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STRICT TRIANGULAR NORMS AND PERFECT MV-ALGEBRAS

Lawrence Peter Belluce — Antonio Di Nola — Salvatore Sessa

ABSTRACT. Abelian ℓ-groups and perfect MV-algebras are categorically equivalent via the functor $\triangle$ of Di Nola and Lettieri [10]. Thus the perfect MV-algebra $\triangle(\mathbb{R})$ corresponds to the classical ℓ-group $\mathbb{R}$ of the reals. Given a strict triangular conorm $s\colon [0,1]^2 \to [0,1]$, we build a perfect MV-algebra $A = (A, \oplus, \cdot, *, 0_A, 1_A)$ such that $(\operatorname{Rad} A, \oplus, 0_A)$ and $([0,1), s, 0)$ are isomorphic Abelian ordered monoids. Further, we prove that A is isomorphic to $\triangle(\mathbb{R})$ and a representation theorem for MV-algebras of functions of $\triangle(\mathbb{R})^X$ type, X being any nonempty set, is also given.

1. Preliminaries

If Boolean algebras are the algebraic counterpart of set theory, it is a deep open question to establish in general which algebras are the algebraic counterpart of fuzzy set theory. A first attempt, in this direction, was due to Belluce [1] who showed that the fuzzy sets of Zadeh [18] and the MV-algebras of Chang [9] are intimately related. Since the triangular norms and the related dual concept of conorms are used heavily in fuzzy set theory (see, e.g., [8], [11]), it is natural to study the relationship between triangular norms and MV-algebras. This investigation began only some years ago [2], [6], [8], [12], [13]. We continue the investigation here showing some surprising connections between the class of strict (and hence Archimedean) triangular conorms and the perfect MV-algebra $\triangle(\mathbb{R})$, corresponding via the categorical equivalence $\triangle$ of Di Nola and Lettieri [10], to the classical Abelian lattice-ordered group $\mathbb{R}$ of the reals.

In other words, we show that this MV-algebra $\triangle(\mathbb{R})$ captures the action of a triangular strict conorm on the unit interval $[0,1]$.

This paper follows a companion paper [6], where we showed that the MV-algebra $\mathscr{L} = ([0,1], \oplus, \cdot, *, 0, 1)$, obtained using the usual Łukasiewicz operations $x \oplus y = \min\{1, x+y\}$, $x \cdot y = \max\{0, x+y-1\}$ and $x^* = 1-x$ for any x,

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$y \in [0,1]$, seems to capture the action of a non-strict Archimedean triangular conorm.

For all the notions involved here we refer to the book of Schweizer and Sklar [17] for triangular norms and to [9], [16] for MV-algebras. Below, some definitions are recalled. A triangular conorm $s\colon [0,1]^2 \to [0,1]$ is a real function which is increasing in each variable and is commutative, associative and having 0 as unit. s is Archimedean if it is continuous and $s(x,x) > x$ for any $x \in (0,1)$, s is strict if it is strictly increasing in $(0,1)^2$. By an important representation theorem [14], we know that if s is strict, then there exists a strictly increasing and continuous function $g\colon [0,1] \to [0,\infty)$ with $g(0) = 0$, $g(x) \to +\infty$ if $x \to 1$ and such that $s(x,y) = g^{-1}\big(g(x)+g(y)\big)$ for all x, $y \in [0,1]$. g is usually called the additive generator of s.

In the sequel, since misunderstandings can arise, we sometimes emphasize the zeros and the units of the algebraic structure involved by identifying $0 = 0_A$, $1 = 1_G$ and so on. The context will make clear the notation.

Let $A = (A, \oplus, \cdot, *, 0, 1)$ be an MV-algebra, i.e., an algebra such that

(MVi) $(A, \oplus, 0_A)$ is an Abelian monoid,

(MVii) $a \oplus 1_A = 1_A$,

(MViii) $a^{**} = a$,

(MViv) $0_A^* = 1_A$,

(MVv) $a \cdot b = (a^* \oplus b^*)^*$,

(MVvi) $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$,

for any a, $b \in A$. It is well known that the structure $L(A) = (A, \vee, \wedge, \leq, 0_A, 1_A)$ is a bounded distributive lattice with the operations $a \vee b = (a \cdot b^*) \oplus b$, $a \wedge b = (a \oplus b^*) \cdot b$, $a \leq b$ iff $a \wedge b = a$. As usual, we write ab instead of $a \cdot b$ and we put $na = a \oplus \ldots \oplus a$ (n times) for any integer $n \geq 1$. If $L(A)$ is linearly ordered, then A is said to be linearly ordered.

The variety of the MV-algebras is generated by the MV-algebra $\mathscr{L}$.

Let $I \subseteq A$ be such that $0_A \in I$, I is closed under $\oplus$ and $a \in I$, $b \in A$ implies $ab \in I$. Then I is said to be an ideal of A and P is a prime ideal of A if P is a prime ideal of the underlying lattice $L(A)$. A proper ideal M of A is maximal if $M \subseteq I \subseteq A$ implies $M = I$ or $I = A$.

If $\operatorname{Rad} A$ denotes the intersection of all maximal ideals of A, then A is said to be perfect [5] if $A = \operatorname{Rad} A \cup (\operatorname{Rad} A)^*$, where $(\operatorname{Rad} A)^* = \{x \in A\colon x^* \in \operatorname{Rad} A\}$, $\operatorname{Rad} A$ being the unique maximal ideal of A.

2. Main theorem

Let $s\colon [0,1]^2 \longrightarrow [0,1]$ be a strict triangular conorm with additive generator g.

Let $A = \{(0,x)\colon x \in [0,1)\} \cup \{(1,-x)\colon x \in [0,1)\}$. Order A lexicographically, i.e., $(x,y) \le (x',y')$ iff $x < x'$ or $x = x'$ and $y \le y'$.

Define the following operations on A for any x, y, z, $t \in [0,1)$:

$$(0,x)^* = (1,-x)\,,\ (1,-x)^* = (0,x)\,,$$
$$(0,x) \oplus (0,y) = \big(0, s(x,y)\big)\,,$$
$$(1,-x) \oplus (1,-y) = (1,0) = (1,-y) \oplus (1,-x)\,,$$
$$(0,x) \oplus (1,-y) = (1,-y) \oplus (0,x) = \begin{cases} \big(1,-g^{-1}(gy-gx)\big) & \text{if } x \le y\,, \\ (1,0) & \text{if } y \le x\,, \end{cases}$$
$$(x,y)\cdot(z,t) = \big[(x,y)^* \oplus (z,t)^*\big]^*\,,$$
$$0_A = (0,0)\,,\ 1_A = (1,0)\,.$$

We can now prove the following:

Theorem 1. *With the above operations, A is a linearly ordered perfect MV-algebra.*

P r o o f . (MVi) By construction, $\oplus$ is commutative. Now $(1,-x) \oplus (0,0) = \big(1,-g^{-1}(gx-g0)\big) = (1,-g^{-1}gx) = (1,-x)$ and $(0,x) \oplus (0,0) = \big(0,s(x,0)\big) = (0,x)$ for any $x \in [0,1)$. Thus $(A,\oplus,0_A)$ has $(0,0) = 0_A$ as a neutral element. It remains to show that $\oplus$ is associative.

Indeed, we consider four cases:

(i) $(0,x) \oplus \big[(0,y) \oplus (0,z)\big] = (0,x) \oplus \big(0,s(y,z)\big) = \big(0,s\big(x,s(y,z)\big)\big) = \big(0,s\big(s(x,y),z\big)\big) = \big(0,s(x,y)\big) \oplus (0,z) = \big[(0,x) \oplus (0,y)\big] \oplus (0,z)$.

(ii) $(1,-x)\oplus\big[(1,-y)\oplus(1,-z)\big] = (1,-x)\oplus(1,0) = (1,0) = (1,0)\oplus(1,-z) = \big[(1,-x) \oplus (1,-y)\big] \oplus (1,-z)$.

(iii) $(0,x) \oplus \big[(1,-y) \oplus (1,-z)\big] = (0,x) \oplus (1,0) = (1,0) = (1,\alpha) \oplus (1,-z) = \big[(0,x) \oplus (1,-y)\big] \oplus (1,-z)$, where $\alpha \in \{0,-g^{-1}(gy-gx)\}$.

(iv) Here we consider $\ell = (0,x)\oplus\big[(0,y)\oplus(1,-z)\big]$ and $\ell' = \big[(0,x)\oplus(0,y)\big] \oplus (1,-z)$.

We distinguish two subcases.

The subcase $y \le z$. Then $(0,y) \oplus (1,-z) = \big(1,-g^{-1}(gz-gy)\big)$. Hence

$$\begin{aligned} \ell &= (0,x) \oplus \big(1,-g^{-1}(gz-gy)\big) \\ &= \begin{cases} \big(1,-g^{-1}(gz-gy-gx)\big) & \text{if } x \le g^{-1}(gz-gy)\,, \\ (1,0) & \text{otherwise.} \end{cases} \end{aligned}$$

For ℓ', we have $\ell' = \big(0, s(x,y)\big) \oplus (1,-z)$. Since $g\big(s(x,y)\big) = g(x) + g(y)$ because s is strict, we deduce that

$$\ell' = \begin{cases} \big(1, -g^{-1}(gz - gy - gx)\big) & \text{if } s(x,y) \le z\,, \\ (1,0) & \text{otherwise.} \end{cases}$$

Now $s(x,y) = g^{-1}(gx+gy) \le z$ iff $gx + gy \le gz$ iff $gx \le gz - gy$ iff $x \le g^{-1}(gz - gy)$, therefore $\ell = \ell'$.

The subcase $z \le y$. Then $(0,y) \oplus (1,-z) = (1,0)$. Thus $\ell = (0,x) \oplus (1,0) = (1,0)$. Now $z \le y$ implies $z \le s(x,y)$, thus $\ell' = \big[(0,x) \oplus (0,y)\big] \oplus (1,-z) = \big(0, s(x,y)\big) \oplus (1,-z) = (1,0)$, so again $\ell = \ell'$.

This proves that $(A, \oplus, 0_A)$ is an Abelian monoid.

(MVii) $(x,y) \oplus (1,0) = (1,0)$ since $0 \le y$.

(MViii) $(0,x)^{**} = (1,-x)^* = (0,x)$, $(1,-x)^{**} = (0,x)^* = (1,-x)$.

(MViv) $0_A^* = (0,0)^* = (1,0) = 1_A$.

(MVvi) is true by definition.

(MVvi) For $\alpha, \beta \in A$, $\ell = \alpha \oplus (\alpha \oplus \beta^*)^* = \beta \oplus (\beta \oplus \alpha^*)^* = \ell'$.

Then we distinguish three cases:

(i) $\alpha = (1,-x)$, $\beta = (1,-y)$. Then

$$\begin{aligned} \ell &= (1,-x) \oplus \big[(1,-x) \oplus (0,y)\big]^* \\ &= \begin{cases} (1,-x) \oplus \big(1, -g^{-1}(gx - gy)\big)^* = (1,-x) \oplus \big(0, g^{-1}(gx - gy)\big) & \text{if } y \le x\,, \\ (1,-x) \oplus (1,0)^* = (1,-x) \oplus (0,0) = (1,-x) & \text{if } x \le y\,. \end{cases} \end{aligned}$$

Now $y \le x$ implies $gy \le gx$, so $gx - gy \le gx$, hence $g^{-1}(gx - gy) \le x$, so $\ell = \big(1, -g^{-1}\big(gx - (gx - gy)\big)\big)$; thus $\ell = \big(1, -g^{-1}(gy)\big) = (1,-y) = \big(1, -\max(-x,-y)\big)$. Further, $x \le y$ implies $\ell = (1,-x) = \big(1, -\max(-x,-y)\big)$. On the other hand,

$$\begin{aligned} \ell' &= (1,-y) \oplus \big[(1,-y) \oplus (1,-x)^*\big]^* = (1,-y) \oplus \big[(1,-y) \oplus (0,x)\big] \\ &= \begin{cases} (1,-y) \oplus \big(1, -g^{-1}(gy - gx)\big)^* = (1,-y) \oplus \big(0, g^{-1}(gy - gx)\big) & \text{if } x \le y\,, \\ (1,-y) \oplus (1,0)^* = (1,-y) \oplus (0,0) = (1,-y) & \text{if } y \le x\,. \end{cases} \end{aligned}$$

Now $x \le y$ implies $gx \le gy$, so $gy - gx \le gy$ and we have $g^{-1}(gy - gx) \le y$. Thus $(1,-y) \oplus \big(0, g^{-1}(gy - gx)\big) = (1,-x)$.

Hence

$$\ell' = \big(1, -\max\{-x,-y\}\big) = \begin{cases} (1,-y) & \text{if } y \le x\,, \\ (1,-x) & \text{if } x \le y\,. \end{cases}$$

Thus $\ell = \ell'$.

(ii) $\alpha = (0,x)$, $\beta = (1,-y)$. Then

$$\begin{aligned}\ell &= (0,x) \oplus \big[(0,x) \oplus (1,-y)^*\big]^* = (0,x) \oplus \big[(0,x) + (0,y)\big]^* \\ &= (0,x) \oplus \big(0, s(x,y)\big)^* = (0,x) \oplus (1, -s(x,y)) \\ &= \big(1, -g^{-1}\big(g\big(s(x,y)\big) - gx\big)\big)\end{aligned}$$

since $x = s(x,0) \le s(x,y)$.

Therefore $\ell = (1,-y)$ because $g\big(s(x,y)\big) = gx + gy$. Hence

$$\begin{aligned}\ell' &= (1,-y) \oplus \big[(1,-y) \oplus (0,x)^*\big]^* = (1,-y) \oplus \big[(1,-y) \oplus (1,-x)\big]^* \\ &= (1,-y) \oplus (1,0)^* = (1,-y) \oplus (0,0) = (1,-y) = \ell\,.\end{aligned}$$

(iii) $\alpha = (0,x)$, $\beta = (0,y)$. Then

$$\begin{aligned}\ell &= (0,x) \oplus \big[(0,x) + (0,y)^*\big]^* = (0,x) \oplus \big[(0,x) \oplus (1,-y)\big] \\ &= \begin{cases} (0,x) \oplus \big(1, -g^{-1}(gy - gx)\big)^* = (0,x) \oplus \big(0, g^{-1}(gy-gx)\big) \\ (0,x) \oplus (1,0)^* = (0,x) \oplus (0,0) \end{cases} \\ &= \begin{cases} \big(0, s\big(x, g^{-1}(gy - gx)\big)\big) = (0,y) & \text{if } x \le y\,, \\ (0,x) & \text{if } y \le x\,. \end{cases}\end{aligned}$$

Thus $\ell = \big(0, \max\{x,y\}\big)$. By symmetry, $\ell' = \big(0, \max\{y,x\}\big) = \ell$.

This proves that A is an MV-algebra. Obviously A is linearly ordered and perfect since $\operatorname{Rad} A = \big\{(0,x) \colon x \in [0,1)\big\}$.

This concludes the proof. □

COROLLARY 2. *Let s be a strict triangular conorm. Then there exists a linearly ordered perfect MV-algebra $A = (A, \oplus, \cdot, *, 0_A, 1_A)$ such that $(\operatorname{Rad} A, \oplus, 0_A)$ and $\big([0,1), s, 0\big)$ are order-isomorphic. Indeed, $\big(\operatorname{Rad} A \cup \{1_A\}, \oplus, 0_A\big)$ and $\big([0,1], s, 0\big)$ are order-isomorphic Abelian monoids.*

Proof. The MV-algebra A constructed above satisfies the thesis. Clearly the map $(0,x) \to x$ is an order-isomorphism from $\operatorname{Rad} A$ onto $[0,1)$. If we extend the map to $\operatorname{Rad} A \cup \{1_A\}$ by sending 1_A into 1, we have $1_A \oplus 1_A = 1_A \to 1 = s(1,1)$; $(0_A, x) \oplus 1_A = (0_A, x) \oplus (1,0) = (1,0) = 1_A \to 1 = s(x,1)$. Hence $(\operatorname{Rad} A \cup \{1_A\}, \oplus, 0_A)$ is order-isomorphic to $\big([0,1], s, 0\big)$. □

Thus we see that the perfect MV-algebra A captures the action of the triangular conorm s on $[0,1]$.

3. Abelian ℓ-groups and perfect MV-algebras

For definitions and terms involving the theory of lattice-ordered groups (in the sequel, ℓ-groups), we refer to [7]. Here we use the same terminology as [3].

As in [16], we recall that if $(G,+)$ is an Abelian ℓ-group with strong unit u, then $A = \Gamma(G,u) = \{a \in G\colon 0_G \le a \le u\}$ is an MV-algebra with the operations $a \oplus b = u \wedge (a+b)$, $a^* = u - a$, $a \cdot b = (a^* \oplus b^*)^*$, $a, b \in A$. Here Γ is a categorical equivalence [16].

Now let $\mathbb{Z}$ be the additive linearly ordered ℓ-group of the integers and let $\mathbb{Z} \times G$ be the lexicographic product of $\mathbb{Z}$ by any Abelian ℓ-group G. By choosing the strong unit $(1, 0_G)$, the MV-algebra $\triangle(G) = \Gamma\big(\mathbb{Z} \times G, (1, 0_G)\big)$ is a perfect MV-algebra, whose elements have form $(0,a)$ or $(1,-a)$ with $a \in G^+ = \{a \in G\colon a \ge 0_G\}$, G^+ being the positive cone of G.

Let A be a perfect MV-algebra and Θ be the congruence relation defined on $\operatorname{Rad} A \times \operatorname{Rad} A$ by $(x,y)\Theta(x',y')$ iff $x \oplus y' = x' \oplus y$ for any x, y, x', $y' \in \operatorname{Rad} A$.

Thus, according to [10], the quotient $D(A) = \operatorname{Rad} A \times \operatorname{Rad} A/\Theta$ is an Abelian ℓ-group under the operation $[x,y] + [x',y'] = [x+x', y+y']$ and $[x,y] \le [x',y']$ iff $x + y' \le x' + y$ with respect to the MV-order of A. Here $[x,y]$ denotes the equivalence class of (x,y) under Θ.

Remark. Perfect MV-algebras and Abelian ℓ-groups are categorically equivalent ([10], Thm. 3.5), hence $A \simeq \triangle\big(D(A)\big)$ and $G \simeq D\big(\triangle(G)\big)$.

Let $\mathbb{R}$ be the additive linearly ordered ℓ-group of the reals and let

$$\triangle(\mathbb{R}) = \big\{(0,x)\colon x \in [0,+\infty)\big\} \cup \big\{(1,-x)\colon x \in [0,+\infty]\big\}$$

be the corresponding linearly ordered MV-algebra. Note that the MV-addition in $\triangle(\mathbb{R})$ is defined as $(0,x) \oplus (0,y) = (0, x+y)$, $(1,-x) \oplus (1,-y) = (1,0)$, $(0,x) \oplus (1,-y) = (1, x-y)$ (resp. (1,0)) if $x \le y$ (resp. $x \ge y$). The remaining operations are identical to those of Section 2.

Then we have that:

Theorem 4. *For any strict triangular conorm, the MV-algebra A defined in Section 2 is isomorphic to $\triangle(\mathbb{R})$.*

Proof. Let g be the additive generator of s and for any $x \in [0,1)$, define the map $(0,x) \to (0, gx)$, $(1,x) \to (1, -gx)$. It is easily seen that this map is the required isomorphism. $\square$

Corollary 5. *If s_1 and s_2 are two strict triangular conorms and A_1 and A_2 are the corresponding perfect MV-algebras as in Section 2, then A_1 is isomorphic to A_2.*

Following [3], we know that a perfect MV-algebra is locally Archimedean whenever x, $y \in \operatorname{Rad} A$ and $nx \le y$ for all n, then $x = 0$.

Remark. $\triangle(\mathbb{R})$ is locally Archimedean. Indeed, $\operatorname{Rad}\triangle(\mathbb{R}) = \big\{(0,x)\colon x \in [0,+\infty)\big\}$ and hence $n(0,x) \le (0,y)$ iff $(0,nx) \le (0,y)$ iff $nx \le y$ iff $x = 0$ for all n and x, $y \in [0, +\infty)$. Note that $\operatorname{Rad} \triangle(\mathbb{R})$ is the only non-zero prime ideal of $\triangle(\mathbb{R})$.

We can now prove:

Theorem 7. *Let A be an MV-algebra. The following are equivalent*

(i) *A is a linearly ordered, perfect and locally Archimedean;*

(ii) *A is subalgebra of $\triangle(\mathbb{R})$;*

(iii) *A is perfect and $\operatorname{Rad} A$ is the only non-zero prime ideal of A.*

Proof.

(i) implies (ii). By ([3], Lemma 6), $D(A)$ is a linearly ordered Archimedean (Abelian) ℓ-group. By Hölder's theorem ([7], 2.6.3), $D(A)$ is isomorphic to a ℓ-subgroup H of $\mathbb{R}$, that is $D(A) \simeq H \subseteq \mathbb{R}$. By Remark 3, it follows that $A \simeq \triangle\big(D(A)\big) \simeq \triangle(\mathbb{R})$.

(ii) implies (iii). If A is subalgebra of $\triangle(\mathbb{R})$, we have $\operatorname{Rad} A = A \cap \operatorname{Rad} \triangle(\mathbb{R})$. Thus A is perfect and let P be a non-zero prime ideal of A. Then $P = A \cap P'$ for some non-zero prime ideal P' of $\triangle(\mathbb{R})$, but $P' = \operatorname{Rad} \triangle(\mathbb{R})$ by Remark 6. Hence $P = \operatorname{Rad} A$.

(iii) implies (i). Let $\operatorname{Spec} A$ be the set of prime ideals of A. Since $\{0\} = \cap\{P\colon P \in \operatorname{Spec} A\}$, by hypothesis, we have that $\operatorname{Spec} A = \big\{\{0\}, \operatorname{Rad} A\big\}$. Thus $\{0\}$ is a prime ideal and consequently A is linearly ordered. Now let $\{0\} \neq H$ be a proper convex ℓ-subgroup of $D(A)$. Then $I = \triangle(H^+) = \big\{(0, x)\colon x \in H^+\big\}$ is (isomorphic to) a proper ideal of $A \simeq \triangle\big(D(A)\big)$ by ([3], Prop. 22a).

Since any proper ideal of A is the intersection of all prime ideals of A which contain it and $I \neq \operatorname{Rad} A$ necessarily (otherwise $H \simeq D(\operatorname{Rad} A) = \big\{[x, y]\colon x, y \in \operatorname{Rad} A\big\} = D(A)$), then $I = \{0\}$. Hence $H = \{0\}$, a contradiction.

Thus $D(A)$ has no proper convex (non-trivial) ℓ-subgroups and this means that $D(A) \subseteq \mathbb{R}$ by ([7], 2.6.4). Then $D(A)$ is an Archimedean ℓ-group, hence $A \simeq \triangle\big(D(A)\big)$ is locally Archimedean by ([3], Lemma 8). $\square$

Let $\emptyset \neq X$ be a set and let $\triangle(\mathbb{R})^X$ be the MV-algebra of all the functions from X into $\triangle(\mathbb{R})$ obtained under pointwise operations in the usual way. Then, if we define an ideal P of an MV-algebra A to be quasi-maximal if the quotient MV-algebra A/P is subalgebra of $\triangle(\mathbb{R})$, we have the following representation theorem:

Theorem 8. *Let X be the set of all the quasi-maximal ideals of an MV-algebra A and let $\cap\{P\colon P \in X\} = \{0\}$. Then A is subdirectly representable as a subalgebra of $\triangle(\mathbb{R})^X$.*

We recall [4] that a subset $S = \{a_i\}_{i \in I}$ of an MV-algebra A is a basis of A if $a_i \neq 0$ for any $i \in I$, $a_i \wedge a_j = 0$ for $i \neq j$, the set $\{x \in A\colon x \leq a_i\}$ is linearly ordered for any $i \in I$ and S is maximal (with respect to the inclusion).

The following Lemma is well known (see, e.g., [15]):

LEMMA 9. *Let A be an MV-algebra. Then the set $\{x \in A\colon x \leq a\}$ is linearly ordered iff $a^{\perp} = \{b \in A\colon a \wedge b = 0\}$ is a (minimal) prime ideal.*

THEOREM 10. *Let A be a perfect and locally Archimedean MV-algebra having a basis $S = \{a_i\}_{i \in I}$. Then A is subalgebra of $\triangle(\mathbb{R})^I$.*

Proof. By Lemma 7, $a_i^{\perp}$ is a prime ideal of A for any $i \in I$. Thus $A/a_i^{\perp}$ is linearly ordered. By ([3], Prop. 3), $A/a_i^{\perp}$ is locally Archimedean and of course it is perfect (cf. [10], Thm. 5.1). Then $A/a_i^{\perp}$ is subalgebra of $\triangle(\mathbb{R})$ by Theorem 5. Since S is maximal, we have that $\cap_{i \in I} a_i^{\perp} = \{0\}$ which implies that A is (subdirectly) a subalgebra of the direct product $B = \Pi_{i \in I} A/a_i^{\perp}$ and hence the result, since $A \subseteq B \subseteq \triangle(\mathbb{R})^I$. ☐

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L. P. Belluce
University of British Columbia
Department of Mathematics
Vancouver, B. C.
V6T 1Y4
CANADA

Istituto di Matematica
Facoltá di Architettura
Universitá di Napoli
Via Monteoliveto 3
I–80134 Napoli
ITALY
E-mail: dinola@unina.it
sessa@unina.it