Tatra Mt. Math. Publ. **17** (1999), 295–300

TATRA MOUNTAINS
Mathematical Publications

ROBUSTNESS OF THE TEST IN MULTIVARIATE STANDARD MODEL

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ABSTRACT. In this paper two fixed, linear multivariate models, standard MS=$(U, XB, \sigma^2\Sigma \otimes I)$ and Aitken MA= $(U, XB, \sigma^2\Sigma \otimes V)$, are considered. The following problem was stated: under which conditions tests for linear hypothesis $ABC = 0$ are identical in models MS and MA, respectively. Solution of this problem is obtained in two ways: by developing the model onto univariate one and by means of the projector operators.

1. Introduction

Let us consider the fixed, linear, multivariate standard model MS of the form:

$$(U, XB, \sigma^2\Sigma \otimes I), \tag{1.1}$$

where U is a random matrix of n observations of p features with expected value XB and covariance matrix $\sigma^2\Sigma\otimes I$. X denotes a known matrix of experimental plan with dimension nxq; B — unknown matrix of parameters with dimensions qxp and $\Sigma \geq 0$ correlation matrix of features with dimensions pxp. Symbol $\otimes$ stands for Kronecker product of a matrix.

Simultaneously with the model MS we will consider multivariate fixed, linear Aitken model MA of the form:

$$(U, XB, \sigma^2\Sigma \otimes V). \tag{1.2}$$

Here $V > 0$ stands for the known correlation matrix of observations and other symbols are the same as in (1.1).

Now let the linear hypothesis H_0: $ABC = 0$ testable in models MS and MA be given. Let test for verifying this hypothesis be L_I and L_V, respectively. The main problem of this paper is to find conditions under which the equality $L_I = L_V$ holds.

2000 Mathematics Subject Classification: 62H15.
Keywords: multivariate linear model, development of a matrix, Kronecker product, likelihood ratio test.

2. Notations, lemmas and auxiliary facts

In the subsequent part of this paper the following notations will be used:

P'	transpose of matrix P,
$r(P)$	rank of a matrix P,
P^-	g-inverse to a matrix P,
$\lvert P \rvert$	determinant of a matrix P,
$R(P)$	column space of a matrix P,
$P^{\perp}$	orthogonal complement of a matrix P,
I_n	identity matrix with dimension nxn,

$X_0 = X(I - A^- A)$,

$Z = X^{\perp}$,

$Z_0 = (Z \vdots Z_1) = X_0^{\perp}$,

$P_{H,G} = H(H'GH)^- H'G$, where $G > 0$,

$P_H = P_{H,I} = H(H'H)^- H'$.

Definition 2.1. Let $P = [p_1, p_2, \ldots, p_p]$ be a matrix with dimensions $n \times p$, whose columns are vectors $p_1, p_2, \ldots, p_p$. Then $\operatorname{vec}(P)$ (development of P) denotes vector with dimensions $np \times 1$ of the form:

$$\operatorname{vec} P = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_p \end{bmatrix} . \tag{2.1}$$

The properties of the operation of development of a matrix and Kronecker product of a matrix give the following lemma:

Lemma 2.1. (Neudecker [3]) *Let Q, R, S, T denote such matrices that suitable matrix products exist. Then the following equalities hold:*

$$\operatorname{vec}(QRS) = (S' \otimes Q) \operatorname{vec} R \,, \tag{2.2}$$

$$(Q \otimes R)(S \otimes T) = (QS \otimes RT), \tag{2.3}$$

$$(Q \otimes R)' = Q' \otimes R' \,, \tag{2.4}$$

$$(Q \otimes R)^- = Q^- \otimes R^- , \tag{2.5}$$

$$k(S \otimes T) = (kS) \otimes T = S \otimes (kT) \,, \text{ where } k \text{ denotes a constant} \,, \tag{2.6}$$

$$P_{S \otimes T} = P_S \otimes P_T \,. \tag{2.7}$$

3. Development of models

We will consider the multivariate models MS and MA and the hypothesis $H_0\colon ABC = 0$ testable in both models. Taking the hypothesis into consideration leads us to the models MSH and MAH, respectively:

$$(U, XB, \sigma^2\Sigma \otimes I \,|\, ABC = 0)\,, \tag{3.1}$$

$$(U, XB, \sigma^2\Sigma \otimes V \,|\, ABC = 0)\,. \tag{3.2}$$

Let us apply the operation of development of matrix to the models (3.1) and (3.2). Then we get the following univariate models

$$\mathrm{MSDH} = (Y_D, X_D\beta_D, \sigma^2 V_D \,|\, A_D\beta_D = 0)\,, \tag{3.3}$$

$$\mathrm{MADH} = (Y_D, X_D\beta_D, \sigma^2 V_{1D} \,|\, A_D\beta_D = 0)\,. \tag{3.4}$$

In the formulas (3.3) and (3.4) we accept the following notations:

$$Y_D = \operatorname{vec} U\,, \qquad X_D = I \otimes X\,, \qquad \beta_D = \operatorname{vec} B\,, \tag{3.5}$$

$$V_D = \Sigma \otimes I\,, \qquad V_{1D} = \Sigma \otimes V\,, \qquad A_D = C' \otimes A\,. \tag{3.6}$$

Moreover, let us assume that:

$$X_{0D} = X_D(I - A_D^- A_D)\,, \tag{3.7}$$

$$Z_D = X_D^\perp\,, \qquad Z_{0D} = X_{0D}^\perp = (Z_D \,\vdots\, Z_{1D})\,. \tag{3.8}$$

4. Main results

Theorem 4.1. *Let the multivariate models* MS (1.1) *and* MA (1.2) *be given. Tests for hypothesis $H_0\colon ABC = 0$ are identical in both models if and only if any of the three following conditions is fulfilled*

$$(I - P_{X_{0D}})(V_D - kV_{1D})(I - P_{X_{0D}}) = 0\,, \quad \text{for some} \quad k > 0\,, \tag{4.1}$$

$$\begin{pmatrix} I - P_{X_{D,E}} \\ LP_{X_{D,E}} \end{pmatrix}(V_D - kV_{1D})(I - P'_{X'_{D,E}} \,\vdots\, P'_{X'_{D,E}} L') = 0 \ \text{for some} \ k > 0\,, \tag{4.2}$$

where L is any matrix satisfying condition $LX_D = A_D$ and E any g-inverse to the matrix $(V_{1D} + X_D X'_D)$:

$$\begin{aligned} V_{1D} = {} & X_D D_1 X'_D + kV_{1D}Z_D(Z'_D V_{1D} Z_D)^- Z'_D V_{1D} \\ & + X_{0D} D_3 Z'_D V_{1D} + V_{1D} Z_D D'_3 X'_{0D}\,, \end{aligned} \tag{4.3}$$

where D_1 *and* D_3 *are any matrices and* k *any real values such that:*

$$V_D \geq 0\,; \tag{a}$$

$$Z'_{1D}X_D D_1 X'_D Z_{1D} = kZ'_{1D}(V_{1D} - V_{1D}Z_D(Z'_D V_{1D} Z_D)^- Z'_D V_{1D})Z_{1D}\,; \tag{b}$$

$$R(X_D D_1 X'_D) \subset R(X_{0D} \vdots V_{1D})\,. \tag{c}$$

P r o o f. We get the assertion of this theorem by straightforward application of Theorem 3.2 from paper by M a t h e w and B h i m a s a n k a r a m [2] to univariate model MSDH and MSAH. □

Conditions given in the theorem (4.1) are very complicated. The simplest of them is (4.1). Further on we will examine it in detail.

Corollary 4.1. *Let the multivariate models* MS (1.1) *and* MA (1.2) *be given. Tests for hypothesis* H_0: $ABC = 0$ *are identical in both models if and only if the following condition is satisfied:*

$$\big(I - P_{I\otimes X - C'^- C'\otimes XA^- A}\big)\big(\Sigma \otimes (I - kV)\big)\big(I - P_{I\otimes X - C'^- C'\otimes XA^- A}\big) = 0 \quad \text{for some} \quad k > 0\,. \tag{4.4}$$

P r o o f. Applying (3.6), (3.7) and Lemma 2.1 we get successively:

$$V_D - VkV_{1D} = (\Sigma \otimes I - k\Sigma \otimes V) = \Sigma \otimes I - \Sigma \otimes kV = \Sigma \otimes (I - kV)\,.$$

Further, (3.5), (3.6), (3.7) and Lemma 2.1 give:

$$\begin{aligned} X_{0D} &= X_D(I - A_D^- A_D) = (I \otimes X)\big[I - (C' \otimes A)^-(C' \otimes A)\big] \\ &= (I \otimes X)\big[I - (C'^- \otimes A^-)(C' \otimes A)\big] \\ &= (I \otimes X)\big[I - C'^- C' \otimes A^- A\big] = I \otimes X - C'^- C' \otimes XA^- A\,. \end{aligned} \tag{4.5}$$

□

Now we will show that for special hypothesis $AB = 0$ ($C = I_p$) condition (4.4) may be much more simplified.

Corollary 4.2. *Let the multivariate models* MS (1.1) *and* MA (1.2) *be given. Tests for hypothesis* H_0: $AB = 0$ *are identical in both models if and only if the following condition is satisfied:*

$$(I - P_{X_0})(I - kV)(I - P_{X_0}) = 0 \quad \text{for some} \quad k > 0\,. \tag{4.6}$$

P r o o f. Putting $C = I_p$ we get from (4.5):

$$\begin{aligned} X_{0D} &= I \otimes X - C'^- C' \otimes XA^- A = I \otimes X - I \otimes XA^- A \\ &= I \otimes X(I - A^- A) = I \otimes X_0\,. \end{aligned} \tag{4.7}$$

From (2.7) and (4.7) we have $P_{X_{0D}} = P_{I\otimes X_0} = P_I \otimes P_{X_0}$.
Condition (4.1) acquires now the following form:

$$\begin{aligned}
&(I_{np} - P_{X_{0D}})\big(\Sigma \otimes (I - kV)\big)(I_{np} - P_{X_{0D}}) = 0\,,\\
&(I_{np} - I_p \otimes P_{X_0})\big(\Sigma \otimes (I - kV)\big)(I_{np} - I_p \otimes P_{X_0}) = 0\,,\\
&(I_p \otimes I_n - I_p \otimes P_{X_0})\big(\Sigma \otimes (I - kV)\big)(I_p \otimes I_n - I_p \otimes P_{X_0}) = 0\,,\\
&\big[I_p \otimes (I_n - P_{X_0})\big]\big[\Sigma \otimes (I - kV)\big]\big[I_p \otimes (I_n - P_{X_0})\big] = 0\,,\\
&\Sigma \otimes (I_n - P_{X_0})(I - kV)(I_n - P_{X_0}) = 0\,.
\end{aligned} \tag{4.8}$$

Because $\Sigma \neq 0$, (4.6) results from (4.8). □

Let us note that condition (4.6) does not depend on matrix Σ and means identity of tests for hypothesis $A\beta = 0$ in univariate models $(y, X\beta, \sigma^2 I)$ and $(y, X\beta, \sigma^2 V)$.

Lemma 4.1. *Let the multivariate models* MS (1.1) *and* MA (1.2) *be given. If tests for hypothesis* $H_0\colon AB = 0$ *are identical in both models* $((L_I = L_V) = 1)$ *then* BLUE *for* ABC*in model* MS *is the same as in model* MA.

P r o o f . After developing the models MS and MA and applying Lemma 3.1 from M a t h e w and B h i m a s a n k a r a m 's paper [2] to univariate models we get the required result. □

5. Operators approach

Now we will try to solve stated problem in a different way. It is well known that the classical multivariate tests such as Hotelling's, Wilk's, Roy's and Pillay's tests are based on two quadratic forms: for hypothesis S_H and for error S_E, respectively. Starting with this fact we get the following theorem.

Theorem 5.1. *Tests for hypothesis* $AB = 0$ *are identical in models* MS *and* MA *if the following conditions are satisfied:*

$$I - P_X = V^{-1}\big(I - P_{X,V^{-1}}\big)\,, \tag{5.1}$$

$$P_{X,Z^*} = V^{-1} P_{XZ,V^{-1}}\,, \tag{5.2}$$

where Z^* *and* Z *are any matrices such that:*

$$A' = X'XZ^*\,, \tag{5.3}$$

$$A' = X'V^{-1}XZ\,. \tag{5.4}$$

P r o o f. O k t a b a and T a r a s i ń s k a [4] (see also C h r i s t e n s e n [1]) have proved that quadratic forms S_H and S_E for hypothesis and for error, respectively, may be expressed by means of projector operators. They gave the following results for MS:

$$S_{E,I} = U'(I - P_X)U\,, \tag{5.5}$$

$$S_{H,I} = U'P_{X,Z^*}U\,. \tag{5.6}$$

For MA we have, respectively:

$$S_{E,V} = U'V^{-1}(I - P_{X,V^{-1}})U\,, \tag{5.7}$$

$$S_{H,V} = U'V^{-1}P_{XZ,V^{-1}}U\,, \tag{5.8}$$

where Z^* and Z satisfy (5.3) and (5.4).

By comparing (5.5) with (5.7) and (5.6) with (5.8) we get the requested results. □

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Received May 4, 1998

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