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ON THE USE OF HIGH BREAKDOWN POINT ESTIMATORS IN THE IMAGE ANALYSIS

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ABSTRACT. For eliminating noise from an image, Meer *et al.* (1990, 1991) have proposed to use high breakdown point estimators and in particular to use the least median of squares (LMS) estimator for linear regression. But, the highest breakdown points are attained by other estimators as the Cauchy estimator and some least trimmed squares (LTS) estimators. Therefore, in this paper the behavior of the Cauchy estimator and the LTS estimator in image analysis is studied and compared with that of the LMS estimator and the least squares (LS) estimator. For that, test images with 0 %, 14 %, 30 %, 44 % and 49 % noise are used. It turns out that the LTS estimator can eliminate a high amount of noise and preserves discontinuities while the Cauchy estimator behaves similarly to the nonrobust LS estimator.

1. Introduction

An image is given by $T \times U$ pixel values $y(t,u) \in [0,M]$, $t = 1,\dots,T$, $u = 1,\dots,U$, where often $M = 255$ is the maximum pixel value. We assume that with some probabilities the pixel values contain some error (noise) so that we have

$$y(t,u) = y^*(t,u) + \varepsilon(t,u)\,, \qquad t = 1,\dots,T, \quad u = 1,\dots,U\,,$$

where $y^*(t,u)$ is the original signal. To eliminate the noise the pixel values $y(t,u)$ can be filtered by averaging over a $(2v+1) \times (2v+1)$ window $W(t,u)$ centered at (t,u), i.e.,

$$W(t,u) := \big\{(t+i, u+j)\colon \quad i,j = -v,\dots,v\big\}\,.$$

For example, if the pixels are filtered (smoothed) by a mean smoother, then the filtered values $\hat{y}(t,u)$ are given by

$$\hat{y}(t,u) = \sum_{i=-v}^{v} \sum_{j=-v}^{v} w_{ij}\, y(t+i, u+j)$$

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for $t = v+1, \ldots, T-v$, $u = v+1, \ldots, U-v$, where w_{ij} are some weights. A special mean smoother is the following

$$\hat{y}(t,u) = x(0,0)\,(X^\top X)^{-1}\,X^\top y_{t,u}\,,$$

where $x\colon \{-v,\ldots,v\} \times \{-v,\ldots,v\} \to \mathbb{R}^r$ is some regression function, $X = \big(x(-v,-v),\ldots,x(-v,v),\ldots,x(v,v)\big)^\top$ and $y_{t,u} = \big(y(t-v,u-v),\ldots,y(t-v,u+v),\ldots,y(t+v,u+v)\big)^\top$. I.e., $\hat{y}(t,u)$ is the Gauss-Markov estimator for estimating the surface at the centre (t,u) of the window $W(t,u)$ if the surface is given by the function $x(\tilde{t},\tilde{u})^\top\beta_{t,u}$ for $(\tilde{t},\tilde{u}) \in W(t,u)$. To assume such a linear model within a window makes in particular sense, since images are usually representations of real surfaces and surfaces are piecewise polynomial, even linear. Thus, for most of the windows $W(t,u)$ we can assume that the original signal has the form

$$y^*(t+i,u+j) = x(i,j)^\top\beta_{t,u} \qquad \text{for} \quad i,j = -v,\ldots,v\,.$$

If some deviations from the linearity assumptions appear within a window, then these deviations can be put in the error term.

Hence, we will work with filters of the following form:

$$\hat{y}(t,u) = x(0,0)^\top\,\hat{\beta}_{t,u}\,,$$

where $\hat{\beta}_{t,u}$ is some estimator for $\beta_{t,u}$ in the window $W(t,u)$. Traditionally the least squares (LS) estimator is used for $\hat{\beta}_{t,u}$. But, the LS estimator is known to be very sensitive against outliers and model deviations. Therefore, M e e r *et al.* (1990, 1991) proposed to use for $\hat{\beta}_{t,u}$ a high breakdown point estimator and in particular they proposed the least median of squares (LMS) estimator. M e e r *et al.* assumed that the LMS estimator has a breakdown point of approximately 50 %. But this holds only if the regressors $x(-v,-v),\ldots,x(-v,v),\ldots,x(v,v)$ are in general position (see R o u s s e e u w and L e r o y 1987). In Section 2 it is shown that the regressors are not in general position and that the highest breakdown point is attained by some other estimators as the Cauchy estimator and some least trimmed squares (LTS) estimators. In Section 3 the LTS, the Cauchy, the LMS and the LS estimator are applied to some test images with and without noise.

2. Breakdown points of LTS and Cauchy estimators

Let $y = X\beta + z$, where $y = (y_1,\ldots,y_N)^\top \in \mathbb{R}^N$, $X = (x_1,\ldots,x_N)^\top \in \mathbb{R}^{N\times r}$, $\beta \in \mathbb{R}^r$, and $z = (z_1,\ldots,z_N)^\top \in \mathbb{R}^N$. If $\hat{\varphi}\colon \mathbb{R}^N \times \mathbb{R}^{N\times r} \to \mathbb{R}^s$ is an estimator for $\varphi(\beta) = L\beta$, $L \in \mathbb{R}^{s\times r}$, then the breakdown point $\varepsilon^*(\hat{\varphi},y,X)$ of $\hat{\varphi}$ at (y,X) is defined by (see H e *et al.* 1990)

$$\varepsilon^*(\hat{\varphi},y,X) := \frac{1}{N}\min\big\{M \in \mathbb{N}\cup\{0\}\colon B_M(\hat{\varphi},y,X) = \infty\big\}\,,$$

where

$$B_M(\hat{\varphi}, y, X) := \max\{\,|\hat{\varphi}(y, X) - \hat{\varphi}(\overline{y}, X)|\colon \overline{y} \in \mathcal{Y}_M(y)\}\,,$$

and

$$\mathcal{Y}_M(y) := \{\overline{y} = (\overline{y}_1, \ldots, \overline{y}_N)^\top \in \mathbb{R}^N\colon \operatorname{card}\{n\colon y_n \neq \overline{y}_n\} \leq M\}\,.$$

To avoid the dependence on the observation vector y we also define

$$\varepsilon^*(\hat{\varphi}, X) := \min\{\varepsilon^*(\hat{\varphi}, y, X)\colon y \in \mathbb{R}^N\}\,.$$

Contrary to other definitions of the breakdown point (see Donoho and Huber 1983, Rousseeuw and Leroy 1987), in the definition of He *et al.* (1990) the regressors x_n are unchanged. Only the observations y_n can be exchanged by outliers in this definition. Therefore this definition is appropriate for situations where the regressors are fixed. In particular in image analysis this is the case because the regressors are given by the pixel positions.

In Müller (1997), p. 58, the following upper bound for the breakdown point of a regression equivariant estimator $\hat{\varphi}$ was shown:

$$\varepsilon^*(\hat{\varphi}, y, X) \leq \frac{1}{N}\left\lfloor \frac{N - \mathcal{N}_\varphi(X) + 1}{2} \right\rfloor\,,$$

where $\mathcal{N}_\varphi(X)$ is the maximum cardinality of a subset V of $\{x_1, \ldots, x_N\}$ so that $\varphi(\beta)$ is not identifiable at V. In particular, we have

$$\mathcal{N}_\beta(X) = \max\left\{\sum_{n=1}^{N} 1_V(x_n)\colon V \text{ subspace of } \mathbb{R}^r\right\}.$$

Estimators which are attaining this upper bound are, for example, estimators based on the Cauchy estimator and special least trimmed squares (LTS) estimators. A Cauchy estimator for β is defined by

$$\hat{\beta}_C(y, X) \in \arg\min_\beta \sum_{n=1}^{N} \rho\big(y_n - x_n^\top \beta\big)\,,$$

where $\rho(z) = \log(1 + z^2)$. A LTS estimator for β is defined by

$$\hat{\beta}_h(y, X) \in \arg\min_\beta \sum_{n=l}^{h} r_{(n)}(y, X, \beta)^2\,,$$

where $r_n(y, X, \beta) := \big|y_n - x_n^\top \beta\big|$ and $r_{(1)}(y, X, \beta) \leq r_{(2)}(y, X, \beta) \leq \cdots \leq r_{(N)}(y, X, \beta)$. If $h = \lfloor \frac{N+2}{2} \rfloor$ and $l = \lfloor \frac{N+1}{2} \rfloor$, then the LTS estimator is the least median of squares (LMS) estimator, i.e.,

$$\hat{\beta}_h(y, X) \in \arg\min_\beta \operatorname*{med}_{n=1,\ldots,N} \big(y_n - x_n^\top \beta\big)^2\,.$$

Estimators attaining the upper bound for estimating $\varphi(\beta) = L\beta$ are of the form $\varphi(\hat{\beta})$, where $\hat{\beta}$ is a Cauchy estimator or a special LTS estimator. This is shown by the following theorem.

Theorem 1. *If $\hat{\beta}$ is the Cauchy estimator or a* LTS *estimator with* $\left\lfloor \frac{N+\mathcal{N}_\varphi(X)+1}{2} \right\rfloor \le h \le \left\lfloor \frac{N+\mathcal{N}_\varphi(X)+2}{2} \right\rfloor$ *and if* $\mathcal{N}_\varphi(X) = \mathcal{N}_\beta(X)$, *then*

$$\varepsilon^*(\varphi(\hat{\beta}), y, X) = \frac{1}{N} \left\lfloor \frac{N - \mathcal{N}_\varphi(X) + 1}{2} \right\rfloor .$$

P r o o f . For linear aspects $\varphi(\beta) = L\beta$ we always have

$$\varepsilon^*(\hat{\beta}, y, X) \le \varepsilon^*\big(\varphi(\hat{\beta}), y, X\big) \le \frac{1}{N} \left\lfloor \frac{N - \mathcal{N}_\varphi(X) + 1}{2} \right\rfloor .$$

On the other hand it was shown in M ü l l e r (1995, 1997, p. 66) for the LTS estimator and in M i z e r a and M ü l l e r (1996) for the Cauchy estimator that we have

$$\varepsilon^*(\hat{\beta}, y, X) = \frac{1}{N} \left\lfloor \frac{N - \mathcal{N}_\beta(X) + 1}{2} \right\rfloor \tag{2.1}$$

so that the assumption $\mathcal{N}_\varphi(X) = \mathcal{N}_\beta(X)$ provides the assertion. □

There are other trimmed estimators and other M-estimators which satisfy (2.1) (see M ü l l e r 1995, 1997, Section 4.3, and M i z e r a and M ü l l e r 1996, respectively). But the LTS and the Cauchy estimators are the most popular estimators.

A crucial assumption in Theorem 1 is that $\mathcal{N}_\varphi(X) = \mathcal{N}_\beta(X)$ is satisfied. At least for two-dimensional linear regression, i.e., for $x(t,u) = (1,t,u)^\top$, we can show for the setup in image analysis that $\mathcal{N}_\varphi(X) = \mathcal{N}_\beta(X)$ holds.

Proposition 1. *If* $x(t,u) = (1,t,u)$, $X = \big(x(-v,-v), \dots, x(-v,v), \dots, x(v,v)\big)^\top$, *and* $\varphi(\beta) = x(0,0)^\top\beta$, *then* $\mathcal{N}_\varphi(X) = \mathcal{N}_\beta(X) = 2v+1$.

P r o o f . Without loss of generality we can assume $v \ge 1$. The aspect $\varphi(\beta)$ is not identifiable at $\big\{x(-1,-v), x(-1,-v+1), \dots, x(-1,v-1), x(-1,v)\big\}$ so that $\mathcal{N}_\beta(X) \ge \mathcal{N}_\varphi(X) \ge 2v+1$. If we have $\big\{x(t_1,u_1), \dots, x(t_{2v+2},u_{2v+2})\big\} \subset \big\{x(-v,-v), \dots, x(-v,v), \dots, x(v,v)\big\}$, then there exist i,j,k with $t_i = t_j$, $u_i \ne u_j$ and $t_j \ne t_k$, $u_j = u_k$, or i,j,k,l with $t_i = t_j$, $u_i \ne u_j$ and $t_k \ne t_l$, $u_k = u_l$. In the first case we have

$$\det\big((x(t_i,u_i), x(t_j,u_j), x(t_k,u_k)\big) = (t_i - t_k)(u_j - u_i) \ne 0 .$$

In the second case set $a_1 := x(t_i,u_i) - x(t_j,u_j)$, $a_2 := x(t_k,u_k) - x(t_l,u_l)$, and $a_3 := x(t_i,u_i) + x(t_j,u_j)$. Then we obtain

$$\det\big((a_1,a_2,a_3)\big) = 2(t_k - t_l)(u_i - u_j) \ne 0 .$$

Hence, β is identifiable in both cases, so that $2v+2 > \mathcal{N}_\beta(X) \ge \mathcal{N}_\varphi(X)$. □

Regressors $x_1, \ldots, x_N$ are in general position if and only if $\mathcal{N}_\beta(X) = r - 1$. Hence, for $x(t,u) = (1,t,u)^\top$ and $v \geq 1$, the proposition provides that the regressors $x(-v,-v), \ldots, x(-v,v), \ldots, x(v,v)$ are not in general position since $\mathcal{N}_\beta(X) = 2v+1 > 2 = r-1$. Moreover, the LMS estimator does not provide the highest breakdown point. If $\hat{\beta}_{\mathrm{LMS}}$ is the LMS estimator and $\varphi(\beta) = x(0,0)^\top \beta$, then the breakdown point of $\varphi(\hat{\beta}_{\mathrm{LMS}})$ is (see M ü l l e r 1997, p. 65, 66)

$$\begin{aligned}\varepsilon^*\left(\varphi(\hat{\beta}_{\mathrm{LMS}}), X\right) &:= \frac{1}{N}\left(\left\lfloor \frac{N}{2} \right\rfloor - \mathcal{N}_\varphi(X) + 1\right) \\ &= \frac{2v^2}{(2v+1)^2} = \begin{cases} 0.2\overline{2} & \text{for } v = 1\,, \\ 0.32 & \text{for } v = 2\,. \end{cases}\end{aligned}$$

If $\hat{\beta}$ is the Cauchy estimator or the LTS estimator with $\left\lfloor \frac{N+\mathcal{N}_\varphi(X)+1}{2} \right\rfloor \leq h \leq \left\lfloor \frac{N+\mathcal{N}_\varphi(X)+2}{2} \right\rfloor$, then according to Theorem 1 and Proposition 1 the breakdown point of $\varphi(\hat{\beta})$ is

$$\begin{aligned}\varepsilon^*\left(\varphi(\hat{\beta}), X\right) &:= \frac{1}{N}\left\lfloor \frac{N - \mathcal{N}_\varphi(X) + 1}{2} \right\rfloor \\ &= \frac{v}{2v+1} = \begin{cases} 0.3\overline{3} & \text{for } v = 1\,, \\ 0.4 & \text{for } v = 2\,. \end{cases}\end{aligned}$$

Hence, the breakdown point of the LMS estimator is not approximately 50 % and significantly smaller than that of the Cauchy and the optimal LTS estimator.

3. Applications on test images

As a test image, an image given by S m i t h and B r a d y (1995) (downloaded from `http://www.robots.ox.ac.uk/~steve/susan/susan.ps.gz`) with 100×100 pixels and pixel values ranging from 0 to 255 was used. Additionally, a picture of a CD-Cover with Lee Perry (downloaded from `http://www.rounder.com/rounder/catalog/bylabel/hbea/76 /76.html`) with 300×300 pixels was studied. The images were disturbed by $p = 0\,\%,\ 14\,\%,\ 30\,\%,\ 44\,\%$ and $49\,\%$ salt and pepper noise, where in each pixel with probability p an error occurs. If an error occurs, then a value was choosen from $[0, 255]$ with uniform distribution. Although the LMS estimator is a special LTS estimator, the abbreviation LTS denotes only the LTS estimator with the highest breakdown point, i.e., the LTS estimator with $h = \left\lfloor \frac{N+\mathcal{N}_\varphi(X)+1}{2} \right\rfloor$. All calculations were done with S-PLUS 3.2, where almost always 5×5 windows were used.

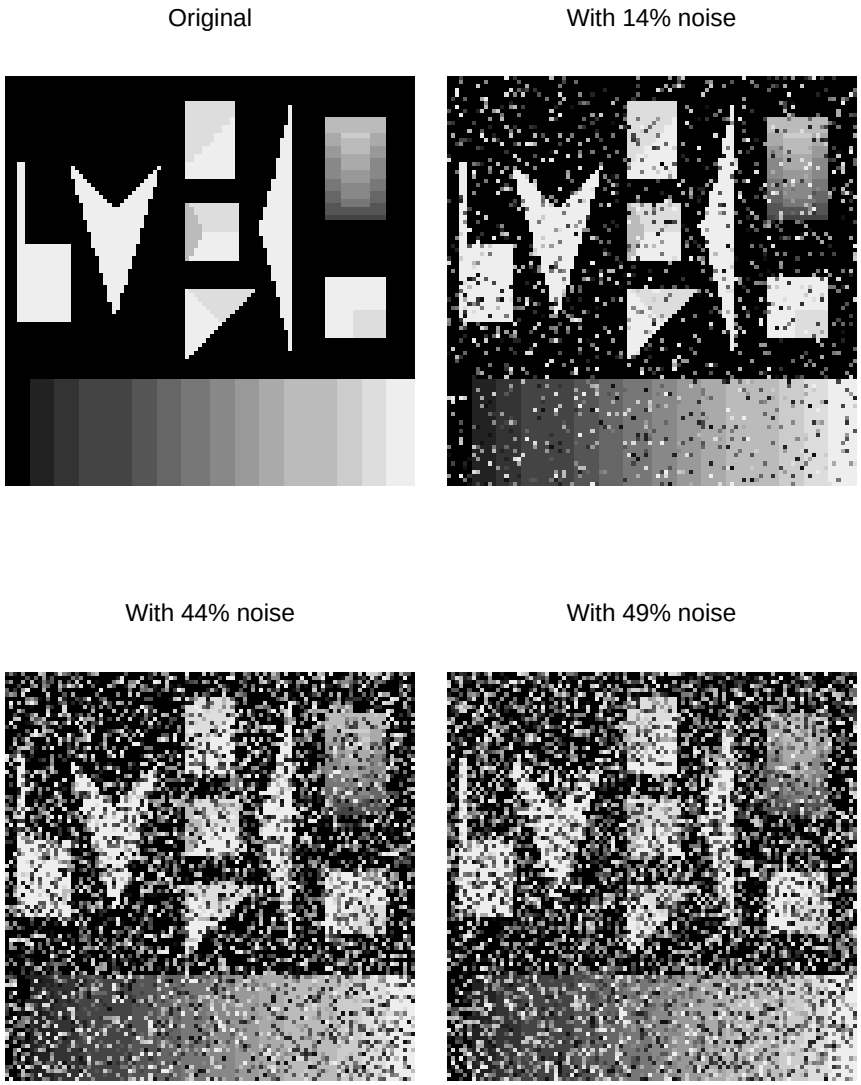

FIGURE 1. Original with and without noise

4. Conclusions

Figures 2, 3 and 5 show that the LMS and the LTS estimators are able to eliminate a high amount of noise. Due to the higher breakdown point of the LTS estimator, the LTS estimator can eliminate more noise than the LMS estimator, but the differences are not large. In the presence of low amount of noise, say 0 % or 14 %, their behavior is also good (see Figure 3 and 4). In particular,

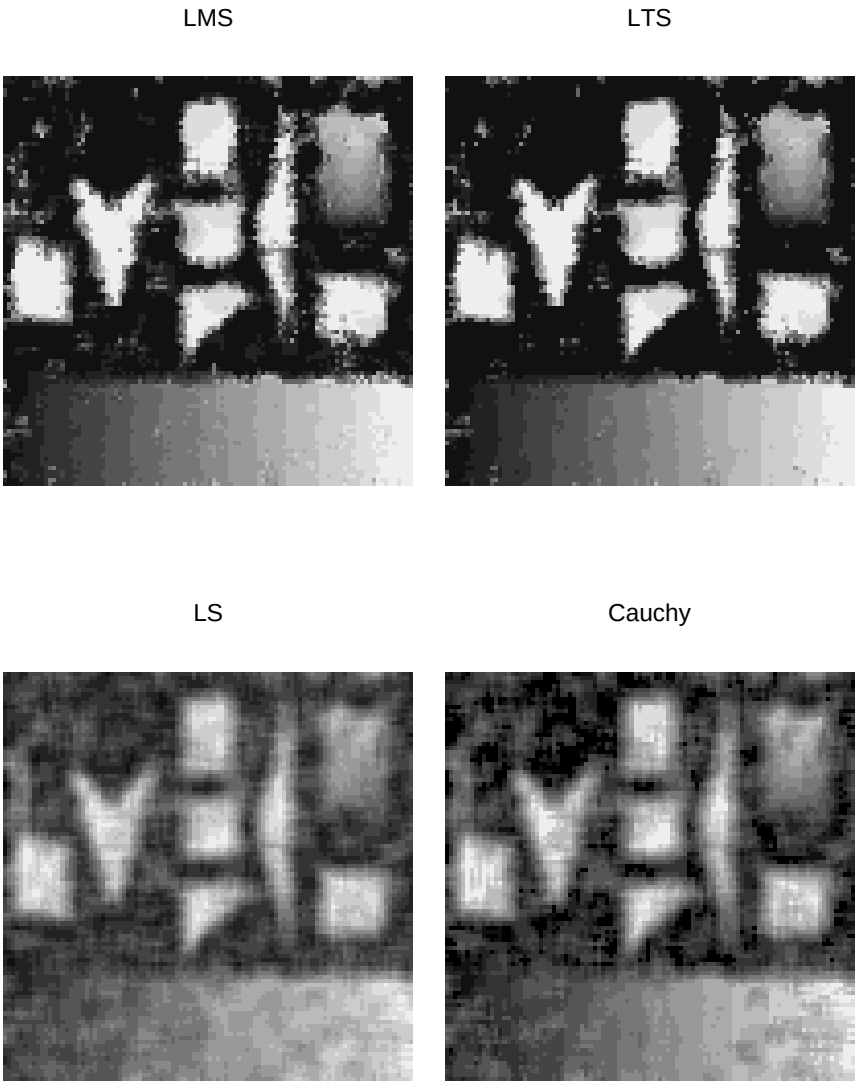

FIGURE 2. Estimators applied on the image with 44 % noise

they are preserving discontinuities. While with 5×5 windows the edges are preserved completely, this is not the case for the LTS estimator used at 3×3 windows (see Figure 4). The reason is that in this case the LTS estimator is not unique. Although in this case the LMS estimator is not unique as well, it preserves the edges. Probably this different behaviour is caused by the different programmes used by S-PLUS. The programme used for the LTS estimator is based on a genetic algorithm while the programme for the LMS estimator is

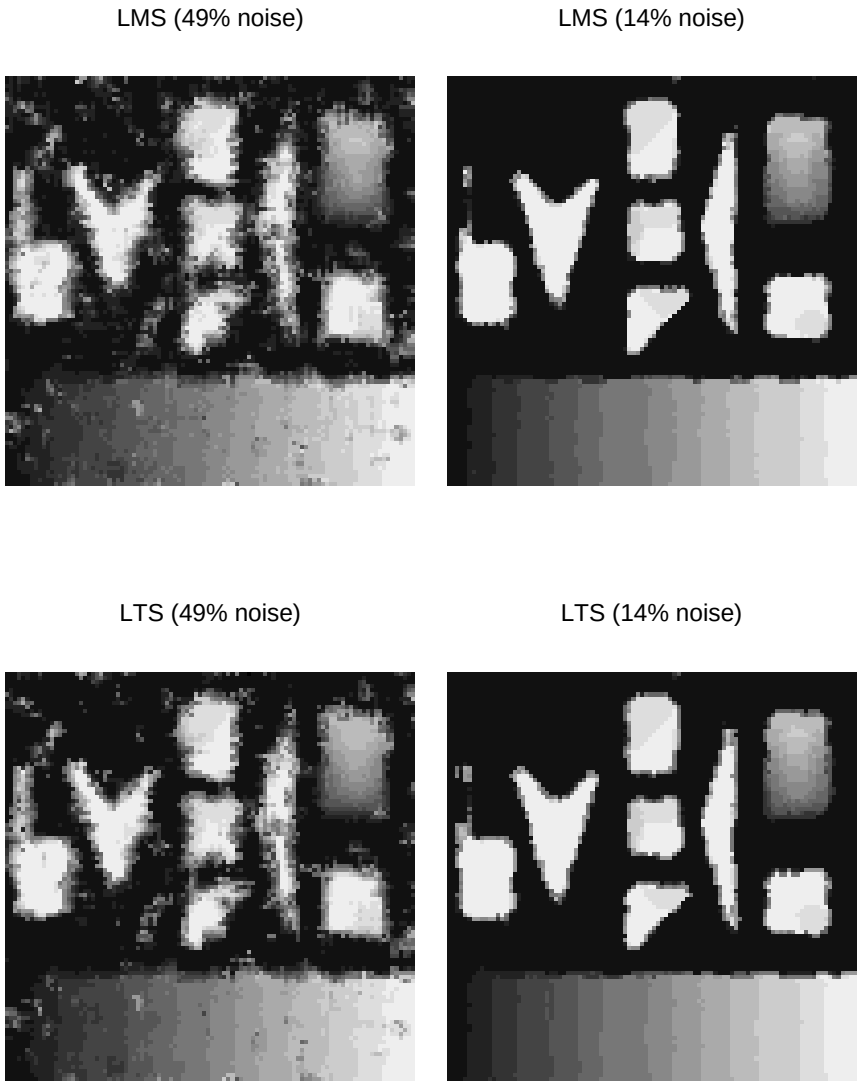

FIGURE 3. LMS and LTS applied on the image with 49 % and 14 % noise

completely deterministic. In opposite to edges, the corners are never preserved. This is due to the fact that the LTS and the LMS estimators are determined by the majority of the data.

Although the Cauchy estimator has the highest breakdown point as well, it behaves not well. Its behaviour is only a little bit better than that of the non-robust LS estimator (see Figure 2). As an LS estimator, it is smoothing discontinuities. This bad behaviour of the Cauchy estimator can have three reasons:

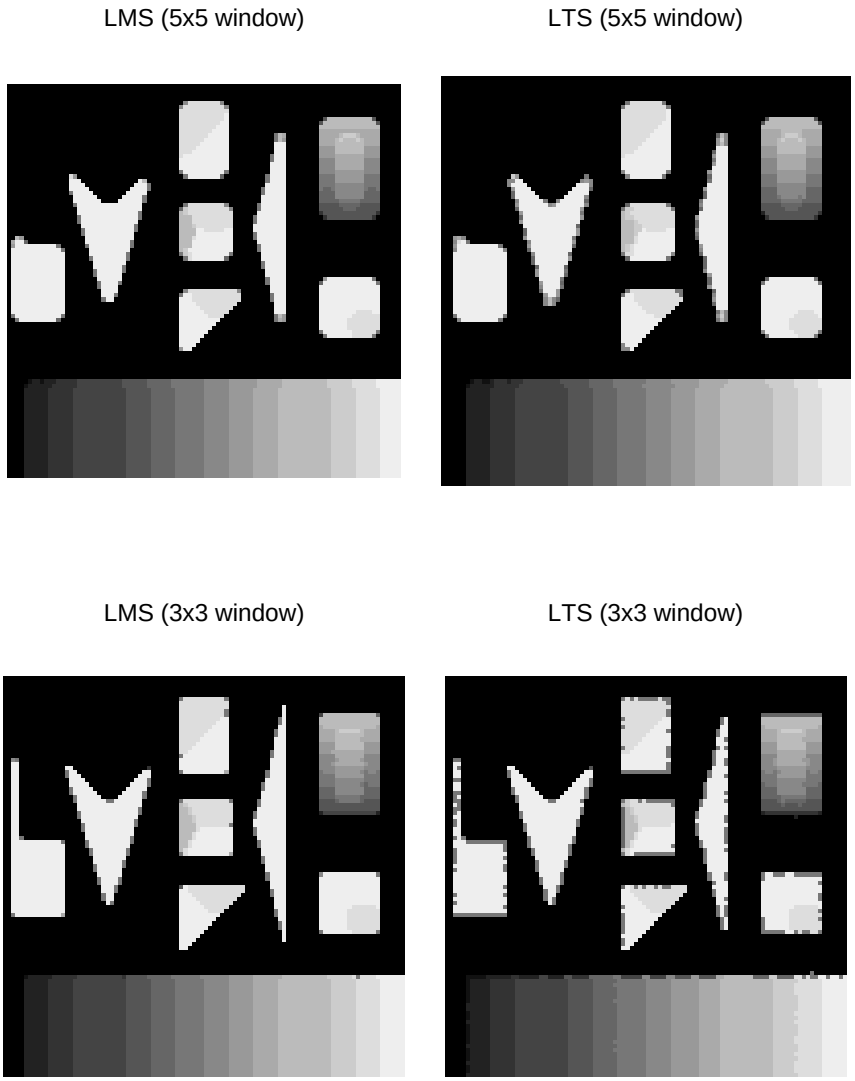

FIGURE 4. LTS and LMS applied on the original

Firstly, it could be that the estimator provided by S-PLUS is not exactly the Cauchy estimator. Secondly, the scale estimator within the Cauchy estimator could overestimate the scale. Thirdly, it could be that a high breakdown point is not the main property providing a good behaviour in image analysis. Perhaps, it is more important for image representations that the estimator depends only on a majority of the data and is independent of a minority.

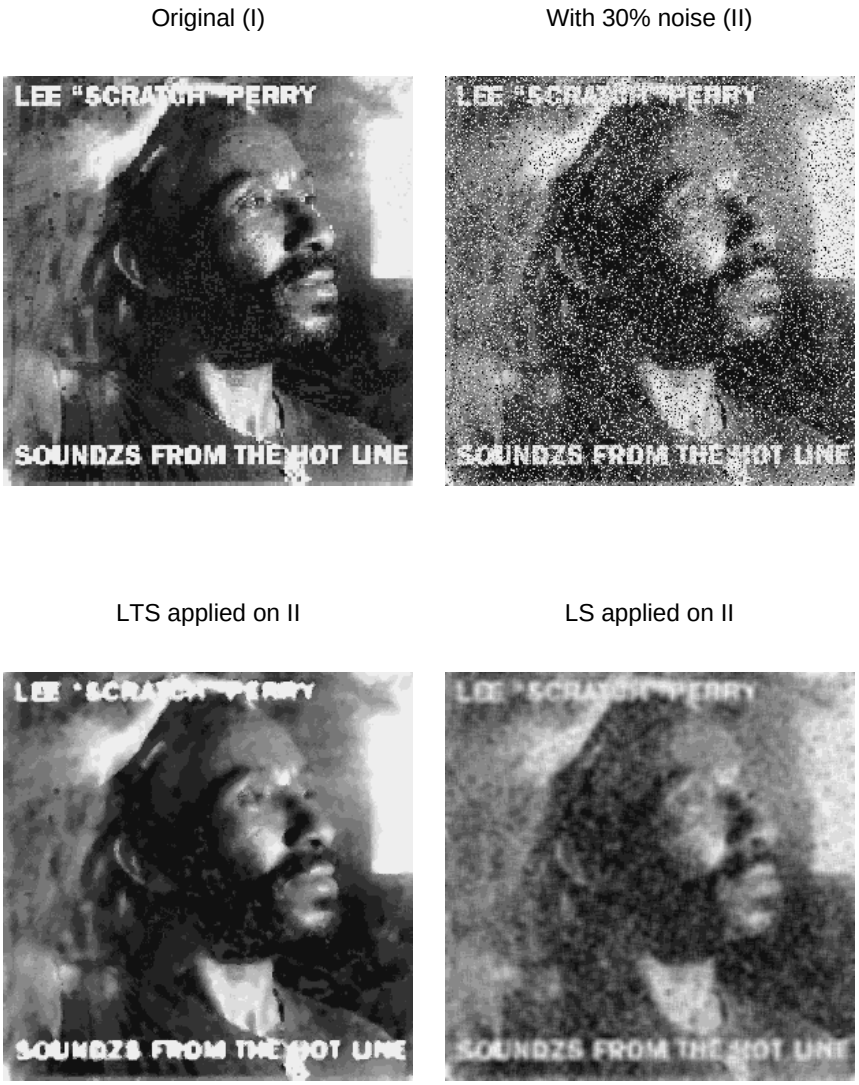

FIGURE 5. LTS and LS applied on the image of Lee Perry with 30 % noise

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