Theorems 3.3 and 6.1, Proposition 3.2 and Corollary 3.2.

Proposition 6.1. *Consider a binary operator $\mathcal{O}$ and a strong negator $\mathcal{N}$. If $\mathcal{O}$ is commutative, then the following statements are equivalent:*

(i) $\mathcal{O} \in \mathscr{F}$ *and $R[\mathcal{O}]$ is contrapositive w.r.t. $\mathcal{N}$;*

(ii) $\mathcal{O} \in \mathscr{G}_{\mathcal{N}}$;

(iii) $\mathcal{O} \in \mathscr{H}_{\mathcal{N}}$;

(iv) $R[\mathcal{O}] = S_{\mathcal{N}}[\mathcal{O}]$;

(v) $C_{\mathcal{N}}[\mathcal{O}] = \mathcal{O}$.

A more restrictive version of Proposition 6.1 is stated in [12] for a t-norm $\mathcal{T} \in \mathscr{F}$. The above result is more general in the sense that only commutative operators are considered, since, for example, associativity is not required.

The following theorem can be proven easily. It applies in particular to $R\big[C_{\mathcal{N}}[\mathcal{O}]\big]$, for any $\mathcal{O} \in \mathscr{F}$ and strong negator $\mathcal{N}$.

Theorem 6.2. *Consider a binary operator $\mathcal{O}$ and a strong negator $\mathcal{N}$. Then $S_{\mathcal{N}}[\mathcal{O}]$ is contrapositive w.r.t. $\mathcal{N}$ if and only if $\mathcal{O}$ is commutative.*

As a consequence of Proposition 3.1 and Theorem 6.1, we also get the following result.

Proposition 6.2. *Consider $\mathcal{O} \in \mathscr{F}$ and a strict negator $\mathcal{N}$, then $R[C_{\mathcal{N}}[\mathcal{O}]]$ is contrapositive w.r.t. $\mathcal{N}$ if and only if $C_{\mathcal{N}}[\mathcal{O}] \in \mathscr{H}_{\mathcal{N}}$.*

Corollary 6.1. *Consider $\mathcal{O} \in \mathscr{F}$ and a strong negator $\mathcal{N}$. Then $\mathcal{O}$ is commutative if and only if $C_{\mathcal{N}}[\mathcal{O}] \in \mathscr{H}_{\mathcal{N}}$.*

Example 6.1. [3] Consider the conjunctor $\mathcal{O}_k \in \mathscr{F}$ defined by

$$\mathcal{O}_k(a,b) = \begin{cases} \frac{a+b-1}{\max(a,b)} & \text{if } a+b>1\,, \\ 0 & \text{elsewhere}\,. \end{cases}$$

The operator $\mathcal{O}_k$ is first introduced in [3] and is derived from another operator in [12] by means of $C_{\mathcal{N}}$, with $\mathcal{N}(x) = 1-x$. The corresponding residual implicator is given by

$$R[\mathcal{O}_k](a,b) = \begin{cases} \max\left(1-a+ab, \frac{1-a}{1-b}\right) & \text{if } a>b\,, \\ 1 & \text{elsewhere}\,. \end{cases}$$

Clearly, $\mathcal{O}_k$ is commutative but $\mathcal{O}_k \notin \mathscr{H}_{\mathcal{N}}$ as the equivalence

$$\mathcal{O}_k(a,b) \le c \iff \mathcal{O}_k\big(\mathcal{N}(c),b\big) \le \mathcal{N}(a)$$

does not hold for $a=0.5$, $b=0.7$ and $c=0.3$. Therefore, $S_{\mathcal{N}}[\mathcal{O}_k]$ is contrapositive w.r.t. $\mathcal{N}$, but $R[\mathcal{O}_k]$ is not.

Theorem 6.3. [1] *Consider a border implicator $\mathcal{I}$ and a negator $\mathcal{N}$. If $\mathcal{I}$ is contrapositive w.r.t. $\mathcal{N}$, then $\mathcal{N} = \mathcal{N}_{\mathcal{I}}$ and $\mathcal{N}_{\mathcal{I}}$ is strong.*

The foregoing theorem is important, since it shows that for a border implicator $\mathcal{I}$ it does not make sense to talk about contrapositivity w.r.t. a negator $\mathcal{N}$: it can only be contrapositive w.r.t. its induced negator $\mathcal{N}_{\mathcal{I}}$. We can therefore simply talk about contrapositive border implicators.

Corollary 6.2. *Consider $\mathcal{O} \in \mathscr{F}$ such that $R[\mathcal{O}]$ is a border implicator and a negator $\mathcal{N}$. If $R[\mathcal{O}]$ is contrapositive w.r.t. $\mathcal{N}$, then $\mathcal{N} = \mathcal{N}_{\mathcal{I}}$ and $\mathcal{N}_{\mathcal{I}}$ is strong.*

7. Exchange principle

In classical propositional logic, the tautology "$\big(p \Rightarrow (q \Rightarrow r)\big) \iff \big(q \Rightarrow (p \Rightarrow r)\big)$" is known as the *exchange principle*. It is extended to the multivalued framework, for a binary operator $\mathcal{O}$, in the following way.

Definition 7.1. A binary operator $\mathcal{O}$ is said to *satisfy the exchange principle* if

$$(\forall\,(a,b,c)\in[0,1]^3)\Big(\mathcal{O}\big(a,\mathcal{O}(b,c)\big)=\mathcal{O}\big(b,\mathcal{O}(a,c)\big)\Big)\,.$$

Clearly, any commutative and associative binary operator $\mathcal{O}$ satisfies the exchange principle. Of course, this is not helpful when dealing with implicators, since they are not commutative.

Theorem 7.1. [11] *Consider $\mathcal{O}\in\mathscr{F}$. Then $R[\mathcal{O}]$ satisfies the exchange principle if and only if $\mathcal{O}$ satisfies the exchange principle.*

The following theorem again follows easily.

Theorem 7.2. *Consider a binary operator $\mathcal{O}$ and a strong negator $\mathcal{N}$. Then $S_{\mathcal{N}}[\mathcal{O}]$ satisfies the exchange principle if and only if $\mathcal{O}$ satisfies the exchange principle.*

It is easy to see that Corollary 7.1 immediately follows from Theorems 7.1 and 7.2, Proposition 3.1 and Theorem 3.2.

Corollary 7.1. *Consider $\mathcal{O}\in\mathscr{F}$ and a strong negator $\mathcal{N}$. Then the following statements are equivalent:*

(i) *$\mathcal{O}$ satisfies the exchange principle;*

(ii) *$R[\mathcal{O}]$ satisfies the exchange principle;*

(iii) *$S_{\mathcal{N}}[\mathcal{O}]=R\big[C_{\mathcal{N}}[\mathcal{O}]\big]$ satisfies the exchange principle;*

(iv) *$C_{\mathcal{N}}[\mathcal{O}]$ satisfies the exchange principle.*

Classically, the exchange principle can be reformulated as follows:

$$\big(p\Rightarrow(q\Rightarrow r)\big)\iff\big((p\wedge q)\Rightarrow r\big)\,.$$

We investigate the role of this tautology in the multivalued case.

Proposition 7.1. *Consider $\mathcal{O}\in\mathscr{F}$. Then $R[\mathcal{O}]$ satisfies*

$$(\forall\,(a,b,c)\in[0,1]^3)\Big(R[\mathcal{O}]\big(a,R[\mathcal{O}](b,c)\big)=R[\mathcal{O}]\big(\mathcal{O}(a,b),c\big)\Big)$$

if and only if $\mathcal{O}$ satisfies $(\forall\,(a,b,c)\in[0,1]^3)\Big(\mathcal{O}\big(b,\mathcal{O}(a,c)\big)=\mathcal{O}\big(\mathcal{O}(a,b),c\big)\Big)$.

Proof. Let $(a,b,c)\in[0,1]^3$. Since $\mathcal{O}\in\mathscr{F}$, it holds that

$$\big\{x\colon\mathcal{O}\big(b,\mathcal{O}(a,x)\big)\le c\big\}=\big\{x\colon\mathcal{O}(a,x)\le R[\mathcal{O}](b,c)\big\}=\big[0,R[\mathcal{O}]\big(a,R[\mathcal{O}](b,c)\big)\big]$$

and

$$\big\{x\colon\mathcal{O}\big(\mathcal{O}(a,b),x\big)\le c\big\}=\big[0,R[\mathcal{O}]\big(\mathcal{O}(a,b),c\big)\big]\,.$$

We will, for instance, give the proof from left to right. Suppose that

$$(\forall\,(a,b,c)\in[0,1]^3)\Big(R[\mathcal{O}]\big(a,R[\mathcal{O}](b,c)\big)=R[\mathcal{O}]\big(\mathcal{O}(a,b),c\big)\Big)\,,$$

then it also holds that

$$(\forall\,(a,b,c,d)\in[0,1]^4)\Big(\mathcal{O}\big(b,\mathcal{O}(a,d)\big)\le c \iff \mathcal{O}\big(\mathcal{O}(a,b),d\big)\le c\Big)\,.$$

Applying the above equivalence, we easily obtain that

$$(\forall\,(a,b,c)\in[0,1]^3)\Big(\mathcal{O}\big(b,\mathcal{O}(a,c)\big)=\mathcal{O}\big(\mathcal{O}(a,b),c\big)\Big)\,.$$

□

A similar result holds for the operator $S_{\mathcal{N}}[\mathcal{O}]$. It applies in particular to $R\big[C_{\mathcal{N}}[\mathcal{O}]\big]$, for any $\mathcal{O}\in\mathscr{F}$ and strong negator $\mathcal{N}$.

PROPOSITION 7.2. *Consider a binary operator $\mathcal{O}$ and a strong negator $\mathcal{N}$. Then $S_{\mathcal{N}}[\mathcal{O}]$ satisfies*

$$(\forall\,(a,b,c)\in[0,1]^3)\Big(S_{\mathcal{N}}[\mathcal{O}]\big(a,S_{\mathcal{N}}[\mathcal{O}](b,c)\big)=S_{\mathcal{N}}[\mathcal{O}]\big(\mathcal{O}(a,b),c\big)\Big)$$

if and only if $\mathcal{O}$ is associative, i.e., $(\forall\,(a,b,c)\in[0,1]^3)\Big(\mathcal{O}\big(a,\mathcal{O}(b,c)\big)=\mathcal{O}\big(\mathcal{O}(a,b),c\big)\Big)$.

Notice the slight difference between Propositions 7.1 and 7.2: in Proposition 7.1 the characterizing property is $\mathcal{O}(b,\mathcal{O}(a,c))=\mathcal{O}\big(\mathcal{O}(a,b),c\big)$, while in Proposition 7.2 it is $\mathcal{O}\big(a,\mathcal{O}(b,c)\big)=\big(\mathcal{O}(a,b),c\big)$.

THEOREM 7.3. *Consider $\mathcal{O}\in\mathscr{F}$. If $\mathcal{O}$ is commutative, then the following statements are equivalent:*

(i) *$\mathcal{O}$ is associative;*

(ii) *$R[\mathcal{O}]$ satisfies the exchange principle;*

(iii) *$R[\mathcal{O}]$ satisfies* $(\forall\,(a,b,c)\in[0,1]^3)\Big(R[\mathcal{O}]\big(a,R[\mathcal{O}](b,c)\big)=R[\mathcal{O}]\big(\mathcal{O}(a,b),c\big)\Big)$.

Proof. We prove the chain (iii)$\Rightarrow$(ii)$\Rightarrow$(i)$\Rightarrow$(iii).

(iii)$\Rightarrow$(ii): Since $\mathcal{O}$ is commutative, it follows that

$$\begin{aligned}R[\mathcal{O}]\big(a,R[\mathcal{O}](b,c)\big)&=R[\mathcal{O}]\big(\mathcal{O}(a,b),c\big)=R[\mathcal{O}]\big(\mathcal{O}(b,a),c\big)\\&=R[\mathcal{O}]\big(b,R[\mathcal{O}](a,c)\big)\,.\end{aligned}$$

(ii)$\Rightarrow$(i): From Theorem 7.1, we have $\mathcal{O}\big(a,\mathcal{O}(b,c)\big)=\mathcal{O}\big(b,\mathcal{O}(a,c)\big)$. Applying the commutativity of $\mathcal{O}$, we obtain $\mathcal{O}\big(\mathcal{O}(b,c),a\big)=\mathcal{O}\big(b,\mathcal{O}(c,a)\big)$, which shows that $\mathcal{O}$ is also associative.

(i)$\Rightarrow$(iii): For $\mathcal{O}$ is commutative and associative, it also holds that $\mathcal{O}\big(b,\mathcal{O}(a,c)\big)=\mathcal{O}\big(\mathcal{O}(a,b),c\big)$. Then the result follows by applying Proposition 7.1. □

A similar result holds for the operator $S_{\mathcal{N}}[\mathcal{O}]$. It applies in particular to $R\big[C_{\mathcal{N}}[\mathcal{O}]\big]$, for any $\mathcal{O}\in\mathscr{F}$ and strong negator $\mathcal{N}$.

Theorem 7.4. *Consider a binary operator $\mathcal{O}$ and a strong negator $\mathcal{N}$. If $\mathcal{O}$ is commutative, then the following statements are equivalent:*

(i) *$\mathcal{O}$ is associative;*

(ii) *$S_{\mathcal{N}}[\mathcal{O}]$ satisfies the exchange principle;*

(iii) *$S_{\mathcal{N}}[\mathcal{O}]$ satisfies $(\forall\,(a,b,c)\in[0,1]^3)\big(S_{\mathcal{N}}[\mathcal{O}]\big(a,S_{\mathcal{N}}[\mathcal{O}](b,c)\big)$ $= S_{\mathcal{N}}[\mathcal{O}]\,\big(\mathcal{O}(a,b),c\big)\big)$.*

Example 7.1. [5] Consider the conjunctor $Y \in \mathscr{F}$ defined by

$$Y(a,b) = \begin{cases} b^{\frac{1}{a}} & \text{if } a>0 \text{ and } b>0\,,\\ 0 & \text{elsewhere}\,. \end{cases}$$

The operator Y is used in [10] to generate Y a g e r 's implicator [15] given by

$$R[Y](a,b) = \begin{cases} b^{a} & \text{if } a>0 \text{ and } b<1\,,\\ 1 & \text{elsewhere}\,. \end{cases}$$

Consider a strong negator $\mathcal{N}$, then the border implicator $S_{\mathcal{N}}[Y] = R\big[C_{\mathcal{N}}[Y]\big]$ is given by:

$$S_{\mathcal{N}}[Y](a,b) = \begin{cases} \mathcal{N}\big(\mathcal{N}(b)^{\frac{1}{a}}\big) & \text{if } a>0 \text{ and } b<1\,,\\ 1 & \text{elsewhere}\,. \end{cases}$$

One easily verifies that Y satisfies the exchange principle, and therefore also $R[Y]$ and $S_{\mathcal{N}}[Y]$. Note that Y is neither commutative, nor associative.

The property $Y\big(a,Y(b,c)\big) = Y\big(Y(b,a),c\big)$, however, does not hold in general. Indeed, consider $a>0$, $b>0$ and $c>0$, then $Y\big(a,Y(b,c)\big) = c^{\frac{1}{ab}}$ and $Y\big(Y(b,a),c\big) = c^{a^{\frac{1}{b}}}$. Consequently, the property $R[Y]\big(a,R[Y](b,c)\big) = R[Y]\big(Y(a,b),c\big)$ does not hold in general. Similarly, the property $S_{\mathcal{N}}[Y]$ $\big(a,S_{\mathcal{N}}[Y](b,c)\big) = S_{\mathcal{N}}[Y]\big(Y(a,b),c\big)$ does not hold in general.

If the binary operator $\mathcal{O}\in\mathscr{F}$ is such that $(\forall\,a\in[0,1])\ \big(\mathcal{O}(a,1)=a\big)$, then the above results lead to the following interesting characterizations of t-norms.

Theorem 7.5. *Consider $\mathcal{O}\in\mathscr{F}$ such that $(\forall\,a\in[0,1])\ \big(\mathcal{O}(a,1)=a\big)$. Then $R[\mathcal{O}]$ satisfies the exchange principle and*

$$(\forall\,(a,b,c)\in[0,1]^3)\ \Big(R[\mathcal{O}]\big(a,R[\mathcal{O}](b,c)\big) = R[\mathcal{O}]\big(\mathcal{O}(a,b),c\big)\Big)$$

if and only if $\mathcal{O}$ is a t-norm.

P r o o f . The proof from right to left follows immediately from Theorem 7.3. We will now give the proof from left to right. Since $R[\mathcal{O}]$ satisfies the exchange principle and

$$(\forall\,(a,b,c)\in[0,1]^3)\ \Big(R[\mathcal{O}]\big(a,R[\mathcal{O}](b,c)\big) = R[\mathcal{O}]\big(\mathcal{O}(a,b),c\big)\Big)\,,$$

it follows with Theorem 7.1 and Proposition 7.1 that

$$(\forall\,(a,b,c)\in[0,1]^3)\,\Big(\mathcal{O}\big(a,\mathcal{O}(b,c)\big)=\mathcal{O}\big(b,\mathcal{O}(a,c)\big)=\mathcal{O}\big(\mathcal{O}(a,b),c\big)\Big)\,.$$

Hence, $\mathcal{O}$ is associative. We again apply Proposition 7.1 to

$$(\forall\,(a,b,c)\in[0,1]^3)\,\Big(R[\mathcal{O}]\big(b,R[\mathcal{O}](a,c)\big)=R[\mathcal{O}]\big(\mathcal{O}(b,a),c\big)\Big)$$

and get

$$(\forall\,(a,b,c)\in[0,1]^3)\,\Big(\mathcal{O}\big(b,\mathcal{O}(a,c)\big)=\mathcal{O}\big(\mathcal{O}(b,a),c\big)\Big)\,.$$

Thus,

$$(\forall\,(a,b,c)\in[0,1]^3)\,\Big(\mathcal{O}\big(\mathcal{O}(a,b),c\big)=\mathcal{O}\big(\mathcal{O}(b,a),c\big)\Big)\,.$$

Taking $c=1$, it follows that $\mathcal{O}$ is commutative. Since $\mathcal{O}\in\mathscr{F}$ is such that $\big(\forall\,a\in[0,1]\big)$ $\big(\mathcal{O}(a,1)=a\big)$, it then follows that $\mathcal{O}$ is a t-norm. ☐

Similarly, the following theorem holds. It applies in particular to $R\big[C_{\mathcal{N}}[\mathcal{O}]\big]$, for any $\mathcal{O}\in\mathscr{F}$ and strong negator $\mathcal{N}$.

Theorem 7.6. *Consider a binary operator $\mathcal{O}$ such that $\big(\forall\,a\in[0,1]\big)$ $\big(\mathcal{O}(a,1)=a\big)$ and a strong negator $\mathcal{N}$. Then $S_{\mathcal{N}}[\mathcal{O}]$ satisfies the exchange principle and*

$$(\forall\,(a,b,c)\in[0,1]^3)\,\big(S_{\mathcal{N}}[\mathcal{O}]\big(a,S_{\mathcal{N}}[\mathcal{O}](b,c)\big)=S_{\mathcal{N}}[\mathcal{O}]\big(\mathcal{O}(a,b),c\big)\big)$$

if and only if $\mathcal{O}$ is a t-norm.

It is stated in [14] for $S_{\mathcal{N}}[\mathcal{O}]$, with $\mathcal{O}$ a t-norm, and in [10] for $R[\mathcal{O}]$, with $\mathcal{O}\in\mathscr{F}$ a *weak t-norm*, respectively, that commutativity and associativity of $\mathcal{O}$ are sufficient for the fulfillment of the exchange principle and

$$(\forall\,(a,b,c)\in[0,1]^3)\,\Big(S_{\mathcal{N}}[\mathcal{O}]\big(a,S_{\mathcal{N}}[\mathcal{O}](b,c)\big)=S_{\mathcal{N}}[\mathcal{O}]\big(\mathcal{O}(a,b),c\big)\Big)$$

and

$$(\forall\,(a,b,c)\in[0,1]^3)\,\Big(R[\mathcal{O}]\big(a,R[\mathcal{O}](b,c)\big)=R[\mathcal{O}]\big(\mathcal{O}(a,b),c\big)\Big)\,,$$

respectively. With our analysis, we have shown that commutativity and associativity of $\mathcal{O}\in\mathscr{F}$ such that $\big(\forall\,a\in[0,1]\big)$ $\big(\mathcal{O}(a,1)=a\big)$ are also necessary for the fulfillment of the exchange principle and the above mentioned property.

8. Summary

We have considered a broad class of binary operators on the unit interval. With each member of this class, we have associated three binary operators. We have identified necessary and sufficient conditions to be imposed on a given operator to ensure a certain property of one of its associated operators. As a result, it is shown that a binary operator $\mathcal{O} \in \mathscr{F}$ does not need to be a conjunctor for its associated residual operator $R[\mathcal{O}]$ to be an implicator. However, it is necessary and sufficient for a binary operator $\mathcal{O} \in \mathscr{F}$ to be a conjunctor for its associated residual operator $R\big[C_{\mathcal{N}}[\mathcal{O}]\big]$ to be an implicator, for any strict negator $\mathcal{N}$.

Furthermore, properties such as the neutrality principle, contrapositivity and the exchange principle have been investigated. For example, it is shown that a binary operator $\mathcal{O} \in \mathscr{F}$ does not need to be commutative and associative for its associated residual operators $R[\mathcal{O}]$ and $R\big[C_{\mathcal{N}}[\mathcal{O}]\big]$ to satisfy, respectively,

$$(\forall\,(a,b,c) \in [0,1]^3)\ \Big(R[\mathcal{O}]\big(a, R[\mathcal{O}](b,c)\big) = R[\mathcal{O}]\big(\mathcal{O}(a,b),c\big)\Big) \tag{1}$$

$$(\forall\,(a,b,c) \in [0,1]^3)\ \Big(R\big[C_{\mathcal{N}}[\mathcal{O}]\big]\big(a, R\big[C_{\mathcal{N}}[\mathcal{O}]\big](b,c)\big) = R\big[C_{\mathcal{N}}[\mathcal{O}]\big]\big(\mathcal{O}(a,b),c\big)\Big) \tag{2}$$

or the exchange principle. However, if $R[\mathcal{O}]$ is to satisfy both the exchange principle and (1) or if $R\big[C_{\mathcal{N}}[\mathcal{O}]\big]$ is to satisfy both the exchange principle and (2), for any binary operator $\mathcal{O} \in \mathscr{F}$ such that $\big(\forall\, a \in [0,1]\big)\ \big(\mathcal{O}(a,1) = a\big)$, then this is only possible if $\mathcal{O}$ is a t-norm. This complements known characterizations of t-norms such as the following [1]: $S_{\mathcal{N}}[\mathcal{O}]$ is a contrapositive border implicator that satisfies the exchange principle if and only if $\mathcal{O}$ is a t-norm.

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