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ROBUST ESTIMATION IN LINEAR MODEL FOR SPATIALLY LOCATED SENSORS AND RANDOM INPUT

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ABSTRACT. A random linear normal model for measurements of an intensity of a source of signals measured by a spatially located sensors is considered. Robust estimators, given by Fisher consistent and Fréchet differentiable von Mises functional, are proposed. The estimate minimizes a properly modified log-likelihood function.

1. Introduction and notation

In this paper we consider a problem of robust estimation in a random linear model frequently assumed for measurements of an intensity of a source of signals by a number of sensors.

Denote by v the intensity of the source of random signals, which is not directly measurable, but we can measure it, with some possible random errors independent of v, by a sensor. So for a single observed random variable y measuring the intensity of the source by the sensor, we assume the following structure: $y = kv + e$, where k denotes the influence of the source with the unit intensity on the sensor, while e is an unobservable random error. The value k can be obtained from theoretical considerations or can be measured directly in earlier carefully controlled experiments. For instance, when a model of diffusion for source of pollutions is considered, the value k is frequently computed as a function of distance r between the source and the sensor given by

$$k(r) = \left(\sqrt{2\pi}a\right)^{-1} \exp\left\{-0.5(r/a)^2\right\}, \tag{1}$$

where a denotes the diffusion coefficient (for more details see Gnot *et al.*, [7], and references therein). Under the additional assumption that v has $N(\mu, \sigma^2)$ distribution and that $e \sim N(0, \sigma_e^2)$, we get that $y \sim N(\mu, \sigma^2 + \sigma_e^2)$. When the

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intensity is measured at the same time by $J > 1$ sensors with given location, then the multivariate model for the observed intensity takes the form

$$Y = Kv + \varepsilon\,,$$

where components of Y, K and ε are formed with the corresponding values y, k and e for successive sensors. Under standard assumptions we get that

$$Y \sim N\big(K\mu, \sigma^2 KK^T + \sigma_e^2 I_J\big)\,, \tag{2}$$

where K^T denotes the transposed vector and I_J the unit $J \times J$ matrix. Note that for K being the vector of ones, a sample from the above model distribution yields the balanced one way classification random model.

There are not many papers considering the problem of robust analysis in random and mixed linear models, contrary to the case of linear and quadratic estimation. Therefore, for the latter case we mention only the paper by Gnot *et al.* [7], which concerns the model (2). In that paper, an explicit formula for maximum likelihood estimators of $(\mu, \sigma^2, \sigma_e^2)^T$ is presented. Some aspects of non-negative and admissible estimation are also considered. Among papers dealing with robust analysis there can be mentioned papers of Fellner [6], Rocke [14], Huggins [9], Lischer [12], and Bednarski and Zontek [3]. We adopt the approach proposed by the last authors to the considered model.

In Section 2 there is shown that sets of conditions imposed on a modification of the log-likelihood function, given by Bednarski and Zontek [3], yield asymptotically normal robust estimators also for the considered model. The asymptotic covariance matrix is explicitly calculated. In Section 3 results of simulation study are presented, where maximum likelihood estimator and robust estimator are compared with data without and with contamination.

Throughout the paper, $\mathrm{diag}(w)$ stands for a diagonal matrix with ith diagonal element equal to the ith component of w. A block diagonal matrix with blocks $B_1, \ldots, B_a$ is written as $\mathrm{diag}(B_1, \ldots, B_a)$. The a-dimensional vector with all entries unity and the identity $a \times a$ matrix are denoted by $\mathbf{1}_a$ and I_a, respectively. Finally, let $\mathbb{R}_+$ denote the set of positive real numbers.

2. Robust estimators

In model distribution (2), there is the same vector K appearing in the expectation and in the covariance matrix. In our considerations such kind of dependence is not utilized. Therefore we assume slightly more general model given by

$$N\big(A\mu, \sigma^2 KK^T + \sigma_e^2 I_J\big)\,, \tag{3}$$

where A and $K = (k_1, \dots, k_J)^T$ are the known J-vectors, $\mu \in \mathbb{R}$ is an unknown shift parameter, while $\sigma, \sigma_e \in \mathbb{R}_+$ are unknown scale components. The above model makes easier to follow which part of formulas depends directly on the expectation and which one depends on the covariance matrix. We are interested in robust estimation of $\theta = (\mu, \sigma, \sigma_e)^T$ of the model parameters. Denote by $\mathbf{F}_\theta$ the distribution function corresponding to the model distribution given by (3) and by Θ the parameter space $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$.

Denote by $\mathcal{G}$ a set of distribution functions defined on $\mathbb{R}^J$ and let T be a von Mises functional (statistical functional), that is, a function defined on $\mathcal{G}$ and taking values in Θ. An estimator $\hat{\theta}_n$ of θ is defined as $\hat{\theta}_n = T(\hat{\mathbf{F}}_n)$, where $\hat{\mathbf{F}}_n$ denotes the empirical distribution function resulting from a sample $Y_1, \dots, Y_n$. Frequently, a value of the von Mises functional i sdefined as an argument for which a real function on $\mathcal{G}$ (called the objective function) attains the minimum value, or as a solution of a system of equations.

In practice, we do not known whether observed random vectors really satisfy the model assumptions. From this point of view it is reasonable to investigate behaviour of a statistical functional in a neighbourhood, given for example by the supremum norm (denoted by $\|\cdot\|$), of the model distribution function.

Let $\mathsf{P} = (P_1, \dots, P_J)^T$ be an orthogonal matrix such that $P_1 = \left(1/\sqrt{K^T K}\right) K$. Note that for any σ and σ_e, the matrix

$$\Delta^2 = \mathsf{P}^T(\sigma^2 K K^T + \sigma_e^2 I_J)\mathsf{P} \tag{4}$$

is diagonal with diagonal elements $\delta_1^2 = K^T K \sigma^2 + \sigma_e^2$, $\delta_2^2 = \cdots = \delta_J^2 = \sigma_e^2$. These elements play an important role in our considerations. The matrix P is not unique. In simulation study we take $\mathsf{P} = Q^T \operatorname{diag}(s_1, s_1 s_2, \dots, s_{J-1} s_J)^{-1/2}$, where $s_i = \sum_{j=i}^{J} k_j^2$, while

$$Q = \begin{bmatrix} k_1 & k_2 & k_3 & \dots & k_{J-2} & k_{J-1} & k_J \\ -s_2 & k_2 k_1 & k_3 k_1 & \dots & k_{J-2} k_1 & k_{J-1} k_1 & k_J k_1 \\ 0 & -s_3 & k_3 k_2 & \dots & k_{J-2} k_2 & k_{J-1} k_2 & k_J k_2 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -s_{J-1} & k_{J-1} k_{J-2} & k_J k_{J-2} \\ 0 & 0 & 0 & \dots & 0 & -s_J & k_J k_{J-1} \end{bmatrix}.$$

The objective function $\Phi\colon \mathbb{R}^J \times \Theta \to \mathbb{R}$ is defined for $y \in \mathbb{R}^J$ and $\theta \in \Theta$ by

$$\Phi(y|\theta) = \sum_{j=1}^{J} \Big[\ln(\delta_j) + \phi\big(P_j^T(y - A\mu)/(c\delta_j)\big)\Big] \tag{5}$$

with a function $\phi\colon \mathbb{R} \to \mathbb{R}$ properly chosen, $c \in \mathbb{R}$ and $\delta_1 = \sqrt{K^T K \sigma^2 + \sigma_e^2}$, $\delta_2 = \cdots = \delta_J = \sigma_e$.

Note that the function Φ becomes the log-likelihood function when $\phi(x) = 0.5x^2$ and $c = 1$. For a given ϕ we can frequently choose c to make the functional Fisher consistent.

DEFINITION 1. (the functional). Define the functional T at $\mathbf{G} \in \mathcal{G}$ to be the parameter θ in Θ for which

$$\int \Phi(y\,|\,\theta)\, d\mathbf{G}(y) \tag{6}$$

attains the minimum value.

Below stated are assumptions concerning ϕ which imply the Fisher consistency and Fréchet differentiability of T.

(A1) The function ϕ is symmetric about 0, and it has positive derivative for positive arguments.

(A2) The function $x\phi'(x)$ has a nonnegative derivative for $x \geq 0$ and there exists $x_o > 0$ such that $x_o\phi'(x_o) > 1$.

(A3) The functions ϕ' and ϕ'' are bounded.

(A4) The functions $x\phi'(x)$ and $x^2\phi''(x)$ are bounded.

The first two assumptions imply Fisher consistency of a statistical functional for one dimensional shift and scale model. For the same model, assumptions A3 and A4 guarantee Fréchet differentiability. The structure of the objective function allows us to apply results for one dimensional model to the multivariate case. To formulate this result, we need the following notation.

Let $\Psi(\cdot\,|\,\theta) = \big(\Psi_1(\cdot\,|\,\theta),\ \Psi_2(\cdot\,|\,\theta),\ \Psi_3(\cdot\,|\,\theta)\big)^T$, where $\Psi_1(\cdot\,|\,\theta) = \frac{\partial}{\partial\mu}\Phi(\cdot\,|\,\theta)$, $\Psi_2(\cdot\,|\,\theta) = \frac{\partial}{\partial\sigma}\Phi(\cdot\,|\,\theta)$ and $\Psi_3(\cdot\,|\,\theta) = \frac{\partial}{\partial\sigma_e}\Phi(\cdot\,|\,\theta)$, and let

$$D(\cdot\,|\,\theta) = \big[d_{ij}(\cdot\,|\,\theta)\big] = \begin{bmatrix} \frac{\partial^2}{\partial\mu^2}\Phi(\cdot\,|\,\theta) & \frac{\partial^2}{\partial\mu\partial\sigma}\Phi(\cdot\,|\,\theta) & \frac{\partial^2}{\partial\mu\partial\sigma_e}\Phi(\cdot\,|\,\theta) \\ \frac{\partial^2}{\partial\mu\partial\sigma}\Phi(\cdot\,|\,\theta) & \frac{\partial^2}{\partial\sigma^2}\Phi(\cdot\,|\,\theta) & \frac{\partial^2}{\partial\sigma\partial\sigma_e}\Phi(\cdot\,|\,\theta) \\ \frac{\partial^2}{\partial\mu\partial\sigma_e}\Phi(\cdot\,|\,\theta) & \frac{\partial^2}{\partial\sigma\partial\sigma_e}\Phi(\cdot\,|\,\theta) & \frac{\partial^2}{\partial\sigma_e^2}\Phi(\cdot\,|\,\theta) \end{bmatrix}.$$

THEOREM 1. (i) *If* (A2) *and* (A4) *are satisfied, then there is a unique* $c_\phi > 0$ *defining* Φ *for which a function* g *given for* $c > 0$ *by*

$$g(c) = \ln(c) + \int \phi(x/c)\, dF(x)$$

attains the global minimum, where F *is the standard normal distribution function.*

(ii) (Fisher consistency). *If* ϕ *satisfies* (A1) *and* (A2), *then*

$$\int \Phi(y\,|\,\theta)\, d\mathbf{F}_{\theta^o}(y)$$

attains the global minimum if and only if $\theta = \theta^o$.

(iii) (Fréchet differentiability). *Under* (A3) *and* (A4) *we have*

$$T(\mathbf{G}) - T(\mathbf{F}_\theta) = -M(\theta)^{-1} \int \Psi(y\,|\,\theta) d\{\mathbf{G} - \mathbf{F}_\theta\}(y) + o\big(\|\,\mathbf{G} - \mathbf{F}_\theta\|\big)\,,$$

where $M(\theta)$ *is a* 3×3 *matrix given by* $M(\theta) = \int D(y\,|\,\theta)\, d\mathbf{F}_\theta(y)$.

P r o o f . To conclude the uniqueness of c_ϕ, it is sufficient to show that the derivative of g is equal to 0 for exactly one argument, i.e., that the following equation

$$\int (x/c)\phi'(x/c)\, dF(x) = 1 \tag{7}$$

has the unique solution. But this was proved by B e d n a r s k i and Z o n t e k [3].

For the second part, consider first the minimization problem of a single summand given in (5), which can be considered as the minimization problem to the one dimensional shift and scale case. The form of the objective function and the smoothness assumptions about ϕ imply, as one can easily see, that the global minimum is attained for ith summand iff

$$P_i^T A\mu = P_i^T A\mu^o\,,$$

and

$$\begin{aligned} K^T K\sigma^2 + \sigma_e^2 &= K^T K(\sigma^o)^2 + (\sigma_e^o)^2 \quad \text{when} \quad i = 1\,,\\ \sigma_e^2 &= (\sigma_e^o)^2 \qquad\qquad\qquad\quad\;\; \text{when} \quad i = 2, \ldots, J\,. \end{aligned}$$

Thus the minimum of the expected value is attained only when $\mu = \mu^o$ and $(\sigma, \sigma_e)^T = (\sigma^o, \sigma_e^o)^T$, which finishes the proof of part (ii).

Under (A3) and (A4), the function Ψ satisfies the condition A of C l a r k e ([4], see also [5]), which implies that the functional T is Fréchet differentiable.

For Fisher consistent and Fréchet differentiable functional we have

$$\sqrt{n}\{T(\hat{\mathbf{F}}_n) - \theta\} = -M(\theta)^{-1}\sqrt{n} \int \Psi(y\,|\,\theta)\, d\hat{\mathbf{F}}_n(y) + o_{G_n^{\otimes n}}(1)\,,$$

where $\sqrt{n}\|G_n - \mathbf{F}_\theta\|$ stays bounded. By the central limit theorem, the asymptotic distribution is then normal with expectation zero (at the model) and the covariance matrix

$$M(\theta)^{-1} \left\{ \int \Psi(y\,|\,\theta)\Psi(y\,|\,\theta)^T d\mathbf{F}_\theta(y) \right\} M(\theta)^{-1}\,,$$

in the whole infinitesimal model given by the supremum norm. This matrix can be explicitly calculated by using the following technical lemma.

Define for $\sigma, \sigma_e \in \mathbb{R}_+$ the following functions

$$U_{11}(\sigma, \sigma_e) = \sum_{j=1}^{J} (A^T P_j)^2 / \delta_j^2 ,$$

and

$$U_{22}(\sigma, \sigma_e) = (2/\delta_1^4)\big(K^T K\sigma \,,\, \sigma_e\big)^T \big(K^T K\sigma \,,\, \sigma_e\big) + (2/\sigma_e^2)\, \mathrm{diag}(0, J-1) .$$

□

Lemma 1. *Let the constant c_ϕ be defined by* (7) *and let X be a standard normal random variable. If* (A1) *and* (A2) *are satisfied, then we have*

(i) $\int \Psi(y\,|\,\theta)\Psi(y\,|\,\theta)^T d\mathbf{F}_\theta(y) = \mathrm{diag}\big(f_1 U_{11}(\sigma, \sigma_e),\ f_2 U_{22}(\sigma, \sigma_e)\big)$,

(ii) $\int D(y\,|\,\theta)\, d\mathbf{F}_\theta(y) = \mathrm{diag}\big(g_1 U_{11}(\sigma, \sigma_e),\ g_2 U_{22}(\sigma, \sigma_e)\big)$,

where

$$f_1 = c_\phi^{-2} E\{\phi'(X/c_\phi)^2\}, \quad f_2 = 0.5\, E\big[\{(X/c_\phi)\phi'(X/c_\phi) - 1\}^2\big]$$

and

$$g_1 = c_\phi^{-2} E\{\phi''(X/c_\phi)\}, \quad g_2 = 0.5\, E\{(X/c_\phi)^2 \phi''(X/c_\phi) + 1\} .$$

Proof. Let $\theta = (\mu, \sigma, \sigma_e)^T$ be an arbitrary parameter. Define a random vector

$$Z = (Z_1, \dots, Z_J)^T = \Delta^{-1} \mathbf{P}^T (Y - A\mu) ,$$

where Y has the distribution $\mathbf{F}_\theta$, while Δ is given by (4). Then Z has the multivariate standard normal distribution.

Part (i). Note that the random vector $\Psi(Y\,|\,\theta)$ can be represented as

$$\Psi(Y\,|\,\theta) = \begin{bmatrix} -\sum\limits_{j=1}^{J} \big(A^T P_j/(c_\phi \delta_j)\big)\, \phi'(Z_j/c_\phi) \\ K^T K \sigma \delta_1^{-2} \{1 - (Z_1/c_\phi)\phi'(Z_1/c_\phi)\} \\ \sigma_e \sum\limits_{j=1}^{J} \delta_j^{-2} \{1 - (Z_j/c_\phi)\phi'(Z_j/c_\phi)\} \end{bmatrix} .$$

Since ϕ' is an odd function and since $Z_1, \dots, Z_J$ are independent standard normal random variables, we easily see the first part of the lemma.

Part (ii). One can check that $d_{12}(Y\,|\,\theta)$ and $d_{13}(Y\,|\,\theta)$ are the sums of odd functions with respect to $Z_1, \dots, Z_J$. So the integral over the multivariate standard normal distribution is zero. The remaining elements of $D(Y\,|\,\theta)$ are given by

$$d_{11}(Y\,|\,\theta) = \sum_{j=1}^{J} \frac{\big(A^T P_j\big)^2}{c_\phi^2 \delta_j^2}\, \phi''(Z_j/c_\phi) .$$

$$d_{22}(Y|\theta) = \tfrac{(K^T K)^2\sigma^2}{\delta_1^4}\Big\{ \tfrac{Z_1^2}{c_\phi^2}\phi''(Z_1/c_\phi) + 1 + \Big(\tfrac{\delta_1^2}{K^T K\sigma^2} - 3\Big)\Big[1 - \tfrac{Z_1}{c_\phi}\phi'(Z_1/c_\phi)\Big]\Big\},$$

$$d_{23}(Y|\theta) = \tfrac{K^T K\sigma\sigma_e}{\delta_1^4}\Big\{ \tfrac{Z_1^2}{c_\phi^2}\phi''(Z_1/c_\phi) + 1 - 3\Big[1 - \tfrac{Z_1}{c_\phi}\phi'(Z_1/c_\phi)\Big]\Big\},$$

and

$$d_{33}(Y|\theta) = \sum_{j=1}^{J} \tfrac{\sigma_e^2}{\delta_j^4}\Big\{ \tfrac{Z_j^2}{c_\phi^2}\phi''(Z_j/c_\phi) + 1 + \Big(\tfrac{\delta_j^2}{\sigma_e^2} - 3\Big)\Big[1 - \tfrac{Z_j}{c_\phi}\phi'(Z_j/c_\phi)\Big]\Big\}.$$

Applying now equation (7) defining the constant c_ϕ, one can easily derive formula (ii).

It follows from the lemma that the covariance matrix for the Fréchet differentiable estimator generated by (5) and (6) is of the form

$$\begin{bmatrix} w_1\big(U_{11}(\sigma,\sigma_e)\big)^{-1} & 0 \\ 0 & w_2\big(U_{22}(\sigma,\sigma_e)\big)^{-1} \end{bmatrix}, \tag{8}$$

where the positive constants w_1 and w_2 are given by the formulas

$$w_1 = \frac{c_\phi^2 E\big[\phi'(X/c_\phi)^2\big]}{\big[E\phi''(X/c_\phi)\big]^2} \quad \text{and} \quad w_2 = \frac{2E\big\{\big[(X/c_\phi)\phi'(X/c_\phi) - 1\big]^2\big\}}{\big\{E\big[(X/c_\phi)^2\phi''(X/c_\phi) + 1\big]\big\}^2}.$$

The same values appeared in Bednarski and Zontek [3]. □

2. Simulations

In simulation study we used the robust estimator given by (5) and (6) with ϕ proposed by Bednarski *et al.* [2]. The function ϕ fulfils conditions (A1)–(A4) and is defined for $y \in \mathbb{R}$ by

$$\phi(y) = \begin{cases} y^2/2\,, & |y| \le t\,, \\ -y^2/2 + 4t|y| - 2t^2\big\{\ln(|y|/t) + 1.5\big\}, & t < |y| \le 2t\,, \\ 2t^2\big\{\ln(|y|/4t) + 1.5\big\}, & |y| > 2t\,. \end{cases}$$

It is a smoothed modification of Huber's proposition (Huber, [8], p. 137. Example 4.1) for robust estimation of scale. As in Bednarski and Zontek [3], we used the value $t = 1.2535$ together with $c_\phi = 0.8735$ as this yields a consistency of the estimator and efficiency coefficients appearing in (8) $w_1 = 1.1$ and $w_2 = 1.2898$.

We generated the uncontaminated sample for $n = 10$ by taking

(M) $Y = K(10 + y_0) + 0.5(y_1, \ldots, y_5)^T$, where $y_0, \ldots, y_5$ are observations on independent standard normal random variables. So $A = K$ and the model parameter θ_o for the simulations is $\theta_o = (10, 1, 0.5)^T$. We took $K = \big(k(0.1),\ k(0.1),\ k(0.5),\ k(1),\ k(1)\big)^T$, where $k(\cdot)$ is given by (1) with the diffusion coefficient a equal to 3.

We introduced a contamination via parameters. In this case we generated Y as above with probability 0.95 and we took

(C1) $Y = K(20 + y_0) + 0.5(y_1, \ldots, y_5)^T$,

(C2) $Y = K(10 + 5y_0) + 0.5(y_1, \ldots, y_5)^T$,

(C3) $Y = K(10 + y_0) + 2.5(y_1, \ldots, y_5)^T$

with probability 0.05. Also we contaminated observations via the vector K. With probability 0.05 we generated Y according to

(C4) $Y = \kappa(10 + y_0) + 0.5(y_1, \ldots, y_5)^T$,

where $\kappa = \big(k(|r_1|), \ldots, k(|r_5|)\big)^T$ with $a = 3$, while $r_1, \ldots, r_5$ are observations on independent random variable with $N(0, 3)$ distribution.

TABLE 1. Sample averages and sample standard deviations of MLE and RE

model	estimates for μ		estimates for σ		estimates for σ_e	
	MLE	RE	MLE	RE	MLE	RE
M	10.02	10.02	0.87	0.87	0.49	0.50
	(0.38)	(0.40)	(0.38)	(0.43)	(0.053)	(0.062)
	(0.38)	(0.40)	(0.33)	(0.37)	(0.056)	(0.064)
C1	10.49	10.05	1.88	1.05	0.50	0.50
	(0.79)	(0.52)	(1.30)	(0.64)	(0.060)	(0.067)
	(0.38)	(0.40)	(0.33)	(0.37)	(0.056)	(0.064)
C2	10.01	10.03	1.19	0.96	0.50	0.50
	(0.52)	(0.41)	(0.77)	(0.49)	(0.054)	(0.061)
	(0.38)	(0.40)	(0.33)	(0.37)	(0.056)	(0.064)
C3	10.01	10.02	0.79	0.89	0.67	0.54
	(0.45)	(0.42)	(0.54)	(0.46)	(0.269)	(0.099)
	(0.38)	(0.40)	(0.33)	(0.37)	(0.056)	(0.064)
C4	9.60	9.93	1.61	1.05	0.51	0.51
	(0.68)	(0.49)	(1.00)	(0.60)	(0.059)	(0.065)
	(0.38)	(0.40)	(0.33)	(0.37)	(0.056)	(0.064)

Results of 1000 Monte Carlo replications for $J = 5$ sensors and sample size $n = 10$.

For each type of generated data, sample average and sample covariance matrix were computed. In Table 1 presented above, results of 1000 Monte Carlo replications are presented. In the table, rows are divided into five blocks with respect to means of generation. Each block consists of 3 rows: the first one is for the average of the estimates, the second one gives sample standard deviation for the estimates, while the last row shows the "true" asymptotic value for the standard deviation. Columns of the table are divided on three blocks with respect to estimated parameter. Each block consists of 2 columns: the first one for the maximum likelihood estimator (MLE) and the second one for the robust estimator (RE).

Comparison of results presented in tables confirms, as it follow from theory, that MLE is better than RE when data was generated according to the model assumptions. When observations were contaminated, computations shows superiority of RE over MLE for at least these estimated values which were randomly changed. In this case the MLE has essentially bigger bias or higher variability of estimates.

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