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THE CROSSING NUMBER OF $K_5 \times P_n$

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ABSTRACT. There are several known exact results on the crossing numbers of Cartesian products of paths, cycles or stars with "small" graphs. In this paper we extend these results, showing that the crossing number of the Cartesian product $K_5 \times P_n$ is $6n$ for $n \geq 1$.

1. Introduction

The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of pairwise intersections of edges in a drawing of G in the plane. It is well-known that the crossing number of a graph is attained only in a *good* drawings of the graph, which are those drawings where no edge crosses itself, no adjacent edges cross each other, and no two edges intersect more than once. Let D be a good drawing of the graph G. We denote the number of crossings in D by $\mathrm{cr}_D(G)$. Let G_i and G_j be edge-disjoint subgraphs of G. We denote by $\mathrm{cr}_D(G_i, G_j)$ the number of crossings between edges of G_i and edges of G_j in D.

Computing the crossing number of a given graph is in general an elusive problem, and the crossing numbers of few families of graphs are known. Most of them are Cartesian products of special graphs. (For a definition of Cartesian product, see [5].) Let C_n and P_n be the *cycle* and the *path* with n edges. The crossing numbers of the Cartesian products of all 4-vertex graphs with cycles are determined in [2], [3] and [6], and with paths and stars in [7] and [8]. There are several known exact results on the crossing numbers of Cartesian products of paths with 5-vertex graphs. In [7] and [10] it is shown that $\mathrm{cr}(K_{1,4} \times P_n) = 2(n-1)$ and $\mathrm{cr}(K_{2,3} \times P_n) = 2n$. Let G_1, G_2, and G_3 be three graphs of order five defined by removing from K_5 the edges of an elementary subdivision of $K_{1,3}$, the edges of K_3, and the edges of $K_{1,2}$, respectively. In [9] it is shown that the crossing numbers of $G_i \times P_n$ are $2n-2$ for $i = 1$, and $3n-1$ for

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$i = 2$ and 3. In this paper we extend these results to the Cartesian product of a complete graph on five vertices with a path P_n, showing that its crossing number is $6n$.

2. The main result

We assume $n \geq 1$ and find it convenient to consider the graph $K_5 \times P_n$ in the following way: it has $5(n+1)$ vertices and edges that are the edges in the $n+1$ copies K_5^i, $i = 0, 1, 2, \dots, n$, and in the five paths of length n (Figure 1). Furthermore, we call the former edges red and the latter ones blue. For $i = 1, 2, \dots, n$, let P^i denote the subgraph of $K_5 \times P_n$ induced by five blue edges joining K_5^{i-1} to K_5^i.

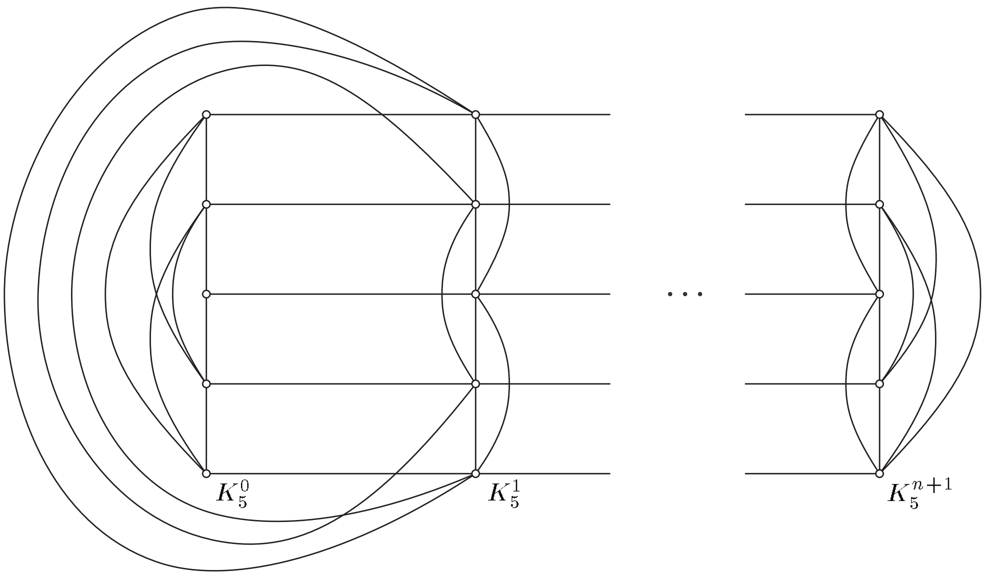

FIGURE 1

Our next lemma establishes the crossing number of the graph obtained by removing one edge from K_7. We shall need it in the proof of Lemma 4.

LEMMA 1. $\mathrm{cr}(K_7 - e) = 6$.

P r o o f . The graph $K_7 - e$ has five vertices of degree 6 and two vertices of degree 5. Let us denote the vertices of degree 5 by x and y, and the vertices of degree 6 by v_i for $i = 1$, 2, 3, 4, and 5. Let T^x and T^y denote the subgraphs induced by the edges incident with x and y, respectively. It is easy to see that the vertices v_i for $i = 1, 2, 3, 4$, and 5 induce K_5. Thus we have

$$K_7 - e = K_5 \cup T^x \cup T^y .$$

The graph $K_7 - e$ can be obtained from the graph $K_5 \times P_2$ by contracting the edges of K_5^0 and the edges of K_5^2. In Figure 1 it is easy to see that $\mathrm{cr}(K_7 - e) \leq 6$. Thus, it remains to prove the reverse inequality. Suppose that there is a good drawing of $K_7 - e$ with five crossings and let D be such a drawing. First we show that in D there is at least one crossing between the edges of T^x and the edges of T^y. Otherwise $\mathrm{cr}_D(T^x, T^y) = 0$, and since in the good drawing two edges incident with the same vertex cannot cross, the subdrawing of $T^x \cup T^y$ induced by D induces the map in the plane without crossings, as shown in Figure 2. In this case, in D, at least five edges $\{v_1, v_3\}$, $\{v_1, v_4\}$, $\{v_2, v_4\}$, $\{v_2, v_5\}$, and $\{v_3, v_5\}$ cross the edges of $T^x \cup T^y$. Moreover, in D, at least two edges of K_5 induced by v_1, v_2, v_3, v_4, and v_5 cross each other, which contradicts our assumption. Hence $\mathrm{cr}_D(T^x, T^y) \geq 1$.

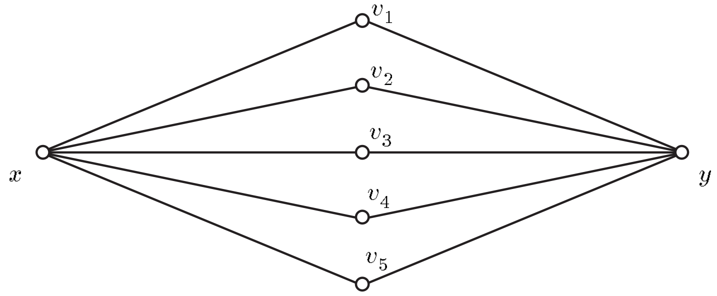

FIGURE 2

It is known (see [1]) that every good drawing of K_5 has an odd number of crossings and that $\mathrm{cr}(K_5) = 1$. Thus, we have three cases to deal with: Assume $\mathrm{cr}_D(K_5) = 1$. As $K_5 \cup T^x$ and $K_5 \cup T^y$ form the complete graphs on six vertices, and since $\mathrm{cr}(K_6) = 3$ (see [4]), $\mathrm{cr}_D(K_5, T^x) \geq 2$ and $\mathrm{cr}_D(K_5, T^y) \geq 2$. This, together with $\mathrm{cr}_D(T^x, T^y) \geq 1$, contradicts our assumption. Assume now $\mathrm{cr}_D(K_5) = 3$. Since no drawing of K_5 with fewer than five crossings has all five vertices on the boundary of the same "region", $\mathrm{cr}_D(K_5, T^x) \geq 1$ and $\mathrm{cr}_D(K_5, T^y) \geq 1$. Moreover, $\mathrm{cr}_D(T^x, T^y) \geq 1$, and D has more than five crossings. Finally assume that $\mathrm{cr}_D(K_5) \geq 5$. As $\mathrm{cr}_D(T^x, T^y) \geq 1$, D has at least six crossings and this contradiction completes the proof. □

Consider a graph obtained by joining all vertices of K_5 to some vertices of a connected graph G. In every good drawing of the graph there are at least three crossings on the edges of K_5, since contracting the edges of G results in a graph isomorphic to K_6. Thus we have the following observation.

LEMMA 2. *In any good drawing of $K_5 \times P_n$, $n \geq 1$, there are at least three crossings on edges of every K_5^i for $i = 0, 1, 2, \ldots, n$.*

Lemma 3. $\operatorname{cr}(K_5 \times P_1) = 6$.

P r o o f . The graph $K_5 \times P_1$ consists of two subgraphs K_5^0 and K_5^1 and of five edges joining K_5^0 to K_5^1. Suppose a good drawing D of the graph $K_5 \times P_1$ is given. If $\operatorname{cr}_D(K_5^0, K_5^1) \neq 0$, then $\operatorname{cr}_D(K_5^0, K_5^1) \geq 4$, because the graph K_5 is 4-connected. Besides, $\operatorname{cr}_D(K_5^0) \geq 1$ and $\operatorname{cr}_D(K_5^1) \geq 1$, and D has at least six crossings. Assume now $\operatorname{cr}_D(K_5^0, K_5^1) = 0$. As we have shown in Lemma 2, there are at least three crossings on both K_5^0 and K_5^1 and D has at least six crossings again. The reverse inequality follows from Figure 1. $\square$

The following Lemma 4 is the heart of the proof of the main result.

Lemma 4. *If D is a good drawing of $K_5 \times P_n$, $n \geq 2$, in which every K_5^i has at most five crossings on its edges, then D has at least $6n$ crossings.*

P r o o f . First we show that in D two different K_5^i and K_5^j do not cross each other and that no K_5^i is crossed by an edge of P^l for $l \neq i$ and $l \neq i+1$. We know that if two different complete graphs on five vertices cross each other, then they cross at least four times. Suppose that for some $i \neq j$, $\operatorname{cr}_D(K_5^i, K_5^j) = 4$. If the edges of K_5^i are deleted, the resulting graph is either homeomorphic to $K_5 \times P_{n-1}$ or contains a subgraph isomorphic to $K_5 \times P_{n-1}$ and in the resulting drawing there is at most one crossing on the edges of K_5^j. This contradicts Lemma 2.

Suppose now that in D a blue edge of P^l not incident with a vertex of K_5^i crosses K_5^i. The endpoints of this blue edge are not in different "regions" of the subdrawing $D(K_5^i)$ induced from D by K_5^i, otherwise, in D, the subdrawing $D(K_5^i)$ separates two other complete subgraphs on five vertices, and K_5^i is crossed by all five blue edges of P^l. Thus, the endpoints of the blue edge of P^l are in the same "region" of $D(K_5^i)$ and this edge crosses K_5^i at least four times. In this case D contains a subdrawing of a graph isomorphic to $K_5 \times P_1$ induced by $V(K_5^{i-1}) \cup V(K_5^i)$ or $V(K_5^i) \cup V(K_5^{i+1})$ with at most one crossing on the edges of K_5^i. This contradicts Lemma 2 again.

For $i = 1, 2, \dots, n-1$, let Q^i denote the subgraph of $K_5 \times P_n$ induced by $V(K_5^{i-1}) \cup V(K_5^i) \cup V(K_5^{i+1})$. Thus, Q^i has 30 red edges in K_5^{i-1}, K_5^i and K_5^{i+1} and 10 blue edges in P^i and P^{i+1}. Clearly, Q^i is isomorphic to $K_5 \times P_2$. Let us denote by Q_c^i the subgraph of Q^i obtained by removing six edges of K_5^{i-1} and six edges of K_5^{i+1} in such a way that Q_c^i is homeomorphic to $K_7 - e$. In a good drawing of $K_5 \times P_n$, we define the *force* $f(Q^i)$ of Q^i ($f(Q_c^i)$ of Q_c^i) to be the total number of crossings of the following types:

(1) a crossing of a blue edge in $P^i \cup P^{i+1}$ with an edge in K_5^i,

(2) a crossing of a blue edge in P^i with a blue edge in P^{i+1},

(3) a self-intersection in K_5^i.

The *total force* of the drawing is the sum of $f(Q^i)$. It is readily seen that a crossing contributes at most one to the total force of the drawing.

Consider now a subdrawing D_c^i of Q_c^i induced by D. As we have shown above, in D_c^i, no two edges belonging to different K_5^p and K_5^r, $p, r \in \{i-1, i, i+1\}$, cross each other, no edge of P^i crosses an edge of K_5^{i+1} and no edge of P^{i+1} crosses an edge of K_5^{i-1}. Thus, one can easily see that in any optimal drawing D_c^i of Q_c^i there are only crossings of types (1), (2) or (3). This implies that, in D, for every i, $i = 1, 2, \dots, n-1$, $f(Q^i) \geq f(Q_c^i) \geq \mathrm{cr}(K_7 - e) = 6$, and that the total force of D is at least $6(n-1)$.

By Lemma 2, in D there are at least three crossings on the edges of K_5^0 and at least three crossings on the edges of K_5^n. None of these crossings is counted in the total force of D. Therefore, in D there are at least $\sum\limits_{1 \leq i \leq n-1} f(Q^i) + 6 \geq 6n$ crossings. $\square$

THEOREM 1. *$\mathrm{cr}(K_5 \times P_n) = 6n$ for $n \geq 1$.*

Proof. The drawing in Figure 1 shows that $\mathrm{cr}(K_5 \times P_n) \leq 6n$ for $n \geq 1$. We prove the reverse inequality by induction on n. By Lemma 3, $\mathrm{cr}(K_5 \times P_1) = 6$, so the result is true for $n = 1$. Assume it is true for $n = k$, $k \geq 1$, and suppose that there is a good drawing of $K_5 \times P_{k+1}$ with fewer than $6(k+1)$ crossings. By Lemma 4, some K_5^i must then be crossed at least six times. By the removal of all edges of this K_5^i, we obtain a graph homeomorphic to $K_5 \times P_k$ or one that contains the subgraph $K_5 \times P_k$ and has drawing with fewer than $6k$ crossings. This contradicts the induction hypothesis. $\square$

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