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NON-HAMILTONIAN MAXIMAL PLANAR GRAPHS WITH HIGH TOUGHNESS

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ABSTRACT. Non-hamiltonian maximal planar graphs with toughness greater than $\frac{3}{2} - \delta$, where δ is an arbitrarily small positive number, are constructed and it is shown that the family of such graphs has a shortness coefficient less than one.

1 Introduction

We consider only finite undirected graphs without loops or multiple edges. For a graph $G = (V, E)$, $v(G) = |V(G)|$, the order of G, and $h(G)$ denotes the length of a longest cycle in G. A planar graph is *maximal* if the insertion of any extra edge makes it non-planar. A graph is *polyhedral* if it is planar and 3-connected, which, by Steinitz's Theorem, implies that it is the graph of some polyhedron. If G is a maximal planar graph, then every face of G is triangular and G is polyhedral, unless it is just a 3-cycle. For definitions and notation not explained here, see, for example, Bondy and Murty [2].

The *toughness*, $t(G)$, of any graph G is defined, following Chvátal [3], by

$$t(G) = \min\left\{ \frac{|S|}{\omega(G-S)} : S \subseteq V(G)\,,\ \omega(G-S) > 1 \right\},$$

where $\omega(G-S)$ is the number of components of $G-S$. Toughness and connectivity are related by the inequality $t(G) \leq \frac{1}{2}\kappa(G)$. If $t(G) < 1$, then G is non-hamiltonian by [2, Theorem 4.2].

Chvátal [3] conjectured that, for some t_0, every graph with toughness at least t_0 is hamiltonian and, later, the value $t_0 = 2$ was proposed. However, Bauer, Broersma and Veldman [1] have shown that there are non-hamiltonian graphs with toughness arbitrarily close to $\frac{9}{4}$ so, if t_0 exists, $t_0 \geq \frac{9}{4}$.

By a theorem of Tutte [8], if G is planar and 4-connected, then G is hamiltonian. It follows that, if G is polyhedral and non-hamiltonian, then $\kappa(G) = 3$ and $t(G) \leq \frac{3}{2}$.

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Non-hamiltonian regular polyhedral graphs of degrees 3, 4 and 5 and with toughness $\frac{3}{2}$ were constructed by Harant [5]. In Harant and Owens [7], non-hamiltonian maximal planar graphs with toughness $\frac{5}{4}$ were constructed and the shortness exponent of the family of such graphs was shown to be less than one. We recall that, for any infinite family of graphs $\mathcal{G}$, the *shortness exponent*, $\sigma(\mathcal{G})$, and *shortness coefficient*, $\rho(\mathcal{G})$, are defined as in Grünbaum and Walther [4] by

$$\sigma(\mathcal{G}) = \liminf_{G\in\mathcal{G}} \left(\frac{\log h(G)}{\log v(G)}\right), \qquad \rho(\mathcal{G}) = \liminf_{G\in\mathcal{G}} \left(\frac{h(G)}{v(G)}\right).$$

In the present paper we prove the following

THEOREM. *Let integers $k \geq 1$, $r \geq 1$ be given. There exists a non-hamiltonian maximal planar graph $G = G(k,r)$ such that*

$$(1) \quad h(G) \leq v(G) - r\,,$$

$$(2) \quad t(G) = \frac{18k+8}{12k+6}\,.$$

In Section 2 we construct $G(k,r)$ and prove part (1) of the theorem. Section 3 contains the proof of part (2). Since (as will be seen below) $v(G)$ is linear in r, the theorem has the following immediate corollary:

COROLLARY. *Given any $\delta > 0$, the family of maximal planar graphs with toughness $> \frac{3}{2} - \delta$ not only contains non-hamiltonian members but has shortness coefficient less than 1.*

2 Construction and non-hamiltonicity

We construct $G(k,r)$ in the form of a plane embedding and it is obtained by joining together a number of fragments and inserting some extra edges. By a *fragment* we mean a connected plane map whose faces, other than the *exterior* (infinite) face are all triangular. We distinguish *boundary vertices*, which are on the boundary of the exterior face, from *interior vertices*. Fragments are joined by identifying boundary vertices and, if necessary, inserting extra edges to form larger fragments or, eventually, $G(k,r)$.

Let T denote the complete graph on four vertices when used as a fragment. It has three boundary vertices s, t and u (see Figure 1) and one interior vertex. In later diagrams it is represented by a labelled triangle.

Next, we define a fragment H by means of Figure 2. It is the main fragment used in our construction and it has been used previously in studying toughness

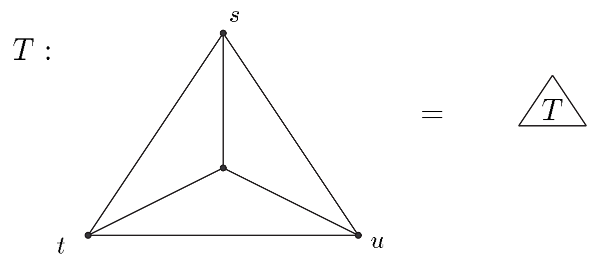

FIGURE 1. The fragment T.

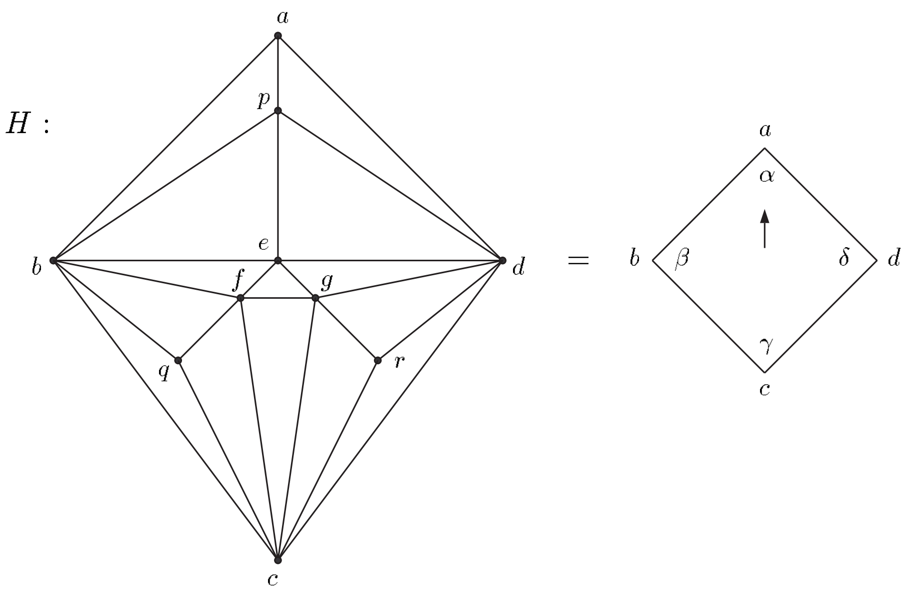

FIGURE 2. The fragment H.

by J. Harant and J. van den Heuvel [6]. An *H-fragment* is any fragment isomorphic to H and it is represented in later diagrams by a quadrilateral containing an arrow that points towards the boundary vertex a (see Figure 2).

We use K to mean either H or T. When a planar graph G has a K-fragment as an induced subgraph and C is a cycle in G that intersects both K and $G-K$, we define the *edge numbers* of K (for the cycle C) to be the numbers of edges of $C \cap K$ incident at the boundary vertices of K.

The importance of H depends on the following lemma.

LEMMA 1. *Let G be a planar graph with H as an induced subgraph. Let C by a cycle in G which contains every interior vertex of H and also intersects $G-H$. Let $\alpha, \beta, \gamma, \delta$ be the edge numbers of H, as shown in Figure 2. If any one of the edge numbers β, γ, δ is 0, then*

$$(1) \quad \alpha = 1, \qquad (2) \quad \{\beta, \gamma, \delta\} = \{0, 1, 2\}.$$

Proof. By trial. □

The arrangements of edge numbers allowed by the lemma are those shown in Figure 3, together with reflections in the line ac.

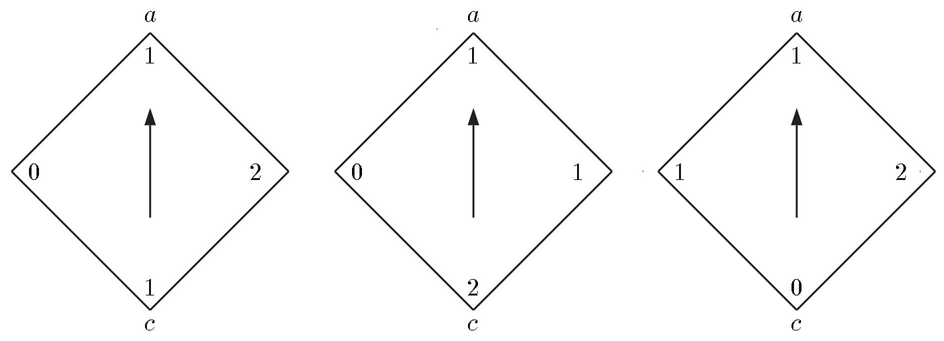

FIGURE 3. Edge numbers of H.

Any fragment other than a K-fragment will be called a *big fragment.* We form a big fragment $P(k)$ by joining $4k+1$ H-fragments and two T-fragments, together with some extra edges to make all interior faces triangular, as shown in Figure 4. $P(k)$ is symmetrical about the line a_0x.

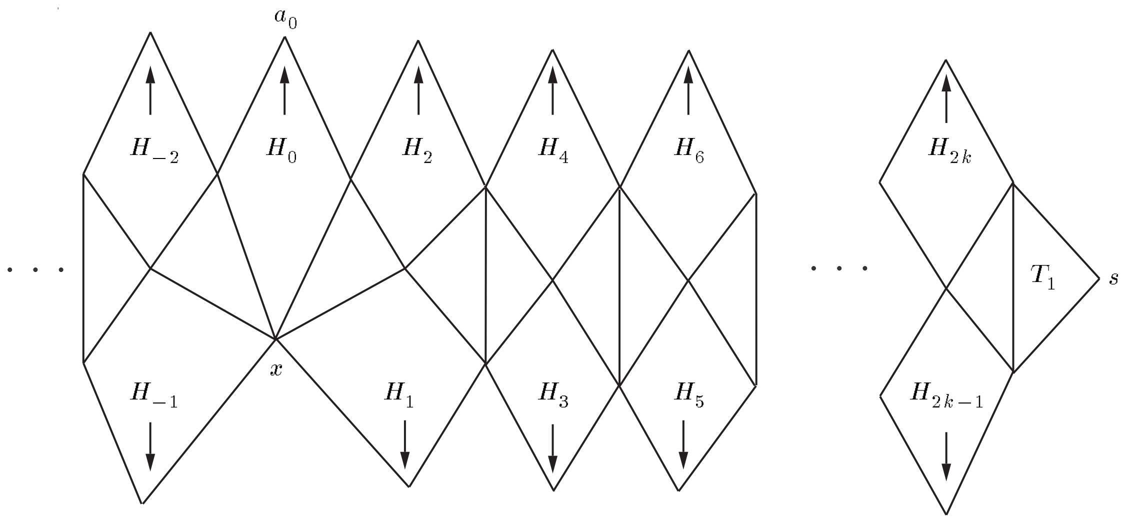

FIGURE 4. The large fragment $P(k)$.

Any big fragment isomorphic to $P(k)$ is a $P(k)$*-fragment.* We join $r+2$ $P(k)$-fragments, by identifying the $r+2$ vertices x, to form a big fragment

$Q(k,r) = Q$. This fragment has $(4k+3)(r+2)$ boundary vertices labelled a or s, alternating with boundary vertices of higher degree. Define $n = (4k+3)(r+2)$.

We need one more big fragment R, which is constructed as follows. Surround a vertex z successively by non-intersecting n-cycles C_3, C_2, C_1, C_0. Insert an edge from z to each vertex of C_3 and, for $i = 3, 2, 1$, two edges from each vertex of C_i to two adjacent vertices of C_{i-1} in such a way that R is planar and all its interior faces are triangular. Part of R is shown in Figure 5.

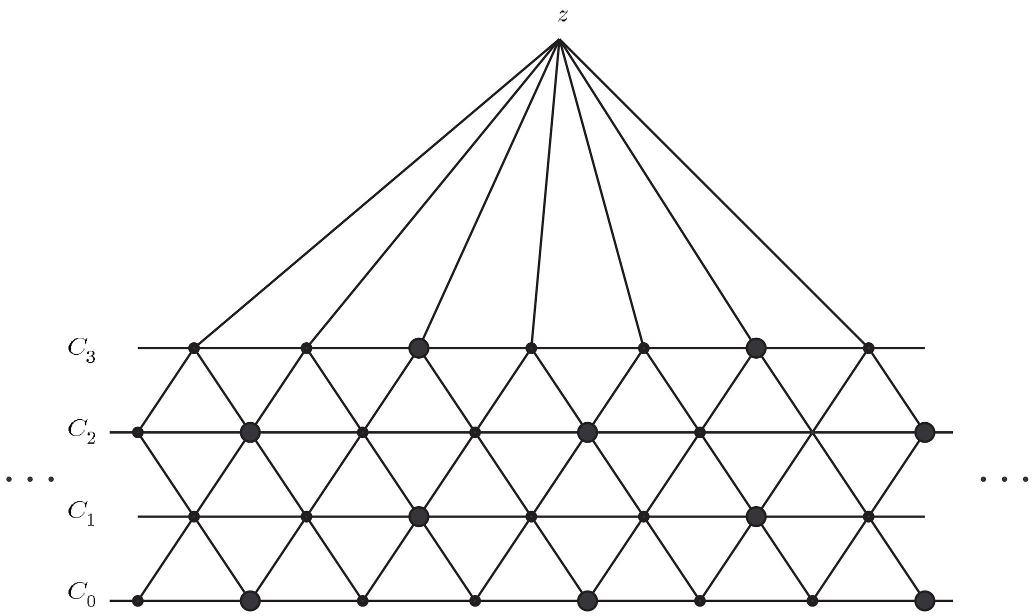

FIGURE 5. Part of R.

We complete the construction by taking $Q(k,r)$ and R and identifying the n boundary vertices, a and s, of degree 3 in $Q(k,r)$ with the n boundary vertices of R, namely the vertices of C_0, to form a maximal planar graph $G(k,r)$.

Before we prove part (1) of our theorem, which we restate as Proposition 1, we state an obvious but useful lemma.

LEMMA 2. *Let v be a common boundary vertex of two K-fragments. Then the sum of their edge numbers at v is at most 2.*

Proof. No cycle can pass through v more than once. □

PROPOSITION 1. *$G(k,r)$ is non-hamiltonian and, in fact, every cycle in it misses at least r vertices.*

Proof. Consider any cycle C in $G(k,r)$. Since C passes at most once through x, at least r of the $P(k)$-fragments do not contain any edges of C

incident at x. Let P_1 be one such $P(k)$-fragment. We suppose that C contains all the interior vertices of P_1 and obtain a contradiction by repeated use of Lemmas 1 and 2.

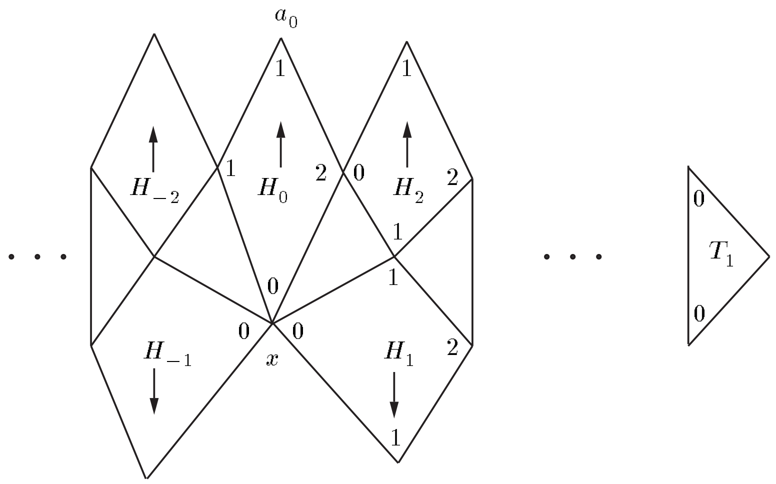

FIGURE 6. Edge numbers in P_1.

Denote the edge numbers of H_i by $\alpha_i, \beta_i, \gamma_i, \delta_i$ for all i. Starting at x, we have $\beta_{-1} = \gamma_0 = \delta_1 = 0$, by the choice of P_1. Since $\gamma_0 = 0$, Lemma 1 gives $\alpha_0 = 1$ and (without loss of generality) $\beta_0 = 1$, $\delta_0 = 2$. Since $\delta_0 = 2$, Lemma 2 gives $\beta_2 = 0$ so, by Lemma 1, $\alpha_2 = 1$ and $\{\gamma_2, \delta_2\} = \{1, 2\}$. Similarly, in H_1 we have $\{\beta_1, \gamma_1\} = \{1, 2\}$. By Lemma 2 the only possibility is $\gamma_1 = \gamma_2 = 1$, so $\beta_1 = \delta_2 = 2$. Similar reasoning can be applied to H_3, H_4, and so on. When we reach T we find that two of its edge numbers are 0, so C cannot contain the interior vertex of T, a contradiction.

Since there were r choices for P_1, the lemma follows. $\square$

3. Proof of the main result

For brevity, write $G(k, r) = G$. We start with the easier half of the proof, restated as a proposition.

PROPOSITION 2. $t(G) \leq \frac{18k+8}{12k+6}$.

P r o o f. Choose one of the $P(k)$-fragments in G, say P_0. Define $S_0 \subseteq V(G)$ to be the set of all boundary vertices of the K-fragments in P_0, together with the vertices e, f (see Figure 2) of all the H-fragments in P_0. Then $|S_0| = 18k + 8$.

The graph $G - S_0$ has 3 components within each H-fragment of P_0 and 1 component in each T-fragment of P_0. There is also a large component $G - P_0$. Hence $\omega(G - S_0) = 12k + 6$. The proposition follows. □

It will be convenient to define $\varepsilon = 1/(12k+6)$. In this notation, Proposition 2 states that $t(G) \le \frac{3}{2} - \varepsilon$. The rest of this section is devoted to proving that $t(G) \ge \frac{3}{2} - \varepsilon$.

Let S be an arbitrary vertex cutset of G and define

$$f_S(G) = |S| - \left(\tfrac{3}{2} - \varepsilon\right)\omega(G - S).$$

We need to prove that $f_S(G) \ge 0$ for all S. Our strategy is to deal first with single fragments (of type H, T or R) and then consider the interactions between them.

The boundary vertices of K-fragments and of R are shared with other fragments to which they are joined. We assign *weights* to the boundary vertices of a given fragment as follows:

(1) For a K-fragment, the vertices b, c, d (or t, u) have weight $\frac{1}{2}$.

(2) For a K-fragment, the vertex a (or s) has weight 1.

(3) For R, every vertex on its boundary C_0 has weight 0.

All interior vertices of fragments have weight 1.

Let $|S|_K$, $|S|_R$ denote the weighted numbers of vertices of S in K, R, respectively. Then

$$|S| = X + \sum_{K \subseteq G} |S|_K + |S|_R,$$

where X is a correction term relating to the vertex x. Since $3(r+2)$ boundary vertices b, c or d of H-fragments are identified at x, the above rules give x a weight $\frac{3}{2}(r+2)$ instead of 1. Hence we define $X = -\frac{3}{2}r - 2$ if $x \in S$ and $X = 0$ if $x \notin S$.

For any fragment F we write $F - S$ for $F - S \cap F$.

We classify components J_K of $K - S$ as follows:

Type 1: J_K contains only interior vertices of K.

Type 2: J_K contains exactly one of the vertices b, c, d (or t, u) but does not contain a (or s).

Type 3: J_K contains at least two of the vertices b, c, d (or t, u) but does not contain a (or s).

Type 4: J_K contains a (or s).

We assign *weights* $1, \frac{1}{2}, \frac{1}{3}, 0$ to components of types $1, 2, 3, 4$, respectively, and denote the weighted number of components of type i by $\omega_i(K)$ for $1 \le i \le 3$. (Components of type 4 are treated as extensions of R.)

Lemma 3. $\omega(G-S) \le \sum_{K \subseteq G} \big(\omega_1(K)+\omega_2(K)+\omega_3(K)\big) + \omega(R-S)$.

Proof. Let J be a component of $G-S$ that intersects a particular K-fragment and let $J_K = J \cap K$. If J_K is of type i (where $i = 2$ or 3), then J intersects at least $i-1$ other K-fragments so it is counted at least i times in the sum over all K-fragments and still at least once after the weighting factor $\frac{1}{i}$ has been applied. □

For each K-fragment in G, define

$$g_S(K) = |S|_K - \left(\tfrac{3}{2}-\varepsilon\right)\big(\omega_1(K)+\omega_2(K)+\omega_3(K)\big)\,.$$

Also define

$$g_S(R) = |S|_R - \left(\tfrac{3}{2}-\varepsilon\right)\omega(R-S)$$

and

$$g_S(G) = X + \sum_{K \subseteq G} g_S(K) + g_S(R)\,.$$

We now have $f_S(G) \ge g_S(G)$. In order to show that $f_S(G) \ge 0$ for all cutsets S in G, and thereby complete the proof of the theorem, we shall prove the stronger inequality $g_S(G) \ge 0$.

In the above sum defining $g_S(G)$, the value of X has already been given so we deal next with $g_S(K)$, that is, $g_S(H)$ and $g_S(T)$. In Table 1, column 1 specifies which boundary vertices of H are in S and column 4 gives the minimum corresponding value of $g_S(H)$, taken over all possible choices of interior vertices of H to be in S. Columns 2, 3 will be used later to check whether a given subset $S \subseteq V(G)$ is a vertex cutset of G.

Table 1

$S \cap \{a,b,c,d\}$	$\omega_1(H)$	$\omega_2(H)+\omega_3(H)$	$\min\{g_S(H)\}$
$\{a,b,c,d\}, \{b,c,d\}$	3	0	3ε
$\{a,b,c\}, \{a,b,d\}, \{a,c,d\}$	1	$\frac{1}{2}$	$\frac{3}{4}+\frac{3}{2}\varepsilon$
$\{b,c\}, \{c,d\}$	1	0	$\frac{1}{2}+\varepsilon$
$\{a,b\}, \{a,c\}, \{a,d\}$	0	$\frac{1}{3}$	$1+\frac{1}{3}\varepsilon$
$\{a\}$	0	$\frac{1}{3}$	$\frac{1}{2}+\frac{1}{3}\varepsilon$
$\{b,d\}$	0	0	1
$\{b\}, \{c\}, \{d\}$	0	0	$\frac{1}{2}$
$\emptyset$	0	0	0

The entries in Table 1 are easily obtained by trial. Note that there are no negative entries in column 4. This would not have been so if we had assigned the same weight $\frac{1}{2}$ to a as to b, c and d. Four different choices of $S \cap V(H)$ yield the entry 3ε in column 4; in the notation of Figure 2, two of these choices are $\{a, b, c, d, e, g\}$ and $\{p, b, c, d, f, g\}$.

Table 2 gives information about $g_S(T)$.

Table 2

$S \cap \{s, t, u\}$	$\omega_1(T)$	$\omega_2(T) + \omega_3(T)$	$\min\{g_S(T)\}$
$\{s, t, u\}$	1	0	$\frac{1}{2} + \varepsilon$
$\{s, t\}, \{s, u\}$	0	$\frac{1}{2}$	$\frac{3}{4} + \frac{1}{2}\varepsilon$
$\{s\}$	0	$\frac{1}{3}$	$\frac{1}{2} + \frac{1}{3}\varepsilon$
$\{t, u\}$	0	0	1
$\{t\}, \{u\}$	0	0	$\frac{1}{2}$
$\emptyset$	0	0	0

Now consider $g_S(R)$. If $\omega(R - S) = 1$, then $g_S(R) \geq -\frac{3}{2} + \varepsilon$ and this lower bound is attained when $|S|_R = 0$. (Recall that $|S|_R$ is the number of vertices of S in $R - C_0$, since the vertices of C_0 have weight 0 in R).

As a guide to how $|S|_R$ and $\omega(R - S)$ are related in general, consider the case $S = S_1$, where S_1 is such that the components of $R - S_1$ are single vertices that form a maximum independent set in R (see the large dots in Figure 5). If we suppose, for simplicity, that $n \equiv 0 \pmod 3$, then $\omega(R - S_1) = \frac{4n}{3}$ and $|S_1|_R = 1 + 3 \cdot \frac{2n}{3} = 1 + \frac{3}{2}\omega(R - S_1)$, so $g_{S_1}(R) = \varepsilon\omega(R - S_1)$. In order to generalize this result, we first analyse $g_{S_1}(R)$ into equal contributions ε from all components of $R - S_1$. To each $v \in S_1 \cap (R - C_0)$ and to each component of $R - S_1$ to which v is adjacent we assign a *partial weight*. The following conditions must be satisfied when $S = S_1$.

Condition 1: The sum of partial weights used by any vertex $v \in S \cap (R - C_0)$ is ≤ 1.

Condition 2: The sum of partial weights used on any component of $R - S$ is $\geq \frac{3}{2}$.

Table 3 shows an assignment of partial weights that can be used when $S = S_1$. Condition 1 is satisfied (with equality) since the row sums are equal to one. Condition 2 holds (also with equality) for a component of $R - S_1$ lying on any one of the four cycles C_i. For instance, a component on C_1 has two adjacent vertices of S_1 on C_1 and two on C_2, so the sum of partial weights used on it is $2 \cdot \frac{5}{8} + 2 \cdot \frac{1}{8} = \frac{3}{2}$.

TABLE 3

vertex in C_i	component in C_j			
i	$j=0$	$j=1$	$j=2$	$j=3$
1	$\frac{3}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	–
2	–	$\frac{5}{8}$	$\frac{1}{8}$	$\frac{1}{4}$
3	–	–	$\frac{1}{2}$	$\frac{1}{2}$

Now consider any $S \subseteq V(G)$ such that $\omega(R-S) > 1$. Components of $R-S$ need not be single vertices, so Table 3 must be interpreted in a slightly different way.

LEMMA 4. *If $\omega(R-S) > 1$, then $g_S(R) \geq \varepsilon\omega(R-S)$.*

Proof. Suppose at first that $z \in S$ and define $R' = R-z$. Then $\omega(R-S) = \omega(R'-S)$. Let v be a vertex of S in $R'-C_0$ and suppose that v is adjacent to a component W of $R'-S$. Choose any vertex w of W that is adjacent to v and define the partial weight of v for use on W to be the value given in Table 3 for a component $\{w\}$. The sum of partial weights of v used on adjacent components of $R'-S$ is ≤ 1 in all cases. (Note, in particular, that if the chosen vertices w_1, w_2 of two components adjacent to v lie on the same cycle C_i as v, then the sum of partial weights used by v is twice an entry in Table 3 for equal i and j, so is still ≤ 1.)

Let W be any component of $R'-S$. Since W is not the only component, there are vertices of S separating W from the remainder of $R'-S$. We shall show that $|S|_W \geq \frac{3}{2}$, where $|S|_W$ denotes the sum of partial weights used on W by the vertices of S adjacent to it. The proof is almost trivial when there are 12 or more vertices of S in $R'-C_0$ adjacent to W. In fact, every entry in Table 3 is at least $\frac{1}{8}$, so we have $|S|_W \geq 12 \cdot \frac{1}{8} = \frac{3}{2}$. Now, therefore, assume that there are at most 11 vertices of S in $R'-C_0$ that are adjacent to W. Because $n > 11$, the subgraph induced by these vertices cannot contain a cycle separating z from C_0. Hence at least one (and at most three) of the following eight assertions are true:

(1.i) W intersects C_i, for $i = 1, 2$ or 3.

(2.i) W intersects C_{i-1} but not C_i, for $i = 1, 2$ or 3.

(3.i) W intersects C_{i+1} but not C_i, for $i = 1$ or 2.

Corresponding to each true assertion there are at least 2 vertices of S on C_i that are adjacent to W. For these vertices we can use the appropriate partial weights as given in Table 3.

TABLE 4

$\{i\colon W \cap C_i \neq \emptyset\}$	lower bound to $\lvert S\rvert_W$
$\{0,1,2,3\}, \{1,2,3\}$	$2(\frac{1}{8}+\frac{1}{8}+\frac{1}{2})$
$\{0,1,2\}, \{1,2\}$	$2(\frac{1}{8}+\frac{1}{8}+\frac{1}{2})$
$\{2,3\}$	$2(\frac{1}{8}+\frac{1}{2}+\frac{1}{8})$
$\{0,1\}, \{1\}$	$2(\frac{1}{8}+\frac{5}{8})$
$\{0\}$	$2(\frac{3}{4})$
$\{2\}$	$2(\frac{1}{8}+\frac{1}{2}+\frac{1}{8})$
$\{3\}$	$2(\frac{1}{2}+\frac{1}{4})$

In Table 4, column 1 shows which of the cycles C_i, $0 \le i \le 3$, intersect W and column 2 shows the corresponding calculation of a lower bound to $|S|_W$. Each calculation yields the value $\frac{3}{2}$.

As an example, consider row 3 of the table. Here W intersects C_2 and C_3 but not C_1 (or C_0). Assertions (1.2), (1.3) and (3.1) are the true ones and the entries in Table 3 to be used for the corresponding pairs of vertices are $\frac{1}{8}$, $\frac{1}{2}$ and $\frac{1}{8}$, respectively. Hence

$$|S|_W \ge 2\left(\frac{1}{8}+\frac{1}{2}+\frac{1}{8}\right) = \frac{3}{2}.$$

We have shown that, if $z \in S$, then $|S|_W \ge \frac{3}{2}$ for every component of $R-S$. Hence

$$|S|_R \ge 1 + \frac{3}{2}\omega(R-S) \qquad \text{and} \qquad g_S(R) \ge 1 + \varepsilon\omega(R-S).$$

Now suppose that $z \notin S$ and not every vertex of C_3 is in S. Then C_3 intersects at least one component of W of $R'-S$ and z is in the component of $R-S$ containing W. Hence

$$\omega(R-S) \le \omega(R'-S) \qquad \text{and} \qquad |S|_R \ge \frac{3}{2}\omega(R'-S),$$

so

$$g_S(R) \ge |S|_R - \left(\frac{3}{2}-\varepsilon\right)\omega(R-S) \ge \varepsilon\omega(R-S).$$

Finally, if $z \notin S$ and every vertex of C_3 is in S, then $\{z\}$ is a component of $R-S$ and $\omega(R-S) = \omega(R'-S)+1$. No vertex of C_3 is adjacent to a vertex on C_3 of a component of $R'-S$, so a partial weight $\frac{1}{2}$ can be used instead on

$\{z\}$ by every vertex of C_3. As before, $|S|_W \geq \frac{3}{2}\omega(R'-S)$, so

$$g_S(R) \geq n \cdot \frac{1}{2} + \frac{3}{2}\omega(R'-S) - \left(\frac{3}{2} - \varepsilon\right)\omega(R-S) = \frac{n-3}{2} + \varepsilon\omega(R-S).$$

In every case, $g_S(R) \geq \varepsilon\omega(R-S)$, as was to be proved. □

We call an H-fragment *full* if its boundary vertices b, c and d are all in S, that is, if row 1 of Table 1 applies to it. For any subgraph N of G induced by a set of K-fragments define $g_S(N) = \sum_{K \subseteq N} g_S(K)$. Let M denote the subgraph of G induced by all the full H-fragments in a particular $P(k)$-fragment P. Let L denote the subgraph induced by all K-fragments in P that are not full H-fragments but are attached to full H-fragments.

Proposition 3. *Let S be any vertex cutset of G such that $x \notin S$. Then $g_S(G) \geq 0$.*

Proof. Since $x \notin S$, $X = 0$ and so $g_S(G) = g_S(Q) + g_S(R)$. If $\omega(R-S) \geq 2$, then $g_S(R) > 0$ by Lemma 4. Every term in $g_S(Q)$ is non-negative so it follows that $g_S(G) \geq 0$.

In the remainder of the proof, assume that $\omega(R-S) = 1$ and so $g_S(R) \geq -\frac{3}{2} + \varepsilon$. Since S is a vertex cutset of G and $\omega(R-S) = 1$, there is at least one component of $G - S$ that lies in Q and is disjoint from R. Consider three cases:

Case 1. Some $P(k)$-fragment P contains a full H-fragment H_j. Since $x \notin S$, $j \notin \{-1, 0, 1\}$. There may be other full H-fragments besides H_j in P, but in any case L contains at least three K-fragments. By Tables 1 and 2, $g_S(L) \geq 3 \cdot \frac{1}{2}$. Hence $g_S(P) \geq g_S(L) + g_S(H_j) \geq \frac{3}{2} + 3\varepsilon$ and

$$g_S(G) \geq g_S(P) + g_S(R) \geq 4\varepsilon > 0.$$

Case 2. There is no full H-fragment in Q, but there is a fragment K whose interior contains a whole component of $G - S$. Here, row 2 or 3 of Table 1 or row 1 of Table 2 is applicable. Hence $g_S(K) \geq \frac{1}{2} + \varepsilon$ and also two of the boundary vertices b, c and d (or t and u) of K are in S. Therefore at least two other fragments K', K'' are attached to K at different vertices of S. It follows that

$$g_S(Q) \geq g_S(K) + g_S(K') + g_S(K'') \geq \left(\frac{1}{2} + \varepsilon\right) + 2 \cdot \frac{1}{2} = \frac{3}{2} + \varepsilon,$$

and

$$g_S(G) = g_S(Q) + g_S(R) \geq 2\varepsilon > 0.$$

Case 3. Every component of $G - S$ disjoint from R intersects more than one K-fragment. Let J be such a component. It intersects at least one H-fragment, for which, by row 4 or 5 of Table 1, $\omega_3(H) = \frac{1}{3}$. This implies that J intersects at least three K-fragments. For each of these fragments $g_S(K) \geq \frac{1}{2} + \frac{1}{3}\varepsilon$, by rows 4, 5 of Table 1 and rows 2, 3 of Table 2. Hence $g_S(Q) \geq 3\left(\frac{1}{2} + \frac{1}{3}\varepsilon\right) = \frac{3}{2} + \varepsilon$ and

$$g_S(G) = g_S(Q) + g_S(R) \geq 2\varepsilon > 0\,.$$

□

Before Proposition 4 we need a lemma about full H-fragments when $x \in S$. The proof makes free use of Tables 1 and 2.

Lemma 5. *Let P be a $P(k)$-fragment of G that contains at least one full H-fragment and suppose that $x \in S$. Then $g_S(P) \geq 2 - \varepsilon$.*

Proof. Let there be m full H-fragments in P. First suppose that $m = 1$ and let H_j be the full H-fragment where, by symmetry, we may assume that $j \geq 0$. If $j = 0$, then $L = H_{-2} \cup H_{-1} \cup H_1 \cup H_2$; if $j = 1$, then $L = H_{-1} \cup H_0 \cup H_2 \cup H_3$. In either case, $g_S(P) > g_S(L) \geq 4 \cdot \frac{1}{2} = 2$. If $j > 1$, then L consists of just three K-fragments so $g_S(L) \geq 3 \cdot \frac{1}{2}$. However, in this case $H_{-1} \not\subseteq L$ and $g_S(H_{-1}) \geq \frac{1}{2}$ since $x \in S$. Hence $g_S(P) > g_S(H_{-1}) + g_S(L) \geq 2$.

In the remainder of the proof, assume that $m \geq 2$ and let ℓ denote the number of K-fragments in L. Clearly $\ell \geq 2$ and, if $\ell \geq 4$, then $g_S(P) > g_S(L) \geq 4 \cdot \frac{1}{2} = 2$. If $\ell = 2$, then $L = T_{-1} \cup T_1$, $M = \{H_i\colon -2k \leq i \leq 2k\}$ and $m = 4k+1$. Moreover, T_{-1} and T_1 both have their vertices t, u in S, so $g_S(L) \geq 2\left(\frac{1}{2} + \varepsilon\right)$ by rows 1 and 4 of Table 2. Hence $g_S(P) \geq (1 + 2\varepsilon) + (4k + 1) \cdot 3\varepsilon = 2 - \varepsilon$.

Now consider the case $\ell = 3$. At least one of the two T-fragments must be in L. If both are in L, then $L = T_{-1} \cup H_j \cup T_1$ where H_j is attached to either T_{-1} or T_1. (Since H_j is not full, its boundary vertices b, c, d cannot all be shared with full H-fragments.) Here, $m = 4k$ so $g_S(M) \geq 4k \cdot 3\varepsilon$, also $g_S(L) \geq 3 \cdot \frac{1}{2}$. Hence $g_S(P) \geq \frac{3}{2} + 12k\varepsilon > 2$. If only one T-fragment is in L, then, by symmetry, we may assume that $L = H_{j_1} \cup H_{j_2} \cup T_1$ with $j_1 < j_2$. (Either H_{j_1} and H_{j_2} are attached to different boundary vertices of a full H-fragment or they are attached to one another.) Consider three cases:

Case 1. $j_1 = -1$, $j_2 = 0$. Here, $g_S(L) \geq \frac{1}{2} + \left(\frac{1}{2} + \varepsilon\right) + \left(\frac{1}{2} + \varepsilon\right)$, $M = \{H_i\colon 1 \leq i \leq 2k\}$ and $g_S(M) \geq 2k \cdot 3\varepsilon$. Hence

$$g_S(P) \geq \left(\frac{3}{2} + 2\varepsilon\right) + 6k\varepsilon = 2 - \varepsilon\,.$$

Case 2. $0 \leq j_1 < j_2$. Here, $g_S(L) \geq 3 \cdot \frac{1}{2}$ and $g_S(H_{-1}) \geq \frac{1}{2}$ since H_{-1} is not full and $x \in S$. Hence

$$g_S(P) > g_S(H_{-1}) + g_S(L) \geq 2\,.$$

Case 3. $j_1 < j_2 < 0$. Here, $g_S(L) \geq 3 \cdot \frac{1}{2}$ and $m \geq 2k+1$, so $g_S(M) \geq (2k+1) \cdot 3\varepsilon = \frac{1}{2}$. Hence

$$g_S(P) \geq g_S(L) + g_S(M) \geq 2\,.$$

□

The proof of Proposition 4 follows a similar pattern to that of Proposition 3 but also uses Lemma 5.

Proposition 4. *Let S be any vertex cutset of G such that $x \in S$. Then $g_S(G) \geq 0$.*

Proof. Since $x \in S$, $X = -\frac{3}{2}r - 2$ and $g_S(G) = -\frac{3}{2}r - 2 + g_S(Q) + g_S(R)$. Suppose first that $\omega(R - S) \geq 2$, so that $g_S(R) > 0$ by Lemma 4. Let P be any $P(k)$-fragment in G. If P contains at least one full H-fragment then, by Lemma 5, $g_S(P) \geq 2 - \varepsilon > \frac{3}{2}$. Otherwise, $g_S(P) \geq g_S(H_{-1}) + g_S(H_0) + g_S(H_1) \geq \frac{3}{2}$ by Table 1. Hence $g_S(Q) \geq (r+2) \cdot \frac{3}{2}$ and $g_S(G) > 1$.

In the remainder of the proof, assume that $\omega(R - S) = 1$, so $g_S(R) \geq -\frac{3}{2} + \varepsilon$ and $g_S(G) \geq -\frac{3}{2}(r+1) - 2 + \varepsilon + g_S(Q)$. Since S is a vertex cutset of G and $\omega(R - S) = 1$, there is at least one component of $G - S$ which is in Q and disjoint from R.

Consider three cases:

Case 1. Some $P(k)$-fragment P contains a full H-fragment H_j. By Lemma 5, $g_S(P) \geq 2 - \varepsilon$. For any other $P(k)$-fragment P' we have $g_S(P') \geq \frac{3}{2}$ as before. Hence $g_S(Q) \geq 2 - \varepsilon + (r+1) \cdot \frac{3}{2}$ and $g_S(G) \geq 0$.

Case 2. There is no full H-fragment in Q, but there is a fragment K, lying in some $P(k)$-fragment P, whose interior contains a whole component of $G - S$. As in Proposition 2, at least two other fragments K', K'' are attached to K at different vertices of S. Now $\{K, K', K''\} \neq \{H_{-1}, H_0, H_1\}$ since $V(K \cap K' \cap K'') = \emptyset$ and $V(H_{-1} \cap H_0 \cap H_1) = \{x\}$. Thus, for some $i \in \{-1, 0, 1\}, H_i \not\subseteq K \cup K' \cup K''$. Hence

$$g_S(P) \geq g_S(K) + g_S(K') + g_S(K'') + g_S(H_i) \geq \left(\frac{1}{2} + \varepsilon\right) + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\,.$$

For any other $P(k)$-fragment P' we have $g_S(P') \geq \frac{3}{2}$ as before. Hence $g_S(Q) \geq (2 + \varepsilon) + (r+1) \cdot \frac{3}{2}$ and $g_S(G) \geq 2\varepsilon > 0$.

Case 3. Every component of $G - S$ disjoint from R intersects more than one K-fragment. Since $x \in S$, such a component J lies in a single $P(k)$-fragment P. As in Proposition 2, J intersects at least three K-fragments in P.

For each of the fragments H_i, $i \in \{-1, 0, 1\}$, we have $g_S(H_i) \geq 1 + \frac{1}{3}\varepsilon$ if J intersects H_i and $g_S(H_i) \geq \frac{1}{2}$ otherwise, by Table 1, rows 4, 6 and 7. For each other K-fragment that J intersects, $g_S(K) \geq \frac{1}{2} + \frac{1}{3}\varepsilon$, by Table 1, rows 4 and 5. Hence, whichever fragments J intersects, we have $g_S(P) \geq 3\left(1+\frac{1}{3}\varepsilon\right)$. As before $g_S(P') \geq \frac{3}{2}$ for any other $P(k)$-fragment P' so $g_S(Q) \geq (3+\varepsilon)+(r+1)\cdot\frac{3}{2}$ and

$$g_S(G) \geq -\frac{3}{2}(r+1) - 2 + \varepsilon + (3+\varepsilon) + \frac{3}{2}(r+1) = 1 + 2\varepsilon > 0\,.$$

□

Propositions 2, 3 and 4 together imply part (2) of the theorem.

Remark. An open question is whether there exists a non-hamiltonian maximal planar graph with toughness exactly $\frac{3}{2}$.

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