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A NOTE ON A CONJECTURE CONCERNING TREE-PARTITIONING 3-REGULAR GRAPHS

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ABSTRACT. If G is a 4-connected maximal planar graph, then G is hamiltonian (by a theorem of Whitney), implying that its dual graph G^* is a cyclically 4-edge connected 3-regular planar graph admitting a partition of the vertex set into two parts, each inducing a tree in G^*, a so-called tree-partition. It is a natural question whether each cyclically 4-edge connected 3-regular graph admits such a tree-partition. This was conjectured by Jaeger, and recently independently by the first author. The main result of this note shows that each connected 3-regular graph on n vertices admits a partition of the vertex set into two sets such that precisely $\frac{1}{2}n+2$ edges have end vertices in each set. This is a necessary condition for having a tree-partition. We also show that not all cyclically 3-edge connected 3-regular (planar) graphs admit a tree-partition, and present the smallest counterexamples.

1. Preliminaries

For any terminology or notation not defined here we refer to [1].

All graphs considered are finite and without loops, but we allow multiple edges. In the latter case we speak about multigraphs. Given a (multi)graph G and subsets S and T of $V(G)$, we shall use $G[S]$ to denote the subgraph of G induced by S, and $e[S,T]$ to denote the number of edges (with their multiplicities in case of multiple edges) in $[S,T]$ joining vertices of S with vertices of T. As usual $\lambda(K_1) = 0$ and if $G \neq K_1$, $\lambda(G)$ is equal to the minimum size of an edge cut of G and G is *k-edge connected* if $\lambda(G) \geq k$, where k is an integer; $\lambda_e(G)$ is equal to the minimum size of an edge cut $[S,\overline{S}]$ such that both $G[S]$ and $G[\overline{S}]$ have at least one edge if such an edge cut exists and $\lambda(G)$ otherwise. Analogously $\lambda_c(G)$ is equal to the minimum size of an edge cut $[S,\overline{S}]$ such that both $G[S]$ and $G[\overline{S}]$ contain a cycle if such an edge cut exists and $\lambda_e(G)$ otherwise. G is called *essentially k-edge connected* if

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$\lambda_e(G) \ge k$ and *cyclically k-edge connected* if $\lambda_c(G) \ge k$, where k is an integer. Clearly, $\lambda_c(G) \ge \lambda_e(G) \ge \lambda(G)$ for any graph G.

A *tree-partition* of a graph $G = (V, E)$ is a pair (T_1, T_2) such that $(V_1, V_2) := \big(V(T_1), V(T_2)\big)$ is a partition of V and both $T_1 = G[V_1]$ and $T_2 = G[V_2]$ are trees. We will also say that (V_1, V_2) is a tree-partition of G if (V_1, V_2) is a partition of V such that $T_i := G[V_i]$ is a tree ($i = 1, 2$). Note that not every graph has a tree-partition (for example K_6).

The dual graph of a planar graph G will be denoted by G^*.

A natural class of graphs that admit a tree-partition is obtained from the class of 4-connected maximal planar graphs by taking their duals.

Theorem 1. *Every 3-regular cyclically 4-edge connected planar graph has a tree-partition.*

The proof follows from the following two well-known lemmas, and a theorem of Whitney [5].

Lemma 2. *Let G be a 3-regular cyclically 4-edge connected planar graph. Then G^* is 4-connected.*

Lemma 3. *A planar graph is hamiltonian if and only if the dual graph has a tree-partition.*

Theorem 4. *A 4-connected maximal planar graph has a Hamilton cycle.*

Now we can prove Theorem 1. Let G be a 3-regular cyclically 4-edge connected planar graph. Then it is clear that the dual G^* is maximal planar and (Lemma 2) G^* is 4-connected. By Theorem 4, G^* has a Hamilton cycle. Thus by Lemma 3, G has a tree-partition.

It is a natural question whether each cyclically 4-edge connected 3-regular graph admits a tree-partition. This was recently conjectured by the first author, but has been conjectured before by Jaeger [2].

Conjecture 5. *Every 3-regular cyclically 4-edge connected graph has a tree-partition.*

In this note we give some partial results that may turn out to be helpful in proving or disproving the conjecture. We start with the following easy observations.

Proposition 6. *If a 3-regular graph has a tree-partition (V_1, V_2), then $G[V_i]$ is connected ($i = 1, 2$), $e[V_1, V_2] = \frac{1}{2}n + 2$ and $|V_1| = |V_2| = \frac{1}{2}n$.*

Proof. Let $G = (V, E)$ be a 3-regular graph with a tree-partition (V_1, V_2). Then $G[V_i]$ is a tree, so $G[V_i]$ is connected ($i = 1, 2$). Define $n := |V|$ and $T_i := G[V_i]$, $n_i := |V_i|$, $x_i :=$ number of vertices in T_i with degree 1 in T_i, $y_i :=$

number of vertices in T_i with degree 2 in T_i, $z_i :=$ number of vertices in T_i with degree 3 in T_i for $i = 1,2$. Then $n_1 + n_2 = n$, $|E| = \frac{3}{2}n$ and $|E(T_i)| = n_i - 1$ for $i = 1,2$. Because $G[V_i]$ is a tree $(i = 1,2)$, all edges of G not in T_1 or T_2 are between V_1 and V_2. So $e[V_1, V_2] = |E| - (n_1 - 1) - (n_2 - 1) = \frac{1}{2}n + 2$. It is easily seen that the number of edges between T_1 and T_2 is equal to $2x_1 + y_1 = 2x_2 + y_2$. Furthermore, since $n_i \geq 2$, $x_i = 2 + z_i$ for $i = 1,2$, so $n_1 = x_1 + y_1 + z_1 = x_1 + y_1 + x_1 - 2 = 2x_1 + y_1 - 2 = 2x_2 + y_2 - 2 = x_2 + y_2 + x_2 - 2 = x_2 + y_2 + z_2 = n_2$. □

Proposition 7. *Every connected graph on at least two vertices has a partition (V_1, V_2) of V such that $G[V_i]$ is connected $(i = 1,2)$.*

Proof. Let $[V_1, V_2]$ be a minimal edge cut. Then $G[V_i]$ is connected $(i = 1,2)$. □

In the next section we deduce that all connected 3-regular (multi)graphs satisfy the following necessary condition for the existence of a tree-partition, supporting Conjecture 5.

Theorem 8. *Every connected 3-regular multigraph G on n vertices admits a partition (V_1, V_2) of $V(G)$ such that $e[V_1, V_2] = \frac{1}{2}n + 2$.*

For convenience we call such a partition a *fair* partition, and we omit the brackets in the notation.

2. Proof of Theorem 8

We need some auxiliary definitions and results.

We call a K_3 with precisely one edge e duplicated (of multiplicity 2) a *cone*, the vertex of K_3 not incident with e the *end vertex* of the cone, and the other two vertices of K_3 the *internal vertices* of the cone. For any integer $k \geq 2$, a *k-cone* is defined as a 3-regular multigraph obtained from k disjoint cones and a tree with k end vertices and all internal vertices of degree 3, by identifying each end vertex of the tree with an end vertex of one of the cones.

Proposition 9. *Any k-cone $(k > 2)$ can be obtained from a $(k-1)$-cone by subdividing an edge e not belonging to a cone with one new vertex, and making the new vertex adjacent to an end vertex of a newly added cone.*

Proof. Let G be a k-cone $(k > 2)$ and let C be a cone of G with end vertex v. Clearly $G - V(C)$ contains one vertex u of degree 2, and contracting one of the edges incident with u yields a $(k-1)$-cone H. Hence G can be obtained from H by the reverse procedure. □

Lemma 10. *All k-cones ($k \geq 2$) have a fair partition.*

P r o o f . We use induction on k. For the 2-cone G one readily checks that the partition of $V(G)$ in S, T, where S and T both contain exactly one end vertex of a cone and the two other vertices of the other cone, is a fair partition. Assume all j-cones with $2 \leq j < k$ have a fair partition, and consider a k-cone G. Let C be a cone of G and let G be obtained from a $(k-1)$-cone H by adding C to $e = uv$ in the sense of Proposition 9. Suppose S, T is a fair partition of H, which exists by the induction hypothesis. Without loss of generality we may assume that at least one of the vertices u and v belongs to S. Then adding the two internal vertices of C to T and the other two new vertices to S yields a fair partition of G. $\square$

Lemma 11. *Let G be a connected 3-regular multigraph on $n \geq 4$ vertices. If G is not a k-cone, then G contains an edge which is not a bridge and does not belong to a cone of G.*

P r o o f . Clearly any connected 3-regular multigraph contains an edge that is not a bridge. Suppose all nonbridges of G are cone edges. Then G has at least one bridge, because otherwise G cannot have any cones (the edge of $G - E(C)$ incident with the end vertex of a cone C of G is a bridge of G), a contradiction. Now, since all noncone edges of G are bridges of G, the noncone edges of G induce a tree in G in which all internal vertices have degree 3 and all end vertices correspond to end vertices of cones of G. Then G is a k-cone. $\square$

Theorem 12. *Every connected 3-regular multigraph G admits a fair partition.*

P r o o f . By induction on $n = |V(G)|$. The statement of the theorem is clearly true for $n = 2$. Suppose $n \geq 4$, and assume every connected 3-regular multigraph on less than n vertices has a fair partition. If G is a k-cone for some $k \geq 2$, we are done by Lemma 10. Next assume G is not a k-cone and let e be a nonbridge of G which does not belong to a cone of G (such an edge exists by Lemma 11). Let C be the smallest cycle of G containing e.

We first deal with the case that $|V(C)| = 2$. Since $e = xy$ is not in a cone, the two neighbours u and v of respectively x and y in $V(G) \setminus V(C)$ do not coincide. Consider the connected 3-regular multigraph H obtained from $G - V(C)$ by adding an edge uv (possibly $uv \in E(G)$). Clearly $|V(H)| = |V(G)| - 2$, hence H has a fair partition S, T. If u and v belong to different parts, say $u \in S$ and $v \in T$, then clearly $S \cup \{x\}$, $T \cup \{y\}$ is a fair partition of G. Hence assume without loss of generality that $\{u, v\} \subset S$, implying $|V(H)| \neq 2$. Since H is connected and $T \neq \emptyset$, without loss of generality assume there is a path P in $G[S]$ from u to some p (possibly $p = u$) such that $v \notin V(P)$, at least one neighbour of p is in T, and all neighbours of other vertices of P (if any) are in S. If precisely one neighbour of p is in T, then $\big(S \setminus \{p\}\big) \cup \{x, y\}$, $T \cup \{p\}$ is a fair partition of G. Suppose now that two neighbours of p are in T. If $|V(P)| \geq 3$, then let q denote the neighbour of p in P, and note that $S \setminus \{p, q\}$, $T \cup \{p, q\}$ is a fair partition of H which can be used to define a fair partition of G as in the previous argumentation. If $|V(P)| = 2$ ($P = up$), then $\big(S \setminus \{u, p\}\big) \cup \{y\}$, $T \cup \{u, p, x\}$ is a fair partition of G. If $V(P) = 1$, we consider the two neighbours q_1, q_2 of $u = p$ in T (possibly $q_1 = q_2$), and the neighbours r_1, r_2 of q_1 in $V(G) \setminus \{u\}$ (possibly $r_1 = r_2$). If $q_1 = q_2$ and $r_1 = r_2$, then in the case $r_1 \in T$, $S \cup \{q_1\}$, $\big(T \setminus \{q_1\}\big) \cup \{x, y\}$ is a fair partition of G, and in the case $r_1 \in S$, $\big(S \setminus \{u\}\big) \cup \{y, q_1\}$, $\big(T \setminus \{q_1\}\big) \cup \{u, x\}$ is; otherwise, if $r_1, r_2 \in T$, then $S \cup \{x, y, q_1\}$, $T \setminus \{q_1\}$ is a fair partition of G; if $r_1, r_2 \in S$, then $\big(S \setminus \{u\}\big) \cup \{x, q_1\}$, $\big(T \setminus \{q_1\}\big) \cup \{u, y\}$ is; in all other subcases, $S \cup \{q_1\}$, $\big(T \setminus \{q_1\}\big) \cup \{x, y\}$ is a fair partition of G.

For the remaining cases we may assume that, apart from the cones, there is no C_2 in G. Let $e = xy$ and $N(x) \setminus \{y\} = \{a, b\}$, $N(y) \setminus \{x\} = \{c, d\}$ (note that $a \neq b$, $c \neq d$, but possibly $a = c$ etc.). Consider the connected 3-regular multigraph H obtained from $G - \{x, y\}$ by adding the edges ab and cd (possibly $ab \in E(G)$ or $cd \in E(G)$). Clearly H has a fair partition S, T. If not all of $\{a, b, c, d\}$ belong to the same part, it is quite easy to check that $S \cup \{x\}$, $T \cup \{y\}$ is a fair partition of G. We are left with the case that $\{a, b, c, d\} \subset S$ (or T). Since H is connected and $T \neq \emptyset$, without loss of generality assume there is a path P in $G[S]$ from a to some p (possibly $a = p$) such that $V(P) \cap \{b, c, d\} \setminus \{a\} = \emptyset$, at least one neighbour of p is in T, and all neighbours of the other vertices of P (if any) are in S. If precisely one neighbour of p is in T, then $\big(S \setminus \{p\}\big) \cup \{x, y\}$, $T \cup \{p\}$ is a fair partition of G. Suppose now that two neighbours of p are in T. If $|V(P)| \geq 2$, then let q denote the neighbour of p in P, and note that $S \setminus \{p, q\}$, $T \cup \{p, q\}$ is a fair partition of H which can be used to define a fair partition of G as in the previous argumentation. If $|V(P)| = 1$, then $S \setminus \{a\}$, $T \cup \{a, x, y\}$ is a fair partition of G. $\square$

3. Final remarks

Suppose G is a connected 3-regular graph.

It is clear that if $\lambda_c(G) = \lambda(G) = 1$, then G does not admit a tree-partition, since each of the two components of $G - e$, where e is a bridge of G, contains a cycle, and hence e should be in both trees.

Figure 1 shows two graphs with $\lambda_c = \lambda = 2$. The left one has a tree-partition, indicated by the black and white vertices; the right one does not admit a tree-partition: this is easily seen using the following general observation.

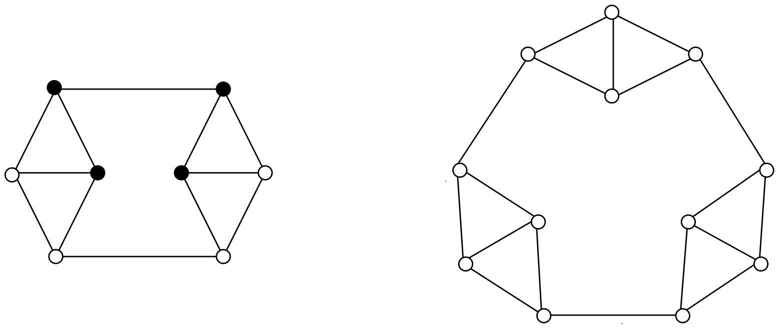

FIGURE 1. The left graph has a tree-partition, the right one has none.

PROPOSITION 13. *If $\{e, f\}$ is a minimum edge cut of G, and S, T is a tree-partition, then $e \in G[S]$ and $f \in G[T]$ or vice versa.*

P r o o f . Both components of $G - \{e, f\}$ contain a cycle, so both $G[S]$ and $G[T]$ must contain at least one vertex of each of these components. □

Using Lemma 3 one can construct (planar) examples with $\lambda_c = \lambda = 3$ not admitting a tree-partition, starting with nonhamiltonian maximal planar graphs with connectivity 3. For example, deleting the five black vertices from the left graph in Figure 2 yields six components, hence G is not hamiltonian, and hence the right graph, which is the dual of the left one, does not admit a tree-partition. A nonplanar example with $\lambda_c = 3$ not admitting a tree-partition is the graph obtained from a $K_{3,3}$ by replacing each vertex v by a triangle $v_1v_2v_3v_1$ and joining each of the v_i's with one of the vertices of each of the three triangles corresponding to the neighbours of v in $K_{3,3}$ (the so-called *inflation* of $K_{3,3}$).

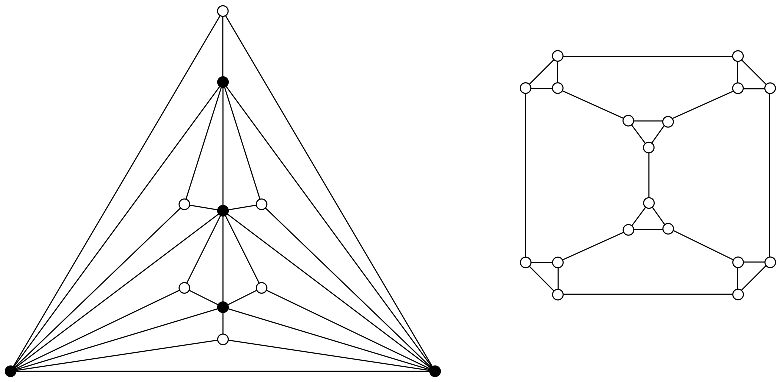

FIGURE 2. The right graph is the dual of the left graph and admits no tree-partition.

The two examples with $\lambda_c = 3$ were the smallest examples found by an exhaustive search by computer, using a self-written program in `Mathematica` and relying heavily on algorithms of the software packages `Combinatorica` and `GENREG`, which were found on the Internet. The `Combinatorica` package due to Skiena [4] is a `Mathematica` package for Combinatorics and Graph Theory with over 230 functions that can be downloaded from WWW `http://www.mathsource.com/cgi-bin/MathSource/Enhancements/DiscreteMath/0200-170`. A description of this package is [4]. The C-program `GENREG` due to Meringer [3] can generate all connected k-regular graphs on n vertices quite fast. The format of the generated graphs can be adapted in such a way that it can be used directly as input for `Combinatorica`. `GENREG` can be downloaded from WWW `http://www.mathe2.uni-bayreuth.de/markus/reggraphs.html`.

REFERENCES

[1] BONDY, J. A.—MURTY, U. S. R.: *Graph Theory with Applications*, Macmillan and Elsevier, London and New York, 1976.

[2] JACKSON, B.: *Personal communication.*

[3] MERINGER, M.: *Erzeugung regulärer Graphen*, Master's Thesis, Universität Bayreuth, Germany, 1996.

[4] SKIENA, S.: *Implementing Discrete Mathematics: Combinatorics and Graph Theory with Mathematica*, Addison-Wesley, New York, 1990.

[5] WHITNEY, H.: *A theorem on graphs*, Ann. Math. **32** (1931), 378–390.

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