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ON AN EBERHARD-TYPE PROBLEM IN CUBIC POLYHEDRAL GRAPHS HAVING PETRIE AND HAMILTONIAN CYCLES

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ABSTRACT. A Hamiltonian cycle in a cubic plane graph which takes alternately left and right edges at each successive vertex is called a Petrie Hamiltonian cycle. In this note we prove a result of Eberhard type for cubic plane 3-connected graphs having Petrie Hamiltonian cycles.

1. Introduction

A cycle $\mathcal{C}$ in a cubic plane graph is said to be a *Petrie cycle* if every two, but no three, consecutive edges of $\mathcal{C}$ (including the last and the first) lie on a common face. A cycle in a cubic plane graph which is Petrie and Hamiltonian is called a *Petrie Hamiltonian cycle* (a *PH-cycle* in the sequel).

A concept of Petrie paths and cycles was introduced over fifty years ago, see, e.g., Coxeter [2] or Grünbaum [5]. Such cycles are also called *L-R*-cycles [4], left-right paths and cycles [19]. Petrie cycles were studied in [16] and *PH*-cycles in [9].

Let $p_k(G)$ denote the number of k-gonal faces of a cubic polyhedral graph G, i.e., (due to Steinitz's theorem [5]), 3-valent plane 3-connected graph. Since Euler, it has been known that if G is a cubic polyhedral graph, then

$$\sum_{k\geq 3}(6-k)p_k(G)=12\,.$$

In 1890, the blind geometer V. Eberhard [3] proved the following theorem concerning the number $p_k(G)$ of k-gonal faces of cubic polyhedral graphs.

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Eberhard's theorem. *If p_3, p_4, p_5, p_7, p_8, $\dots$, p_n are non-negative integers such that*

$$\sum_{k\ge 3}(6-k)p_k = 12\,, \tag{1}$$

then there exists a cubic polyhedral graph G such that $p_k(G) = p_k$ for all $k \ge 3$, $k \ne 6$.

Eberhard's theorem is one of the oldest non-trivial results in the theory of polyhedral graphs. It was, though, practically forgotten for a long time. Grünbaum renewed the interest in Eberhard's theorem, gave a much shorter proof [5] than the original one, proved several ramifications (see, e.g., [6, 7]) and posed the analogous problems for arbitrary situations [6, 7, 8, 17]. There is by now a considerable literature on Eberhard-type theorems; we single out [8, 12, 13, 14, 18] as being more particularly relevant to the present paper, and refer the reader to [10, 11, 12, 14, 15, 17] for further references.

In this note we investigate the following Eberhard-type's problem: A sequence $(p_k\colon k \ge 3)$ of non-negative integers is called *PH-realizable* if there exists a cubic polyhedral graph G with a PH-cycle such that $p_k(G) = p_k$ for all $k \ge 3$. G is then called a *realization* of the sequence. Which sequences of non-negative integers are PH-realizable?

2. The main result

An immediate corollary of Euler's polyhedral formula (1) clearly gives a necessary condition for a sequence to be PH-realizable. In the following assertion there are presented other necessary conditions (see also [5] and [9]).

Lemma 1. *Every PH-realizable sequence $(p_k\colon k \ge 3)$ satisfies*

$$\sum_{k\ge 3} p_k \equiv 0 \pmod 2\,, \tag{2}$$

$$p_k = 0 \qquad \text{for all} \quad k \not\equiv 0 \pmod 3\,. \tag{3}$$

Proof. Let G be a realization of $(p_k\colon k \ge 3)$. Suppose $\mathcal{C}$ is a PH-cycle in G. Let G^* be the dual graph of G (i.e., vertices of G^* correspond to faces of G and two vertices are adjacent in G^* if faces corresponding to them have a common edge in G). Denote by V_1^* (V_2^*) the set of vertices of G^* which correspond to faces of G in the interior (in the exterior, respectively) of $\mathcal{C}$. As $\mathcal{C}$ is a Hamiltonian cycle, the subgraph $\langle V_i^*\rangle$ of G^* induced by V_i^*, $i \in \{1,2\}$, is a tree. Edges of the trees correspond to chords of $\mathcal{C}$. Since $\mathcal{C}$ is a Petrie cycle, the number of chords in the interior and in the exterior of $\mathcal{C}$ is the same. Hence,

$|V_1^*| = |V_2^*|$. The number of faces of G is equal to $|V_1^*| + |V_2^*|$, and so (2) holds.

Edges of $\mathcal{C}$ correspond to edges of a closed trail in G^*. Two consecutive edges of this trail join a vertex of V_i^* with end vertices of an edge of $\langle V_{3-i}^* \rangle$, $i \in \{1,2\}$. Since G is cubic, G^* is a triangulation, and so $\deg_{G^*}(v) = 3\deg_{\langle V_i^* \rangle}(v)$ for every $v \in V_i^*$, $i \in \{1,2\}$. As a k-gonal face of G corresponds to a k-valent vertex of G^*, (3) holds. $\square$

Remark. The above proof suggests a construction of cubic plane graphs having PH-cycles. The construction begins with a plane diagram T of two disjoint trees of the same order. Choose a vertex u in the first tree and an edge vw in the second tree. It is easy to see that there exists only one closed trail $\mathcal{C}$ satisfying:

(a) edges vu and uw belong to $\mathcal{C}$;

(b) every edge of $\mathcal{C}$ joins vertices of distinct trees;

(c) for every two consecutive edges x_1x_2, x_2x_3 of $\mathcal{C}$, the vertices x_1 and x_3 are adjacent in T;

(d) a diagram G, which we obtain from T by adding edges of $\mathcal{C}$, is a triangulation;

(e) every face of G is incident with an edge of T and two consecutive edges of $\mathcal{C}$.

Evidently, the dual graph G^* of G is a cubic plane graph with a PH-cycle satisfying: $p_{3i}(G^*) = \big|\{v \in V(T)\colon \deg_T(v) = i\}\big|$ for all $i \geq 1$. Moreover, G^* is 3-connected if and only if G is a graph without multiple edges.

Let $\Pi = (p_k\colon k \geq 3)$ be a sequence satisfying (1), (2) and (3). Put

$$c(\Pi) := \min_{\substack{(q_{3i}\colon i\geq 3)\\ 0\leq q_{3i}\leq p_{3i}}} \left\{ \left| \sum_{i\geq 3} (i-1)(2q_{3i} - p_{3i}) \right| \right\},$$

when $\sum\limits_{i\geq 3} p_{3i} > 0$, and $c(\Pi) := 2$, otherwise.

The main result of this note is the following

THEOREM 2. *Every sequence $\Pi = (p_k\colon k \geq 3)$ satisfying (1), (2), (3) and*

$$p_6 \geq 2c(\Pi) + \sum_{i\geq 3}(i-1)p_{3i} \tag{4}$$

is PH-realizable.

P r o o f . Suppose $\sum\limits_{i\geq 3} p_{3i} > 0$. Without loss of generality, let $(q_{3i}\colon i \geq 3)$ be a sequence such that $c(\Pi) = \sum\limits_{i\geq 3}(i-1)(2q_{3i} - p_{3i})$. Then, $c(\Pi) + \sum\limits_{i\geq 3}(i-1)\times (p_{3i} - q_{3i}) = \sum\limits_{i\geq 3}(i-1)q_{3i}$. Evidently, there exists a tree T_L' (T_R') having q_{3i}

$(p_{3i} - q_{3i})$ i-valent vertices for all $i \geq 3$, and 0 ($c(\Pi)$, respectively) 2-valent vertices. T'_L and T'_R are of the same order $2 + \sum\limits_{i\geq 3}(i-1)q_{3i} = 2 + c(\Pi) + \sum\limits_{i\geq 3}(i-1)(p_{3i} - q_{3i})$. In the case $\sum\limits_{i\geq 3} p_{3i} = 0$, let T'_L and T'_R be paths of length 2.

Given an integer $t \geq 0$, let T_L (T_R) be a tree which we obtain from T'_L (T'_R) by replacing a terminal edge xv (yw), where $\deg(v) = 1$ ($\deg(w) = 1$), by a path of length $\lfloor t/2 \rfloor$ ($\lceil t/2 \rceil$, respectively). Let T_2 be a tree which we obtain from the disjoint union of T_L and T_R by adding an edge vw. It can be easily seen that T_2 has the same order as a path T_1 of length $1 + c(\Pi) + t + \sum\limits_{i\geq 3}(i-1)p_{3i}$. Now, we denote by T the disjoint union of T_1 and T_2. We choose a 1-valent vertex u in T_1 and the edge vw in T_2, and use the construction described in Remark. The obtained graph G has no multiple edge, because any pair of consecutive edges of the added trail satisfies (c) and two distinct pairs of consecutive edges incident with a 2-valent vertex of T_1 join this vertex with end vertices of two edges, where one of them belongs to T_L and the other to T_R. Thus, G^* is a cubic polyhedral graph with a PH-cycle such that $p_k(G^*) = p_k$, for all $k \neq 6$, and $p_6(G^*) = 2c(\Pi) + 2t + \sum\limits_{i\geq 3}(i-1)p_{3i}$, i.e., for a suitable t, G^* is a realization of Π. $\square$

3. Remarks

Since the condition (1) does not provide any information concerning p_6, one can discuss the quality of the bound (4). By B a r n e t t e [1], the best possible lower bound for p_6 in the case of cubic polyhedral graphs satisfying (3) (except for the graph of a tetrahedron) is

$$p_6 \geq 4 + \sum_{i\geq 3}\left(\frac{i}{2} - 2\right)p_{3i}. \tag{5}$$

Let H_0 be the graph of a tetrahedron and let H_{i+1} be a graph which we obtain from H_i replacing every its vertex by a triangle. It can be easily seen that H_i, $i \geq 1$, is a cubic polyhedral graph with a PH-cycle. Moreover, equality in (5) is attained. Therefore, the bound (5) cannot be improved for the general family of cubic polyhedral graphs having PH-cycles. However, there is a better bound in our special case.

LEMMA 3. *Let $\Pi = (p_k\colon k \geq 3)$ be PH-realizable. If $m(\Pi) := \max\{i\colon p_{3i} > 0\}$, then*

$$p_6 \geq 4\bigl(m(\Pi) - 1\bigr) + \sum_{i\geq 3}(1-i)p_{3i}. \tag{6}$$

Proof. Suppose that G is a realization of Π and $\mathcal{C}$ is its PH-cycle. Let α denote a $(3m(\Pi))$-gonal face of G. Without loss of generality, α is situated in the exterior of $\mathcal{C}$. As G is 3-connected and $\mathcal{C}$ is a PH-cycle, α is adjacent to $2m(\Pi)$ distinct faces in the interior of $\mathcal{C}$. Since there is the same number of faces in the interior and in the exterior of $\mathcal{C}$, G has at least $4m(\Pi)$ faces, i.e., $\sum\limits_{k\geq 3} p_k \geq 4m(\Pi)$. Hence, $\sum\limits_{k\geq 3}(3-k)p_k = \sum\limits_{k\geq 3}(6-k)p_k - 3\sum\limits_{k\geq 3} p_k \leq 12 - 12m(\Pi)$. By (3) we get $-3p_6 + 3\sum\limits_{i\geq 3}(1-i)p_{3i} \leq 12\big(1-m(\Pi)\big)$, which implies the assertion.

$\square$

Note that the bound (6) (and also (4)) is the best possible for a sequence satisfying $2m(\Pi) - c(\Pi) \geq 2 + \sum\limits_{i\geq 3}(i-1)p_{3i}$. However, this condition implies $p_{3m(\Pi)} \leq 2$, and so the problem to find the best possible lower bound for p_6 in general cases is still open.

The following problem seems to be of interest: A sequence $(p_k : k \geq 3)$ of non-negative integers is called *H-realizable* if there exists a cubic Hamiltonian graph G having $p_k(G) = p_k$ for all $k \geq 3$. An open problem is to characterize all H-realizable sequences.

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