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FRACTIONAL $\mathcal{P}$-COLOURINGS AND $\mathcal{P}$-CHOICE-RATIO

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ABSTRACT. The choice ratio of a graph $G = (V, E)$ is the minimum quotient a/b with the following property. For every assignment of a-element sets $L(v)$ to the vertices $v \in V$, there can be chosen b-element subsets $C(v) \subset L(v)$ for all v in such a way that $C(v) \cap C(v') = \emptyset$ holds for every adjacent vertex pair v, v'. Alon, Tuza and Voigt [Discr. Math. 165/166, 1997, 31–38] proved that the choice ratio is equal to the fractional chromatic number, for every graph G. In this paper we prove that the analogous statement holds for the fractional $\mathcal{P}$-chromatic number and the $\mathcal{P}$-choice-ratio of G, where $\mathcal{P}$ is any hereditary graph property. A generalization of these results for hypergraphs is presented, too.

1. Introduction

In this paper we investigate the combination of two subjects of graph theory, both of which have attracted much attention in recent years: hereditary properties, and colourings where the availability of colours is restricted locally, at each vertex. For detailed information on these two large areas, we refer the reader to the extensive surveys [4, 5] and [7], respectively.

The standard concept of chromatic number is based on the notion of proper (vertex) colouring, requiring that each colour class is an independent set. Since "independence" is the simplest graph property (among those without size restrictions) satisfying heredity, in this approach the class of hereditary properties offers a very general framework for a natural generalization.

Here we investigate the problem of choosing "many" colours from "large" colour sets for each vertex of a (finite, simple) graph, under the condition that the set of vertices assigned to any one colour satisfies a given property $\mathcal{P}$. Our main result, formulated in two parts (Theorems 1 and 2), states that the maximum

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size of choosable colour subsets is asymptotically determined by the fractional chromatic number (with respect to $\mathcal{P}$) of the graph in question. The corresponding fact for the particular property of independence was proved by Alon and two of the present authors in [2]; we extend those proof techniques for a more general case. We also introduce the analogous concepts for hypergraphs, and describe an equality that can be viewed as the most general relation between two types of parameters.

The paper is organized as follows. In the next section we give the formal definitions related to colourability with respect to hereditary properties. Some basic properties of the (generalized) fractional chromatic number of a graph are described in Section 3. The main results on graph colourings are presented in Section 4, while Section 5 deals with the more general structure of hypergraphs.

2. Colourings and hereditary properties

In this section we give the necessary definitions on generalized colourings. Let us denote the class of all finite simple graphs by $\mathcal{I}$. A *graph property* is a non-empty, isomorphism-closed, proper subclass of $\mathcal{I}$. If $G \in \mathcal{P}$, we also say that the graph G has the property $\mathcal{P}$. If the subgraph $G[S]$ induced by a set $S \subseteq V(G)$ has the property $\mathcal{P}$, we shall sometimes simply say that S has the property $\mathcal{P}$. (As usual, the vertex set and edge set of graph G is denoted by $V(G)$ and $E(G)$, respectively.)

Recall that a proper vertex colouring of a graph is a partition of the vertex set into subsets which are independent (i.e., consist of mutually nonadjacent vertices). The following generalization, where "independence" is replaced by any graph property $\mathcal{P}$, has been introduced by several authors (cf. the surveys cited above).

Definition 1. Let $\mathcal{P}$ be a graph property. A graph G is called $(\mathcal{P}, k)$*-colourable* if there is a vertex partition $(V_1, V_2, \dots, V_k)$ of G such that the subgraph $G[V_i]$ induced by V_i has the property $\mathcal{P}$ for every $i = 1, 2, \dots, k$. Such a partition is also termed a $\mathcal{P}$*-colouring.*

If the graph G admits at least one $\mathcal{P}$-colouring, then the following parameter can be defined for it.

Definition 2. The $\mathcal{P}$*-chromatic number* $\chi_{\mathcal{P}}(G)$ of G is the smallest integer k such that G is $(\mathcal{P}, k)$-colourable.

To ensure that $\chi_{\mathcal{P}}(G)$ is well-defined for every G, it suffices to assume that the one-vertex graph satisfies $\mathcal{P}$. It is reasonable to require, however, that $\chi_{\mathcal{P}}$ should be monotone (i.e., the chromatic number of a subgraph should not exceed that of the entire graph). Therefore, we shall consider only the following subclass of graph properties. A property $\mathcal{P}$ of graphs is called (*induced*) *hereditary* if it is closed with respect to the relation "to be an induced subgraph"; that is, if H is an induced subgraph of G and $G \in \mathcal{P}$, then $H \in \mathcal{P}$ also holds.

A property $\mathcal{P}$ is called *additive* if it is closed under the vertex-disjoint union of graphs; this is equivalent to assuming that a graph has the property $\mathcal{P}$ if and only if so does each of its connected components.

For any hereditary property $\mathcal{P}$, let the *completeness* of $\mathcal{P}$ be defined as $c(\mathcal{P}) := \sup\{k \colon K_{k+1} \in \mathcal{P}\}$. It is quite easy to see that the unique hereditary additive property whose completeness equals zero is the property to be an *edgeless graph*; it will be denoted by $\mathcal{O}$. In other words, $\mathcal{O}$ means that the vertex set in question is independent.

In the standard definition of chromatic number, any colour may be assigned to any vertex. On the other hand, in "choosability" problems the availability of colours is (or may be) restricted for each vertex. This condition is usually given by a *list* $L(v)$ of admissible colours for every $v \in V(G)$. We shall consider the following generalization of this concept.

Definition 3. A graph G is called $(a,b)_{\mathcal{P}}$*-choosable* if, for every list assignment L with $|L(v)| = a$ for all $v \in V(G)$, it is possible to choose a subset $C(v) \subseteq L(v)$ with $|C(v)| = b$ for all $v \in V(G)$ such that, for every colour $f \in \bigcup_{v \in V(G)} C(v)$, the subgraph G_f induced by $V_f := \{v \in V(G) \colon f \in C(v)\}$ has the property $\mathcal{P}$.

Definition 4. The $\mathcal{P}$*-choice number*, $\mathrm{ch}_{\mathcal{P}}(G)$, of G is the smallest integer k such that G is $(k,1)_{\mathcal{P}}$-choosable.

Definition 5. The value $\mathrm{chr}_{\mathcal{P}}(G) := \inf\left\{\frac{a}{b} \colon G \text{ is } (a,b)_{\mathcal{P}}\text{-choosable}\right\}$ is called the $\mathcal{P}$*-choice ratio* of G.

The next inequalities follow by definition:

$$\mathrm{chr}_{\mathcal{P}}(G) \le \mathrm{ch}_{\mathcal{P}}(G) \qquad \text{and} \qquad \chi_{\mathcal{P}}(G) \le \mathrm{ch}_{\mathcal{P}}(G)\,.$$

Furthermore, it is proved in [BDM] that for every additive and hereditary property with finite completeness, $1 \le c(\mathcal{P}) < \infty$, and for every integer s, there exists a graph $G_s \in \mathcal{P}$ such that

$$\mathrm{ch}_{\mathcal{P}}(G_s) - \chi_{\mathcal{P}}(G_s) > s\,.$$

To investigate the relationship between $\mathrm{chr}_{\mathcal{P}}(G)$ and $\chi_{\mathcal{P}}(G)$ in a greater extent, we shall introduce a further parameter, the *fractional* $\mathcal{P}$*-chromatic number* of G.

Let $\mathcal{S}_P$ be the collection of all subsets $S \subseteq V(G)$ such that the subgraph $G[S]$ induced by S has the property $\mathcal{P}$. (The members of $\mathcal{S}_P$ are the "independent sets" with respect to non-$\mathcal{P}$, i.e., $\mathcal{I} - \mathcal{P}$).

Definition 6. A *fractional $\mathcal{P}$-colouring* of G is a mapping $\varphi\colon \mathcal{S}_P \to [0,1]$ such that

$$\sum_{\substack{S\in\mathcal{S}_P\\ v\in S}} \varphi(S) \geq 1\,, \qquad \forall\, v \in V(G)\,. \tag{1}$$

Definition 7. The *fractional $\mathcal{P}$-chromatic number*, denoted $\chi^*_{\mathcal{P}}(G)$, is the solution of the linear program (1) with objective function

$$\min \sum_{S\in\mathcal{S}_P} \varphi(S)\,.$$

In Section 4 we shall prove that the $\mathcal{P}$-choice ratio of any graph is equal to its fractional $\mathcal{P}$-chromatic number, for every hereditary property $\mathcal{P}$. This result was proved in [2] for the property $\mathcal{O}$ of being edgeless (independent).

3. Properties of $\chi^*_{\mathcal{P}}(G)$

Notice that restricting the range of fractional colourings φ from the interval $[0,1]$ to just $\{0,1\}$, we obtain $\mathcal{P}$-colourings. Therefore, the corresponding two invariants of chromatic number satisfy the inequality $\chi^*_{\mathcal{P}}(G) \leq \chi_{\mathcal{P}}(G)$ for every graph G.

Let φ^* be a fractional $\mathcal{P}$-colouring which realizes $\chi^*_{\mathcal{P}}(G)$. Furthermore, let $\mathcal{S}_0 = \{T_1,\dots,T_t\} \subseteq \mathcal{S}_P$ be the collection of subsets of $V(G)$ (with property $\mathcal{P}$) such that $T \in \mathcal{S}_0$ if and only if $\varphi^*(T) > 0$.

Recall that $\chi^*_{\mathcal{P}}(G)$ is defined to be the solution of a linear optimization problem. Therefore, there exist $q, r_1, \dots, r_t \in \mathbb{N}$ such that $\varphi^*(T_i) = r_i/q$ for all $i = 1,\dots,t$. Consequently, $\chi^*_{\mathcal{P}}(G) = r/q$ with $r := \sum_{i=1}^{t} r_i$. Furthermore, r can be chosen to be not greater than $(n+1)^{\frac{n+1}{2}}$, where $n = |V(G)|$. (For details, cf., e.g., [2].)

The following useful lemma can be concluded from these properties.

Lemma 1. *Let G be a graph with $\chi^*_{\mathcal{P}}(G) = r/q$. Then there exist r (not necessarily distinct) vertex subsets $S_1,\dots,S_r \in \mathcal{S}_P$ with the property $\mathcal{P}$ such that each vertex $v \in V(G)$ belongs to at least q of them.*

Notice that each T_i mentioned above occurs r_i times in the multiset $(S_1,\dots,S_r)$. Moreover, since $\mathcal{P}$ is supposed to be hereditary, one can also ensure that each vertex v belongs to *precisely* q of the sets T_i.

4. Determination of the $\mathcal{P}$-choice ratio

In this section we prove the following two results.

Theorem 1. *For any graph G, the $\mathcal{P}$-choice-ratio $\mathrm{chr}_{\mathcal{P}}(G)$ is equal to its fractional $\mathcal{P}$-chromatic number $\chi^*_P(G)$.*

Theorem 2. *For every integer n there exists a number $f(n) \leq (n+1)^{2n+2}$ such that the following holds. For every graph G with n vertices and with fractional $\mathcal{P}$-chromatic number $\chi^*_{\mathcal{P}}(G)$, and for every integer a which is divisible by all integers up to $f(n)$, G is $(a,b)_{\mathcal{P}}$-choosable where $b = a/\chi^*_{\mathcal{P}}(G)$.*

It follows, in particular, that "infimum" can be replaced by "minimum" in the definition of $\mathcal{P}$-choice ratio.

We prove Theorem 1 in two parts.

Proof of the lower bound $\mathrm{chr}_{\mathcal{P}}(G) \geq \chi^*_{\mathcal{P}}(G)$ **.**

Suppose that G is $(a,b)_{\mathcal{P}}$-choosable. We have to show that $\chi^*_{\mathcal{P}}(G) \leq a/b$. For this purpose, define identical lists

$$L(v) := \{1, \ldots, a\}$$

for all $v \in V(G)$. By assumption, we can choose a subset $C(v) \subseteq L(v)$ with $|C(v)| = b$ for every vertex $v \in V(G)$, such that, for every colour $f \in \{1, \ldots, a\}$, the subgraph G_f induced by $V_f := \{v \in V(G) \colon f \in C(v)\}$ has the property $\mathcal{P}$.

For every $S \in \mathcal{S}_P$, define

$$\varphi(S) := a_i/b,$$

where a_i is the number of occurrences of S in the multiset $(V_1, \ldots, V_a)$. In this way, for every $i \in \{1, \ldots, a\}$, the set V_i gets weight $1/b$. Obviously, φ is a fractional $\mathcal{P}$-colouring of G, because every vertex $v \in V(G)$ belongs to exactly b of the sets $V_1, \ldots, V_a$, and therefore $\sum\limits_{\substack{S \in \{V_1, \ldots, V_a\} \\ v \in S}} \varphi(S) = b/b = 1$ for all $v \in V(G)$.

Furthermore, we have $\sum\limits_{S \in \mathcal{S}_P} \varphi(S) = a/b$, and hence it follows that $\chi^*_{\mathcal{P}}(G) \leq a/b$.

Proof of the upper bound $\mathrm{chr}_{\mathcal{P}}(G) \leq \chi^*_{\mathcal{P}}(G)$ **.**

We have to show that, for every $\varepsilon > 0$, there exists a pair (a,b) such that G is $(a,b)_{\mathcal{P}}$-choosable and $a \leq (1+\varepsilon)\, b \cdot \chi^*_{\mathcal{P}}(G)$. Assume $\chi^*_{\mathcal{P}}(G) = r/q$. Set $a := (1+\varepsilon)rm$ and $b := qm$, for m sufficiently large, where, without loss of generality, $\varepsilon > 0$ is rational and both a and b are integers. Consider arbitrarily given lists $L(v)$ with $|L(v)| = a$ for all $v \in V(G)$, and define $L = \bigcup\limits_{v \in V(G)} L(v)$. By probabilistic arguments (see [2]) we obtain that, for sufficiently large m, there exists a partition

$$L = L_1 \cup \cdots \cup L_r$$

with $L_i \cap L_j = \emptyset$ for all $i \neq j$, such that

$$|L_i \cap L(v)| \geq m$$

for all $i = 1, \dots, r$ and for all $v \in V(G)$.

Now consider the multiset $(S_1, \dots, S_r)$ of Lemma 1, and define

$$C'(v) := \bigcup_{\substack{i \\ v \in S_i}} \bigl(L(v) \cap L_i\bigr).$$

Then $|C'(v)| \geq qm = b$ for all $v \in V$, because every vertex v belongs to at least q of the sets $S_1, \dots, S_r$. Obviously there is $C(v) \subseteq C'(v)$ with $|C(v)| = qm = b$.

Let $f \in C(v)$ be an arbitrary colour of any chosen colour set, and consider $V_f := \bigl\{v \in V(G)\colon f \in C(v)\bigr\}$. There is a unique subscript i such that f belongs to L_i. Thus, all vertices v with $f \in C(v)$ belong to S_i which has the property $\mathcal{P}$ by assumption. Consequently, $V_f \subseteq S_i$ has the property $\mathcal{P}$, and we obtain that G is $(a,b)_{\mathcal{P}}$-choosable. This completes the proof of Theorem 1.

P r o o f o f T h e o r e m 2. Assume that $|L(v)| = a$, where a fulfils the conditions of Theorem 2. Let $L = \bigcup_{v \in V(G)} L(v)$ be the set of all colours occurring in the lists of $V(G)$. Then a result on partitions of hypergraphs (see [6, 1, 2]) yields the following assertion. □

LEMMA 2. *There exist integers n_i, $i \in I$, with $n_i \leq (n+1)^{\frac{n+1}{2}}$, and a partition $L = \bigcup_{i \in I} L_i$ with $L_i \cap L_j = \emptyset$ for all $i \neq j$ such that $|L_i \cap L(v)| = n_i$ for all $i \in I$ and all $v \in V(G)$. Furthermore, for $r \leq (n+1)^{\frac{n+1}{2}}$ and $a = \sum_{i \in I} n_i$, there exists a partition $I = I_1 \cup \dots \cup I_r$ of I into r pairwise disjoint sets such that $\sum_{i \in I_j} n_i = a/r$ holds for every $1 \leq j \leq r$.*

Define $M_j := \bigcup_{i \in I_j} L_i$, and observe that $L = M_1 \cup \dots \cup M_r$, $M_i \cap M_j = \emptyset$ for all $i \neq j$, and $|M_j \cap L(v)| = a/r$ for all $j = 1, \dots, r$ and all $v \in V(G)$.

Consider the multiset $(S_1, \dots, S_r)$ of Lemma 1, and define

$$C'(v) := \bigcup_{\substack{j \\ v \in S_j}} \bigl(L(v) \cap M_j\bigr).$$

Because of Lemma 1, for every vertex $v \in V$ there are at least q sets S_j which contain v. Thus,

$$|C'(v)| \geq q \cdot a/r = a/\chi^*_{\mathcal{P}}(G).$$

Obviously there is $C(v) \subseteq C'(v)$ with

$$|C(v)| = a/\chi^*_{\mathcal{P}}(G).$$

Furthermore, we obtain that every set $V_f := \{v \in V(G)\colon f \in C(v)\}$ satisfies property $\mathcal{P}$, by similar arguments as in the previous proof. Consequently, G is $(a,b)_{\mathcal{P}}$-choosable with $b = a/\chi^*_{\mathcal{P}}(G)$, and the proof is complete.

5. Generalization for hypergraphs

In this last section we extend the previous results for hypergraphs (set systems). Let $\mathcal{H} = (X, \mathcal{F})$ be a hypergraph with finite vertex set X and hyperedge set $\mathcal{F}$.

Definition 8. A hypergraph $\mathscr{H} = (X, \mathcal{F})$ is called (a,b)-*choosable* if, for every list assignment L with $|L(x)| = a$ for all $x \in X$, it is possible to choose subsets $C(x) \subseteq L(x)$ with $|C(x)| = b$ such that

$$\bigcap_{x \in F} C(x) = \emptyset$$

holds for all $F \in \mathcal{F}$.

Definition 9. The quantity $\mathrm{chr}(\mathscr{H}) := \inf\{\frac{a}{b}\colon \mathscr{H} \text{ is (a,b)-choosable}\}$ is called the *choice ratio* of $\mathcal{H}$.

Moreover, for $\mathscr{H} = (X, \mathcal{F})$, define

$$\mathcal{A} := \{A \subseteq X\colon \text{ there is no } F \in \mathcal{F} \text{ with } F \subseteq A\},$$

the collection of independent sets in $\mathscr{H}$.

These definitions are generalizations of the corresponding previous concepts in the following sense. Let there be given a graph $G = (V, E)$ and a hereditary property $\mathcal{P}$. Define a hypergraph $\mathcal{H}$ with vertex set $X = V$ and hyperedge set $\mathcal{F} := \{Y \subseteq V\colon G[Y] \notin \mathcal{P} \text{ but } G[W] \in \mathcal{P} \text{ for all } W \subset Y\}$. Thus, $\mathcal{F}$ corresponds to the set of the minimal vertex subsets of G which do not belong to $\mathcal{P}$.

In this way, the problem of $(a,b)_{\mathcal{P}}$-choosability of the graph G is converted to the problem of (a,b)-choosability of a hypergraph $\mathscr{H}$. Moreover, we note that the $(a,b)_{\mathcal{P}}$-choosability of a hypergraph is the same thing as the (a,b)-choosability of a suitably constructed other hypergraph (i.e., it is irrelevant in this context to introduce "hereditary hypergraph-properties" analogously to the graph properties of the previous sections). Indeed, a hereditary property $\mathcal{P}$ on a *fixed* hypergraph is equivalently described by those subsets S of the vertex set X which satisfy $\mathcal{P}$. Let now $\mathcal{F}$ consist of those $F \subseteq X$ which *do not satisfy*

$\mathcal{P}$. By the assumption that $\mathcal{P}$ is hereditary, it is easily seen that $\mathscr{H} = (X, \mathcal{F})$ satisfies the requirements.

We note further that the set $\mathcal{A}$ in the above construction corresponds to the set $\mathcal{S}_P$ of all subsets of V belonging to $\mathcal{P}$.

In the sequel we recall two definitions from the literature, which, by what has been said, generalize the concepts of fractional $\mathcal{P}$-colouring and fractional $\mathcal{P}$-chromatic number, too.

Definition 10. A *fractional colouring* of $\mathscr{H}$ is a mapping $\varphi\colon \mathcal{A} \to [0,1]$ such that

$$\sum_{\substack{A \in \mathcal{A} \\ x \in A}} \varphi(A) \geq 1\,, \qquad \forall x \in X\,. \tag{2}$$

Definition 11. The *fractional chromatic number*, denoted $\chi^*(\mathscr{H})$, is the solution of the linear program (2) with objective function

$$\min \sum_{A \in \mathcal{A}} \varphi(A)\,.$$

Using analogous arguments as in the previous section, one obtains the following results.

Theorem 3. *The choice ratio* $\operatorname{chr}(\mathscr{H})$ *for any hypergraph* $\mathscr{H}$ *equals its fractional chromatic number* $\chi^*(\mathscr{H})$.

Theorem 4. *For every integer* n *there exists a number* $f(n) \leq (n+1)^{2n+2}$ *such that the following holds. For every hypergraph* $\mathscr{H}$ *with* n *vertices and with fractional chromatic number* $\chi^*(\mathscr{H})$, *and for every integer* a *which is divisible by all integers up to* $f(n)$, $\mathscr{H}$ *is* (a,b)*-choosable where* $b = a/\chi^*(\mathscr{H})$.

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