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DIVISIBILITY IN ADDITIVE HEREDITARY GRAPH PROPERTIES AND UNIQUELY PARTITIONABLE GRAPHS

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ABSTRACT. Let $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$ be graph properties and let G be a graph. A vertex $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$-partition of G is a partition $\{V_1, V_2, \dots, V_n\}$ of $V(G)$ such that the induced subgraph $G[V_i]$ has the property $\mathcal{P}_i$ for each $i = 1, 2, \dots, n$. The property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \cdots \circ \mathcal{P}_n$ is defined as the set of all graphs having a vertex $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$-partition. A graph $G \in \mathcal{P}_1 \circ \mathcal{P}_2 \cdots \circ \mathcal{P}_n$ is said to be uniquely $(\mathcal{P}_1 \circ \mathcal{P}_2 \cdots \circ \mathcal{P}_n)$-partitionable if G has exactly one vertex $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$-partition. In this paper we prove for additive hereditary properties $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$ that a uniquely $(\mathcal{P}_1 \circ \mathcal{P}_2 \cdots \circ \mathcal{P}_n)$-partitionable graph exists if and only if every pair of these properties are coprime.

1. Divisibility in graph properties

We denote by I the class of all finite undirected graphs without loops and multiple edges. For technical reasons we consider the null graph $K_0 = (\emptyset, \emptyset)$ (see [6]) to be a member of $\mathcal{I}$.

In general we follow the notation and terminology of [1]. By the *join* $G_1 + G_2 + \cdots + G_n$ of n graphs $G_1, G_2, \dots, G_n$ we mean the graph which consists of disjoint copies of $G_1, G_2, \dots, G_n$ and all the edges between $V(G_i)$ and $V(G_j)$ for all $i \neq j$. The term *decomposable* graph is used for every graph G which is the join of at least two graphs, where none of them is K_0. A graph that is not decomposable is called *indecomposable*. A graph G is decomposable if and only if its complement $\overline{G}$ is not connected, since obviously every decomposable graph can be uniquely expressed as the join of the complements of the components of $\overline{G}$. The *decomposability number* $\mathrm{dc}(G)$ of a graph G is defined as the number of indecomposable parts of G, i.e., the number of components of $\overline{G}$.

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Let $\mathcal{P}$ be a nonempty, isomorphism closed, proper subset of $\mathcal{I}$. Then $\mathcal{P}$ will also denote the property that a graph is a member of the set $\mathcal{P}$. A property $\mathcal{P}$ is said to be *hereditary* if, whenever $G \in \mathcal{P}$ and H is subgraph of G, then also $H \in \mathcal{P}$. A property $\mathcal{P}$ is called *additive* if for each graph G, all of whose components have property $\mathcal{P}$, it follows that $G \in \mathcal{P}$, too.

The following are frequently used examples of additive hereditary properties:

$$\begin{aligned}
\Theta &= \{K_0\}\,,\\
\mathcal{O} &= \{G \in \mathcal{I}\colon G \text{ is edgeless, i.e., } E(G) = \emptyset\}\,,\\
\mathcal{O}_k &= \{G \in \mathcal{I}\colon \text{ each component of } G \text{ has at most } k+1 \text{ vertices}\}\,,\\
\mathcal{S}_k &= \{G \in \mathcal{I}\colon \text{ the maximum degree } \Delta(G) \le k\}\,,\\
\mathcal{D}_k &= \{G \in \mathcal{I}\colon G \text{ is } k\text{-degenerate, i.e., the minimum degree } \delta(H) \le k\\
&\qquad\qquad \text{for each } H \subseteq G\}\,,\\
\mathcal{I}_k &= \{G \in \mathcal{I}\colon G \text{ does not contain } K_{k+2}\}\,.
\end{aligned}$$

The set of all hereditary properties will be denoted by $\mathscr{L}$ and the set of all additive hereditary properties will be denoted by $\mathscr{L}^a$. The partially ordered sets $(\mathscr{L}, \subseteq)$ and $(\mathscr{L}^a, \subseteq)$ are complete distributive lattices. More comprehensive information about these lattices can be found in [1], [2], [13].

For every hereditary property $\mathcal{P} \ne \mathcal{I}$ there is a nonnegative integer $c(\mathcal{P})$ such that $K_{c(\mathcal{P})+1} \in \mathcal{P}$, but $K_{c(\mathcal{P})+2} \notin \mathcal{P}$, called the *completeness* of $\mathcal{P}$. For example $c(\Theta) = -1$, $c(\mathcal{O}) = 0$ and $c(\mathcal{O}_k) = c(\mathcal{S}_k) = c(\mathcal{D}_k) = c(\mathcal{I}_k) = k$ (see [1]).

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be arbitrary hereditary graph properties. A *vertex* $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$*-partition* of a graph G is a partition $\{V_1, V_2, \ldots, V_n\}$ of $V(G)$ such that the induced subgraph $G[V_i]$ has the property $\mathcal{P}_i$ for each $i = 1, 2, \ldots, n$. The property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ is defined as the set of all graphs having a vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition. It is easy to see that if $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ are additive and hereditary, then $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ is additive and hereditary, too. If $\mathcal{P}_1 = \mathcal{P}_2 = \cdots = \mathcal{P}_n = \mathcal{P}$, then we write $\mathcal{P}^n = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$. An additive hereditary property $\mathcal{R}$ is said to be *reducible* in $\mathscr{L}^a$ if there exist additive hereditary properties $\mathcal{P} \ne \Theta$ and $\mathcal{Q} \ne \Theta$ such that $\mathcal{R} = \mathcal{P} \circ \mathcal{Q}$ and *irreducible* in $\mathscr{L}^a$ otherwise.

By these definitions it immediately follows that the binary operation "$\circ$" on $\mathscr{L}^a$ is associative and commutative and that the property $\Theta = \{K_0\}$ is the only neutral element of "$\circ$" in $\mathscr{L}^a$.

Lemma 1. [2], [9]. *For any reducible property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2$ we have that*

$$c(\mathcal{R}) = c(\mathcal{P}_1) + c(\mathcal{P}_2) + 1\,.$$

Any hereditary property $\mathcal{P}$ can be characterized by the set $\mathbf{M}(\mathcal{P})$ of *$\mathcal{P}$-maximal graphs.* First we define the set $\mathbf{M}(n, \mathcal{P})$ of $\mathcal{P}$-maximal graphs of the given order n:

$$\mathbf{M}(n,\mathcal{P}) = \big\{G \in \mathcal{P}\colon |V(G)| = n \text{ and } G+e \notin \mathcal{P} \text{ for each } e \in E(\overline{G})\big\}.$$

Then we define $\mathbf{M}(\mathcal{P}) = \bigcup_{n=1}^{\infty} \mathbf{M}(n,\mathcal{P})$. It is easy to see that $\mathbf{M}(n,\mathcal{P}) = \{K_n\}$ for $n = 1,2,\ldots,c(\mathcal{P})+1$ and it is useful to define the set $\mathbf{M}^*(\mathcal{P}) = \bigcup_{n=c(\mathcal{P})+1}^{\infty} M(n,\mathcal{P})$. If G is a graph with property $\mathcal{P}$ such that the graph $G+K_1$ does not belong to $\mathcal{P}$, then G is called *$\mathcal{P}$-strict*. It is easy to see that all $\mathcal{P}$-maximal graphs in $\mathbf{M}^*(\mathcal{P})$ are $\mathcal{P}$-strict (see [12]).

A characterization of $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$-maximal graphs is given in the following lemma.

Lemma 2. [1] *Let $n \geq 2$ and suppose that $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ are hereditary properties of graphs. Then a graph G belongs to $M(\mathcal{P}_1 \circ \mathcal{P}_2 \cdots \circ \mathcal{P}_n)$ if and only if for every $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition $\{V_1, V_2, \ldots, V_n\}$ of $V(G)$ the following holds: $G[V_i] \in \mathbf{M}(\mathcal{P}_i)$ for each $i = 1,2,\ldots,n$ and $G = G[V_1] + G[V_2] + \cdots + G[V_n]$.*

Fundamental results on $\mathcal{P}$-maximal graphs are also presented in [4, 7, 15].

Note that every graph $G \in \mathcal{P}$ is a subgraph of some graph $H \in \mathbf{M}(\mathcal{P})$. Following [12], a set $\mathcal{G} \subseteq \mathbf{M}^*(\mathcal{P})$ is said to be a *generating set* of $\mathcal{P}$ if every graph in $\mathcal{P}$ is a subgraph of some graph in $\mathcal{G}$. The members of $\mathcal{G}$ are then called *generators* of $\mathcal{P}$. If $\mathcal{P}$ is a hereditary property and $\mathcal{G}$ is a generating set of $\mathcal{P}$, then the *decomposability number* of $\mathcal{G}$, denoted by $\mathrm{dc}(\mathcal{G})$, is defined as the minimum decomposability number of the graphs belonging to $\mathcal{G}$. In [12] it is shown that all the generating sets of $\mathcal{P}$ have the same decomposability number $\mathrm{dc}(\mathcal{G}) = \mathrm{dc}\big(\mathbf{M}^*(\mathcal{P})\big)$. The *decomposability number* $\mathrm{dc}(\mathcal{P})$ *of a property* $\mathcal{P}$ is then defined as the common value of the decomposability numbers of the generating sets of $\mathcal{P}$, i.e., $\mathrm{dc}(\mathcal{P}) = \mathrm{dc}\big(\mathbf{M}^*(\mathcal{P})\big)$.

We now introduce the well-known notion of divisibility for the binary operation "$\circ$" on $\mathscr{L}^a$ in a natural way: Given any two graph properties $\mathcal{R}$ and $\mathcal{P}$, with $\mathcal{R}$, $\mathcal{P} \in \mathscr{L}^a$, we say that $\mathcal{P}$ is a *divisor* of $\mathcal{R}$, and write $\mathcal{P}|\mathcal{R}$ for it, if $\mathcal{R} = \mathcal{P} \circ \mathcal{Q}$ for some property $\mathcal{Q} \in \mathscr{L}^a$. We also say that $\mathcal{P}$ is a *factor* of $\mathcal{R}$, that $\mathcal{P}$ *divides* $\mathcal{R}$, that $\mathcal{R}$ is *divisible by* $\mathcal{P}$, and that $\mathcal{R}$ is a *multiple* of $\mathcal{P}$. It is easy to see that $\mathcal{P} = \mathcal{P} \circ \Theta$ for every property $\mathcal{P} \in \mathscr{L}^a$ so that Θ and $\mathcal{P}$ are divisors of $\mathcal{P}$; we will call these divisors of $\mathcal{P}$ the *trivial divisors* of $\mathcal{P}$. All others divisors of $\mathcal{P}$ are *nontrivial*.

If $\mathcal{P}$ and $\mathcal{Q}$ are additive hereditary properties, we say that the additive hereditary property $\mathcal{D}$ is a *greatest common divisor* of $\mathcal{P}$ and $\mathcal{Q}$ (and we write $\mathcal{D} = \gcd(\mathcal{P},\mathcal{Q})$), if

(1) $\mathcal{D}$ divides $\mathcal{P}$, and $\mathcal{D}$ divides $\mathcal{Q}$;

(2) if $\mathcal{D}'$ divides $\mathcal{P}$ and $\mathcal{D}'$ divides $\mathcal{Q}$, then $\mathcal{D}'$ divides $\mathcal{D}$.

Note that if $\mathcal{D}$ and $\mathcal{D}'$ both satisfy conditions (1) and (2), it follows that $\mathcal{D}$ divides $\mathcal{D}'$ and $\mathcal{D}'$ divides $\mathcal{D}$. Thus there are two properties $\mathcal{F}$ and $\mathcal{H}$ in $\mathcal{L}^a$ such that $\mathcal{D} = \mathcal{D}' \circ \mathcal{F}$ and $\mathcal{D}' = \mathcal{D} \circ \mathcal{H}$. But then we have that $\mathcal{D} = \mathcal{D} \circ \mathcal{H} \circ \mathcal{F}$, so that, by Lemma 1, $c(\mathcal{H} \circ \mathcal{F}) = -1$, that is, $\mathcal{H} \circ \mathcal{F} = \Theta$. From this equation now follows that $\mathcal{H} = \mathcal{F} = \Theta$ and that $\mathcal{D} = \mathcal{D}'$. Thus the conditions (1), (2) and Theorem 1 (see below) indeed ensure that any two additive hereditary properties have a unique greatest common divisor. If $\gcd(\mathcal{P}, \mathcal{Q}) = \Theta$, then we say that $\mathcal{P}$ and $\mathcal{Q}$ are *coprime*. If an additive hereditary property $\mathcal{P}$ is irreducible, then the only additive hereditary properties which divide $\mathcal{P}$ are Θ and $\mathcal{P}$ itself.

Any additive hereditary property can be expressed as a product of irreducible additive hereditary properties. The existence of such a *factorization* of an additive hereditary property $\mathcal{R}$ into irreducible factors follows, using Lemma 1, by induction on the completeness $c(\mathcal{R})$ of $\mathcal{R}$. The uniqueness of these factorizations is proved in [11].

Theorem 1. [11] *Let $\mathcal{R} \in \mathcal{L}^a$ be a reducible property of graphs and suppose that $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \cdots \circ \mathcal{P}_n$, $n \geq 2$ is a factorization of $\mathcal{R}$ into irreducible factors. Then this factorization is unique (up to the order of the factors).*

Corollary 1. *If $\mathcal{P} \in \mathcal{L}^a$ is an irreducible property and $\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n$ are additive hereditary properties such that $\mathcal{P}$ divides $\mathcal{Q}_1 \circ \mathcal{Q}_2 \circ \cdots \circ \mathcal{Q}_n$, then $\mathcal{P}$ divides $\mathcal{Q}_i$ for some i, $1 \leq i \leq n$.*

For any reducible property $\mathcal{R} \in \mathcal{L}^a$ we can write its factorization into irreducible factors in the following way:

$$\mathcal{R} = \mathcal{P}_1^{e_1} \circ \mathcal{P}_2^{e_2} \circ \cdots \circ \mathcal{P}_r^{e_r},$$

where $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_r$ are distinct irreducible properties and $e_1, e_2, \ldots, e_r$ are positive integers. Using the symbol $\mathcal{P}^0$ to denote the property Θ, one can clearly use this type of factorization to describe the greatest common divisor of any two properties similar to the way it is done in Number Theory [14].

2. Uniquely partitionable graphs

A graph $G \in \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ is said to be *uniquely* $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$*-partitionable* if G has exactly one (unordered) vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition. The set of all uniquely $(\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_n)$-partitionable graphs will be denoted by $\mathbf{U}(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$. The notion of a uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graph appears in [5] and the basic properties of uniquely $\mathcal{P}^n$-partitionable graphs are presented in [8].

Lemma 3. *Let $\mathcal{P}_i \in \mathcal{L}^a$, $\mathcal{P}_i \neq \Theta$ for $i = 1, 2, \ldots n$. Let G be a uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graph and suppose that $\{V_1, V_2, \ldots, V_n\}$ is its* *vertex partition. If G_w is obtained from G by adding a vertex w and edges joining w to every vertex of the set V_i, $i = 2, 3, \ldots, n$, then G_w is uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable and $\{V_1 \cup \{w\}, V_2, \ldots, V_n\}$ is its vertex partition.*

Proof. Let G be any uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graph and let $\{V_1, V_2, \ldots, V_n\}$ be the vertex $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$-partition of $V(G)$. From a known necessary condition for the existence of uniquely partitionable graphs (see [1], Theorem 40) it follows that the subgraphs $G[V_i]$ are $\mathcal{P}_i$-strict, $i = 1, 2, \ldots, n$, that is, the induced subgraph $G[V_i + \{w\}]$ does not belong to property $\mathcal{P}_i$ for $i = 2, 3, \ldots, n$. Note also that $G + K_1 \notin \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$. It now follows that $\{V_1 \cup \{w\}, V_2, \ldots, V_n\}$ is the only vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition of $V(G_w)$. □

THEOREM 2. *Let $\mathcal{P}_i \in \mathscr{L}^a$ for $i = 1, 2, \ldots n$. If $H \in \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ and $\mathbf{U}(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n) \neq \emptyset$, then H is an induced subgraph of some uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graph.*

Proof. Suppose $H \in \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ and let $\{W_1, W_2, \ldots, W_n\}$ be the vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition of $V(H)$. Let F be any uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graph and let $\{V_1, V_2, \ldots, V_n\}$ be its vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition.

Now we construct a graph G in the following way: $V(G) = V(H) \cup V(F)$ and $E(G) = E(H) \cup E(F) \cup E^*$, where E^* is the set of all edges which join every vertex of W_i with every vertex of V_j, $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$. It is easy to see that H is an induced subgraph of G and from the previous lemma it follows that $\{W_1 \cup V_1, W_2 \cup V_2, \ldots, W_n \cup V_n\}$ is the unique vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition of $V(G)$. □

The relationship between the existence of uniquely partitionable graphs and the reducibility of properties leads us to study the existence of uniquely partitionable graphs. A number of necessary conditions for the existence of uniquely $\mathcal{P}^n$-partitionable graphs have been presented by Mihók in [8]. One of them states that if a hereditary property $\mathcal{P}$ is reducible, then $\mathbf{U}(\mathcal{P}^n) = \emptyset$ for every $n \geq 2$. The converse of this result is also true so that we now have

THEOREM 3. [10] *Let $\mathcal{P}$ be an additive and hereditary property of graphs. Then there exist a uniquely $\mathcal{P}^n$-partitionable graph for each $n \geq 2$ if and only if $\mathcal{P}$ is irreducible.*

Another important criterion for irreducibility was presented in [10] and [12].

THEOREM 4. [12] *A hereditary additive property $\mathcal{R}$ is reducible if and only if every $\mathcal{R}$-maximal graph with at least two vertices is decomposable.*

THEOREM 5. [10] *Let $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$, $n \geq 2$, be irreducible additive hereditary properties of graphs. Then there exists a uniquely $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$-partitionable graph.*

These theorems have been proved by analysing the structure of sets of generators of the properties involved. We now briefly summarise this idea for the proof of Theorem 5:

Consider the irreducible additive hereditary properties $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$, $n \geq 2$ and let $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \dots \circ \mathcal{P}_n$. Then the set of $\mathcal{R}$-maximal graphs is a generating set of $\mathcal{R}$. By Theorem 4 it follows that each $\mathcal{R}$-maximal graph is decomposable, hence a set $\mathcal{G}$ of decomposable generators can be found. Using technical lemmas (see [12]), it can now be shown that there exists a subset $\mathcal{G}^*$ of $\mathcal{G}$ which is a generating set of $\mathcal{R}$ and consists of graphs of decomposability number $\mathrm{dc}(\mathcal{R}) = n$. This means that every generator $F \in \mathcal{G}^*$ has a unique partition into n indecomposable graphs so that by Lemma 2 the graph F is uniquely $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$-partitionable.

We now investigate the existence of uniquely $(\mathcal{P}, \mathcal{Q})$-partitionable graphs in terms of divisibility of additive hereditary properties. In this connection, the following results are known.

LEMMA 4. [5] *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are properties of graphs such that at least one of the following holds:*

(1) *$\mathcal{P}$ is a nontrivial divisor of $\mathcal{Q}$.*

(2) *$\mathcal{Q}$ is a nontrivial divisor of $\mathcal{P}$.*

(3) *There exists a property $\mathcal{S}$ such that $\mathcal{S}$ is a nontrivial divisor of both $\mathcal{P}$ and $\mathcal{Q}$.*

Then $\mathbf{U}(\mathcal{P}, \mathcal{Q}) = \emptyset$.

The next theorem shows that these necessary conditions are sufficient, too.

THEOREM 6. *Let $\mathcal{P}$ and $\mathcal{Q}$ be any additive hereditary properties of graphs. Then there exists a uniquely $(\mathcal{P}, \mathcal{Q})$-partitionable graph if and only if $\gcd(\mathcal{P}, \mathcal{Q}) = \Theta$ or $\mathcal{P} = \mathcal{Q}$ is an irreducible property.*

Proof. Suppose there exists a uniquely $(\mathcal{P}, \mathcal{Q})$-partitionable graph. If $\mathcal{P} = \mathcal{Q}$, it follows immediately from Theorem 3 that $\mathcal{P} = \mathcal{Q}$ is an irreducible property. If $\mathcal{P} \neq \mathcal{Q}$, then $\gcd(\mathcal{P}, \mathcal{Q}) = \Theta$ follows from Lemma 4.

To prove the converse, Theorem 3 can be applied again if $\mathcal{P} = \mathcal{Q}$ is an irreducible property. Hence suppose that $\gcd(\mathcal{P}, \mathcal{Q}) = \Theta$. By Theorem 1 we may assume that $\mathcal{P} = \mathcal{P}_1^{\alpha_1} \circ \mathcal{P}_2^{\alpha_2} \circ \dots \circ \mathcal{P}_n^{\alpha_n}$ and $\mathcal{Q} = \mathcal{Q}_1^{\beta_1} \circ \mathcal{Q}_2^{\beta_2} \circ \dots \circ \mathcal{Q}_m^{\beta_m}$ are the unique factorizations of $\mathcal{P}$ and $\mathcal{Q}$ into irreducible properties. From our assumption that $\gcd(\mathcal{P}, \mathcal{Q}) = \Theta$ it then follows that $\mathcal{P}_i \neq \mathcal{Q}_j$ for all i, j, $i \in \{1, 2, \dots, n\}$, $j \in \{1, 2, \dots, m\}$. Thus for each such i and j there exists a graph H_{ij} such that $H_{ij} \in \mathcal{P}_i$ and $H_{ij} \notin \mathcal{Q}_j$ or there exists a graph K_{ji} such that $K_{ji} \in \mathcal{Q}_j$ and $K_{ji} \notin \mathcal{P}_i$. If such H_{ij} (K_{ji}) does not exist, then put $H_{ij} = K_0$ ($K_{ji} = K_0$).

Since the property $\mathcal{R} = \mathcal{P} \circ \mathcal{Q}$ is reducible, we may choose, as explained in our remarks following Theorem 5, a generating set $\mathcal{G}^* \subseteq \mathbf{M}^*(\mathcal{R})$ which consists entirely of $\mathcal{R}$-maximal graphs of decomposability number $s = \mathrm{dc}(\mathcal{R}) = \alpha_1 + \alpha_2 + \cdots + \alpha_n + \beta_1 + \beta_2 + \cdots + \beta_m$. For any generator F of $\mathcal{G}^*$ it then holds that F has a unique decomposition into indecomposable graphs F_k for $k = 1, 2, \ldots, s$. Thus, by the proof of Theorem 5, F is a uniquely $(\mathcal{P}_1, \mathcal{P}_1, \ldots, \mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n, \mathcal{Q}_1, \mathcal{Q}_1, \ldots, \mathcal{Q}_m)$-partitionable graph. (In an example following this proof it is shown why this proof cannot be completed here by taking F as the required uniquely $(\mathcal{P}, \mathcal{Q})$-partitionable graph.)

We now define a supergraph G_k of the indecomposable graph F_k as follows:

If $F_k \in \mathcal{P}_l$, let $G_k^* = F_k \cup \bigcup_{j=1}^{m} H_{lj}$ and if $F_k \in \mathcal{Q}_t$, let $G_k^* = F_k \cup \bigcup_{i=1}^{n} K_{ti}$ for $k = 1, 2, \ldots, s$. Now add new edges to G_k^* to form the graph G_k so that it is a $\mathcal{P}_i$-maximal graph if $F_k \in \mathcal{P}_i$ and a $\mathcal{Q}_j$-maximal graph if $F_k \in \mathcal{Q}_j$.

Now let $G = G_1 + G_2 + \cdots + G_s$. Since the generator F is a subgraph of G, it follows by Lemma 3 and the proof of Theorem 2 that G is a uniquely $(\mathcal{P}_1, \mathcal{P}_1, \ldots, \mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n, \mathcal{Q}_1, \mathcal{Q}_1, \ldots, \mathcal{Q}_m)$-partitionable graph. Suppose that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = t$, $\beta_1 + \beta_2 + \cdots + \beta_m = r$ and that $\{U_1, U_2, \ldots, U_t, V_1, V_2, \ldots, V_r\}$ is the only vertex $(\mathcal{P}_1, \mathcal{P}_1, \ldots, \mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n, \mathcal{Q}_1, \mathcal{Q}_1, \ldots, \mathcal{Q}_m)$-partition of G.

Now consider any vertex $(\mathcal{P}, \mathcal{Q})$-partition of G, say $\{W_1, W_2\}$. Since $G[W_1] \in \mathcal{P} = \mathcal{P}_1^{\alpha_1} \circ \mathcal{P}_2^{\alpha_2} \circ \cdots \circ \mathcal{P}_n^{\alpha_n}$, the set W_1 can be further partitioned into t sets inducing subgraphs which are in the properties $\mathcal{P}_i$. Similarly, the set W_2 can be partitioned into r sets inducing subgraphs which are in the properties $\mathcal{Q}_j$. By the above, these $t + r$ sets are exactly the sets $\{U_1, U_2, \ldots, U_t, V_1, V_2, \ldots, V_r\}$, but not necessarily in this order. We claim, however, that this partition of W_1 is exactly $\{U_1, U_2, \ldots, U_t\}$, since otherwise there are indices k and l such that U_k is not in this partition of W_1 (and hence it is in the partition of W_2), while V_l is in this partition of W_1. This, however, implies that there are indices i and j such that either $H_{ij} \in \mathcal{Q}_j$ (if $H_{ij} \neq K_0$) or $K_{ji} \in \mathcal{P}_i$ (if $K_{ji} \neq K_0$). In both cases this contradicts the choice of these graphs. Finally: Since this partition of W_1 is exactly $\{U_1, U_2, \ldots, U_t\}$, it follows that the partition of W_2 can only be $\{V_1, V_2, \ldots, V_r\}$. $\square$

The following example points out why a uniquely $(\mathcal{P}_1^{\alpha_1}, \mathcal{P}_2^{\alpha_2}, \ldots, \mathcal{P}_n^{\alpha_n}, \mathcal{Q}_1^{\beta_1}, \mathcal{Q}_2^{\beta_2}, \ldots, \mathcal{Q}_m^{\beta_m})$-partitionable graph is not necessarily a uniquely $(\mathcal{P}_1^{\alpha_1} \circ \mathcal{P}_2^{\alpha_2} \circ \cdots \circ \mathcal{P}_n^{\alpha_n}, \mathcal{Q}_1^{\beta_1} \circ \mathcal{Q}_2^{\beta_2} \circ \cdots \circ \mathcal{Q}_m^{\beta_m})$-partitionable graph. This can be shown as follows: Let T_i, $i = 1, 2, 3, 4$ be four pairwise non-isomorphic trees on k vertices and let

$\mathcal{P}_i = \{G \in \mathcal{I} \colon T_i \not\subseteq G\}$ for $i = 1,2,3,4$. Note that each $\mathcal{P}_i$ is irreducible and that $\gcd(\mathcal{P}_i, \mathcal{P}_j) = \Theta$ if $i \neq j$. Now consider the four graphs G_i, $i = 1,2,3,4$ with $G_1 = G_2 = G_3 = K_{k-1} \cup K_1$ and with $G_4 = K_{k-1}$, with vertex sets V_1, V_2, V_3 and V_4, respectively. Then, by the proof of Theorem 3.3 of [3], the graph $G = G_1 + G_2 + G_3 + G_4$ is a uniquely $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4)$-partitionable graph. However, G is not a uniquely $(\mathcal{P}_1 \circ \mathcal{P}_2, \mathcal{P}_3 \circ \mathcal{P}_4)$-partitionable graph, since the two partitions $\{V_1 \cup V_2, V_3 \cup V_4\}$ and $\{V_1 \cup V_3, V_2 \cup V_4\}$ are clearly both vertex $(\mathcal{P}_1 \circ \mathcal{P}_2, \mathcal{P}_3 \circ \mathcal{P}_4)$-partitions of G.

The argument used in the proof of Theorem 6 can now be repeated to yield

THEOREM 7. *Let $\mathcal{Q}_1, \mathcal{Q}_2, \dots, \mathcal{Q}_n$, $n \geq 2$, be any additive hereditary properties of graphs. Then there exists a uniquely $(\mathcal{Q}_1, \mathcal{Q}_2, \dots, \mathcal{Q}_n)$-partitionable graph if and only if for each $i, j \in \{1, 2, \dots, n\}$ with $i \neq j$, we have that $\gcd(\mathcal{Q}_i, \mathcal{Q}_j) = \Theta$ or $\mathcal{Q}_i = \mathcal{Q}_j$ is an irreducible property.*

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