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A NOTE ON MANOVA IN THE EXTENDED GROWTH CURVE MODEL

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ABSTRACT. We consider a multivariate mixed effect (extended growth curve) model of the form $Y = XBZ + \sum_{i=1}^{k-1} X_i B_i Z_i + \varepsilon$, $\mathrm{E}\, B_i = 0\ \forall i$, $\operatorname{var}(\operatorname{vec} B_i) = \Sigma_i \otimes I$ $\forall i$, $\mathrm{E}\varepsilon = 0$, $\operatorname{var}(\operatorname{vec}\varepsilon) = \Sigma \otimes I$, where $B_1, \dots, B_{k-1}$, and ε are independent random variables. A general theorem about the existence of MANOVA in this model under normality is proved and discussed.

The extended growth curve model (EGCM) with covariance components is a multivariate regression model defined by the relations:

$$\begin{aligned} Y &= XBZ + \sum_{i=1}^{k-1} X_i B_i Z_i + \varepsilon\,, \\ \mathrm{E} B_i &= 0\ \ \forall i\,, \quad \operatorname{var}(\operatorname{vec} B_i) = \Sigma_i \otimes I\ \ \forall i\,, \\ \mathrm{E}\,\varepsilon &= 0\,, \quad \operatorname{var}(\operatorname{vec}\varepsilon) = \Sigma \otimes I\,, \end{aligned} \tag{1}$$

where $B_1, \dots, B_{k-1}$, and ε are independent random variables.

Here Y is a $n \times p$ matrix of p-dimensional observations, X, Z, X_1, Z_1, $\dots, X_{k-1}, Z_{k-1}$ are known design matrices of types $n \times m$, $r \times p$, $n \times m_1$, $r_1 \times p$, $\dots$, $n \times m_{k-1}$, $r_{k-1} \times p$ respectively, B is a $m \times r$ matrix of the fixed parameters, $B_1, \dots, B_{k-1}$ are $m_i \times r_i$ matrices of the random parameters, ε is a $n \times p$ random error matrix and the matrices $\Sigma, \Sigma_1, \dots, \Sigma_{k-1}$ are of the type $p \times p$, $r_1 \times r_1, \dots, r_{k-1} \times r_{k-1}$ respectively. The sign $\otimes$ denotes the Kronecker product.

Moreover, throughout this paper we suppose that the B_i's and ε are normally distributed.

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Thus $\mathrm{E}Y = XBZ$ and $\mathrm{vec}(Y)$ has variance matrix

$$\sum_{i=1}^{k-1} Z_i'\Sigma_i Z_i \otimes X_i X_i' + \Sigma \otimes I = \sum_{i=1}^{k} V_i \otimes W_i \,.$$

Suppose that Σ_i's are nonnegative definite matrices and $\sum_{i=1}^{k} V_i \otimes W_i$ is a positive definite matrix.

Let P_X be the projection matrix onto the column space of X and $M_X = I - P_X$.

Let us notice that this model is a multivariate mixed effect model which also includes, as special cases, a multivariate random and/or fixed coefficients regression model, as was shown and discussed by Wesołowska-Janczarek [W]. In the case of the random coefficients regression model we may restrict ourselves to $k = 1$, but since our goal is to study the MANOVA partitions we shall maintain a general k. Different aspects of the present model have been studied by Khatri and Shah [KS] and others; similar models have been studied by von Rosen [R1], [R2], [R3], [R4], [R5], Mathew [M1] and others. This model allows a simultaneous study of MANOVA and (random) regression coefficients; usually the X's contain indicator variables and the Z's regression variables. A slightly less general model was considered by Mathew [M1]. The aim of the present paper is to extend his results concerning MANOVA to our model.

Following Brown [B] and Mathew [M1] we define:

Definition 1. A MANOVA is defined as a partition $Y'Y = \sum_{i=1}^{u} Y'Q_iY$ satisfying:

a) $Q_1, \dots, Q_u$ are nonnegative definite matrices.
b) $Y'Q_1Y, \dots, Y'Q_uY$ are independent.
c) $Y'Q_1Y, \dots, Y'Q_uY$ have Wishart distribution.
d) $Y'Q_1Y, \dots, Y'Q_uY$ have expectations which are different parametric functions apart from being a known scalar multiple.

If Q is a nonnegative definite matrix, then a partition $Y'QY = \sum_{i=1}^{u} Y'Q_iY$ satisfying the above four properties is called a partial MANOVA.

The existence of a MANOVA allows us to make inferences about various parametric functions. The following theorem enables us to find MANOVA partitions in some situations. The proof is carried out in the same manner as in Brown [B] and Mathew [M1].

Theorem 1. *Let the model* (1) *be given with X having full rank and with $B_1, \dots, B_{k-1}$ and ε having normal distribution. Let $P_X = \sum_{i=1}^{t} P_i$, where P_i are* *mutually orthogonal projection matrices. Then there exists the unique partition* $M_X = \sum_{j=1}^{s} M_j$ *such that* M_j *are also mutually orthogonal and*

$$Y'Y = \sum_{i=1}^{t} Y'P_iY + \sum_{j=1}^{s} Y'M_jY$$

is a MANOVA *iff*

(1) $P_1, \dots, P_t, W_1, \dots, W_k$ *commute and*

(2) *there exist nonnegative real numbers* c_{ij} *satisfying* $P_iW_jP_i = c_{ij}P_i$ *for all* $i = 1, \dots, t,\ j = 1, \dots, k$.

P r o o f . *Sufficiency.* Because $P_1, \dots, P_t, W_1, \dots, W_k$ commute, there must exist an orthogonal matrix S such that matrices $S'P_iS$ and $S'W_jS$ are diagonal with the eigenvalues of P_i and W_j on the diagonals $\forall i, j$. Without loss of generality, we can assume that the eigenvalues are ordered and that S is partitioned into $t+1$ matrices in such a way that: $S = (S_1, \dots, S_t, S_o)$, $S_iS_i' = P_i\ \ \forall i$ and $S_oS_o' = M_X$. Let us observe that the matrices $S_1, \dots, S_t, S_o$ are semiorthogonal, orthogonal to each other. Because all P_i's and W_j's commute, P_X and M_X must also commute with all W_j's. It follows that $W_jS_o = S_oS_o'W_jS_o = S_oD_j$, where D_j is a diagonal matrix of dimension $n - r(X)$; and even that $D_k = I_{n-r(X)}$. Let $D^* = (\operatorname{diag}\ D_1, \dots, \operatorname{diag}\ D_k)$ and let s be the number of distinct rows of D^*. Let us denote by $D = \{d_{ij}\}$ the matrix consisting of these rows of D^*. Since the eigenvalues are ordered, there must exist a partition $S_o = (S_{o1}, \dots, S_{os})$ such that $W_jS_o = (d_{1j}S_{o1}, \dots, d_{sj}S_{os})$. Then $W_jS_{ou} = d_{uj}S_{ou}$, $S_{ou}'W_jS_{ou} = d_{uj}I$ and $S_{ou}'W_jS_{ov} = 0\ \ \forall j \forall u \neq v$. In view of the fact that $M_X = \sum_{u=1}^{s} S_{ou}S_{ou}'$, we put $S_{ou}S_{ou}' = M_u$. We shall prove that $M_X = \sum_{u=1}^{s} M_u$ is the desired partition. Let us notice that $P_iW_jP_i = c_{ij}P_i$ implies that $S_i'W_jS_i = c_{ij}I$. Since Y is normally distributed, $S_i'Y$ is also normally distributed with mean value $S_i'XBZ$ and $\operatorname{var}(\operatorname{vec} S_i'Y) = \sum_{j=1}^{k} V_j \otimes c_{ij}I = \left(\sum_{j=1}^{k} c_{ij}V_j\right) \otimes I$. This means that $Y'P_iY$ has a Wishart distribution with associated variance matrix $\sum_{j=1}^{k} c_{ij}V_j$ and noncentrality matrix $S_i'XBZ$. Similarly, since $S_{ou}'X = 0\ \ \forall u$, $S_{ou}'Y$ is also normally distributed with mean value 0 and $\operatorname{var}(\operatorname{vec} S_{ou}'Y) = \sum_{j=1}^{k} V_j \otimes d_{uj}I = \left(\sum_{j=1}^{k} d_{uj}V_j\right) \otimes I$ and $Y'M_uY$ has a central Wishart distribution with associated variance matrix $\sum_{j=1}^{k} d_{uj}V_j$. One can easily verify that the above partition also satisfies the other properties of MANOVA from Definition 1.

Necessity. Since Y has a normal distribution, Ya also has a normal distribution with mean value $XBZa$ and variance matrix $\sum_{i=1}^{k}(a'V_ia)W_i$ for an arbitrary vector a. Because $Y'Y = \sum_{i=1}^{t} Y'P_iY + \sum_{j=1}^{s} Y'M_jY$ is a MANOVA, a constant multiple of $a'Y'P_jYa$ must have χ^2 distribution. It follows that there must exist a constant c_a such that $\sum_{i=1}^{k}(a'V_ia)P_jW_iP_j = c_aP_j$. This is true for every a and every V_i; consequently, P_j must be a scalar multiple of $P_jW_iP_j$, or $\exists c_{ij}\colon P_jW_iP_j = c_{ij}P_j$.

Using independence we get $P_jW_iP_l = 0 \ \forall j \neq l$. Similarly we get that $\exists d_{ij}\colon M_jW_iM_j = d_{ij}M_j$, $M_jW_iM_l = 0 \ \forall j \neq l$ and $P_jW_iM_l = 0$. All these equations imply that the W_i's must be linear combinations of the P_j's and M_l's. Now it is trivial to see that all P_j and W_i commute.

Uniqueness. Let $M_X = \sum_{i=1}^{w} R_i$ be another MANOVA. According to the proof of the sufficiency, $W_jM_X = \sum_{i=1}^{w} g_{ij}R_i$ with $g_{ij} \geq 0$. Since the R_i's are perpendicular projectors and $Y'R_iY$ and $Y'R_jY$ have different expected values for $i \neq j$, the k-tuples $(g_{i1}, \ldots, g_{ik})$ must be distinct (and non-null). This immediately implies that $w = s$ and $R_i = M_i \ \forall i$. $\square$

One important consequence of the existence of a MANOVA is that we can find an explicit expression for the inverse of the variance matrix. According to the previous proof, we have $W_j = \sum_{i=1}^{t} c_{ij}P_i + \sum_{i=1}^{s} d_{ij}M_i$. Then $\sum_{j=1}^{k} V_j \otimes W_j = \sum_{i=1}^{t} \Gamma_i \otimes P_i + \sum_{i=1}^{s} \Delta_i \otimes M_i$, where $\Gamma_i = \sum_{j=1}^{k} c_{ij}V_j$ and $\Delta_i = \sum_{j=1}^{k} d_{ij}V_j$. Since for each i, Γ_i and Δ_i are positive definite, we have:

$$\left[\sum_{j=1}^{k} V_j \otimes W_j\right]^{-1} = \sum_{i=1}^{t} \Gamma_i^{-1} \otimes P_i + \sum_{i=1}^{s} \Delta_i^{-1} \otimes M_i\,.$$

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