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DIGRAPHS WITH HIGHEST HAMILTONICITY EXPONENT

Antoni Marczyk — Günter Schaar

ABSTRACT. For a strongly connected digraph D on $n \geq 2$ vertices, the hamiltonicity exponent $e_H(D)$ — i.e., the least integer $k \geq 1$ such that D^k is hamiltonian — is at most $\lceil \frac{n}{2} \rceil$. In this paper the digraphs D with $e_H(D) = \lceil \frac{n}{2} \rceil$ are characterized in the open case that n is even.

1. Introduction

All digraphs we deal with are supposed to be finite and simple. For a digraph $D = (V, A)$ with vertex set $V(D) = V$ and arc set $A(D) = A \subset V \times V$, the *distance* from a vertex $x \in V$ to a vertex $y \in V$, denoted by $d_D(x, y)$, is the length of a shortest (directed) path from x to y in D if there is any, and ∞ otherwise. For $k = 1, 2, \dots$, the *k-th power* of D, denoted by D^k, is defined to be the digraph with $V(D^k) := V(D)$ and $A(D^k) := \{(x, y) \in V \times V : 1 \leq d_D(x, y) \leq k\}$. Clearly, $D^1 = D$. If D is a strongly connected digraph on $n \geq 2$ vertices then there exists a $k \leq n-1$ such that D^k has a hamiltonian (directed) cycle, and

$$e_H(D) := \min\{k \geq 1 : D^k \text{ is hamiltonian}\}$$

is called the *hamiltonicity exponent* of D (cf. [2]). Corollary 3 in [3] implies the following proposition.

Proposition 1. *For a strongly connected digraph D on $n \geq 2$ vertices, $e_H(D) \leq \lceil \frac{n}{2} \rceil$. (Here, for a real number a, $\lceil a \rceil$ means the least integer $\geq a$.)*

Let $\mathcal{F}(n)$, $n \geq 2$, denote the class of all strongly connected digraphs D on n vertices with the property $e_H(D) = \lceil \frac{n}{2} \rceil$. The question for determining $\mathcal{F}(n)$ goes back to Wojda (cf. [1]), and in [1] the first of the authors has characterized this class in the case that n is odd. In this paper we will describe $\mathcal{F}(n)$ for even n.

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We use standard terminology and standard notation. Especially, $\alpha(D)$ denotes the maximum number of independent vertices in a digraph D. Furthermore, for a given digraph D_0 and a given subset $M \subseteq \{(x,y) \in V(D_0) \times V(D_0)\colon x \neq y \wedge (x,y) \notin A(D_0)\}$ let $\langle D_0, M\rangle$ denote the class of all digraphs D with the property that D is isomorphic to a digraph $D_0 + T$ (i.e., $V(D_0+T) := V(D_0)$, $A(D_0+T) := A(D_0) \cup T$) for some $T \subseteq M$. The digraph D with $V(D) := V(D_0)$ and $A(D) := \{(x,y)\colon (y,x) \in A(D_0)\}$ is denoted by $\overleftarrow{D}_0$; analogously, $\overleftarrow{M} := \{(x,y)\colon (y,x) \in M\}$.

2. Preliminaries and result

Let $r, p \geq 1$ be given integers and $x_1, \ldots, x_r, y_1, \ldots, y_p, x_0, y_0$ the vertices of a simple complete digraph K on $r+p+2$ vertices. Now we will form special subdigraphs D_0 of K, special subsets $M \subseteq A(K) - A(D_0)$, and the relevant classes $\langle D_0, M\rangle$.

(1) Let $D_{r,p}$ denote the digraph with $V(D_{r,p}) := \{x_1, \ldots, x_r, y_1, \ldots, y_p\}$, $A(D_{r,p}) := \{(x_i, x_{i+1})\colon 1 \leq i \leq r-1\} \cup \{(y_i, x_1), (x_r, y_i)\colon 1 \leq i \leq p\}$, and let $M_r := \{(x_i, x_j)\colon 1 \leq j < i \leq r\}$. Then define

$$\mathcal{F}(r,p) := \langle D_{r,p}, M_r\rangle .$$

(2) Let $D'_{r,p}$ denote the digraph with $V(D'_{r,p}) := V(D_{r,p}) \cup \{x_0, y_0\}$, $A(D'_{r,p}) := A(D_{r,p}) \cup \{(y_0, x_1), (x_r, y_0), (x_0, y_0), (y_0, x_0)\}$, and let $M'_r := M_r \cup \{(x_0, x_1), (x_r, x_0)\}$. Then form

$$\mathcal{F}'(r,p) := \langle D'_{r,p}, M'_r\rangle .$$

(3) Let $D''_{r,p}$ denote the digraph with $V(D''_{r,p}) := V(D'_{r,p})$, $A(D''_{r,p}) := A(D_{r,p}) \cup \{(y_0, x_1), (x_r, x_0), (x_0, y_0), (y_0, x_0)\}$. Then define

$$\mathcal{F}''(r,p) := \langle D''_{r,p}, M_r\rangle .$$

(4) For any t with $2 \leq t \leq r$ let $D_{r,p,t}$ denote the digraph with $V(D_{r,p,t}) := V(D_{r,p}) \cup \{y_0\}$, $A(D_{r,p,t}) := A(D_{r,p}) \cup \{(y_0, x_t), (x_r, y_0)\}$, and let $M_{r,t} := M_r \cup \{(y_0, x_j)\colon t < j \leq r\}$. Then we define

$$\mathcal{F}(r,p,t) := \langle D_{r,p,t}, M_{r,t}\rangle$$

and

$$\mathcal{F}^*(r,p,r+1-t) := \langle \overleftarrow{D}_{r,p,t}, \overleftarrow{M}_{r,t}\rangle ;$$

furthermore, let

$$\mathcal{F}_0(r,p) := \bigcup \left\{\mathcal{F}(r,p,t) \colon 2 \le t \le r\right\},$$

and

$$\mathcal{F}_0^*(r,p) := \bigcup \left\{\mathcal{F}^*(r,p,t') \colon 1 \le t' \le r-1\right\}.$$

(5) For any s,t with $1 \le s,t \le p$ and $s+t \ge p$ let $D_{r,p,s,t}$ denote the digraph with $V(D_{r,p,s,t}) := V(D_{r,p}) \cup \{x_0\}$, $A(D_{r,p,s,t}) := \{(x_i,x_{i+1}) \colon 0 \le i \le r\} \cup \{(x_r,y_j) \colon 1 \le j \le p\} \cup \{(y_i,x_0) \colon 1 \le i \le s\} \cup \{(y_i,x_1) \colon p-t+1 \le i \le p\}$, and let $N_r := \{(x_i,x_j) \colon 0 \le j < i \le r\}$. Then let

$$\mathcal{F}_1(r,p,s,t) := \langle D_{r,p,s,t}, N_r \rangle$$

and

$$\mathcal{F}_1^*(r,p,s,t) := \left\langle \overleftarrow{D}_{r,p,s,t}, \overleftarrow{N}_r \right\rangle.$$

Now we form

$$\mathcal{F}_1(r,p) := \bigcup \left\{\mathcal{F}_1(r,p,s,t) \colon 1 \le s,t \le p \wedge s+t \ge p\right\},$$

$$\mathcal{F}_1^*(r,p) := \bigcup \left\{\mathcal{F}_1^*(r,p,s,t) \colon 1 \le s,t \le p \wedge s+t \ge p\right\}.$$

(6) Let D_4 denote the digraph with $V(D_4) := \{x_0,x_1,y_0,y_1\}$, $A(D_4) := \{(x_0,x_1),(x_0,y_0),(y_0,x_0),(y_1,x_0),(y_1,x_1),(x_1,y_0),(x_1,y_1)\}$; then

$$\mathcal{F}'''(r,p) := \left\langle D_4, \{(x_1,x_0)\} \right\rangle$$

if $r=p=1$, and $\mathcal{F}'''(r,p) := \emptyset$ otherwise.

In [1] it was proved

Theorem 1. *Let $n \ge 3$ be an odd integer and $m := \lceil \frac{n}{2} \rceil - 1$. Then $\mathcal{F}(n) = \mathcal{F}(m,m+1)$. (I.e., if D is a strongly connected digraph of odd order $n = 2m+1$, $m \ge 1$, then D^m is hamiltonian iff $D \notin \mathcal{F}(m,m+1)$.)*

This result was obtained as a corollary of

Theorem 2. (cf. [1]) *Let $n \ge 3$ be an integer and $m := \lceil \frac{n}{2} \rceil - 1$. If D is a strongly connected digraph on n vertices such that $\alpha(D^m) \le m$, then D^m is hamiltonian.*

Now we can give the following characterization of the class $\mathcal{F}(n)$ in case that n is even.

Theorem 3. *Let $n \ge 4$ be an even integer and $m := \lceil \frac{n}{2} \rceil - 1$. Then*

$$\begin{aligned}\mathcal{F}(n) = \bigcup \{&\mathcal{F}(m,m+2),\ \mathcal{F}(m+1,m+1),\ \mathcal{F}'(m,m),\ \mathcal{F}''(m,m),\\ &\mathcal{F}'''(m,m),\ \mathcal{F}_0(m,m+1),\ \mathcal{F}_0^*(m,m+1),\ \mathcal{F}_1(m,m+1),\\ &\mathcal{F}_1^*(m,m+1)\}.\end{aligned} \tag{$*$}$$

The case $n=2$ is trivial.

3. Proof of Theorem 3

For each class which is a member of the union on the right hand side of formula $(*)$, it can be easily checked that every digraph D belonging to this class is strongly connected of order $n = 2m+2$, but its mth power D^m cannot contain any hamiltonian cycle; therefore, $e_H(D) = m+1 = \lceil \frac{n}{2} \rceil$ and $D \in \mathcal{F}(n)$.

To complete the proof of Theorem 3 assume that there exists a digraph $D = (V, A) \in \mathcal{F}(n)$ with some even $n \geq 4$ such that D does not belong to the union on the right hand side of $(*)$. Then $e_H(D) = \lceil \frac{n}{2} \rceil$, $n = 2m+2$, $m \geq 1$, and D^m is not hamiltonian. Setting $\alpha := \alpha(D^m)$ we get $\alpha \geq m+1$ because of Theorem 2. In D^m we can choose a (maximal) independent set $Y \subseteq V$ with $|Y| = \alpha$, say $Y = \{y_1, \ldots, y_\alpha\}$. For any two distinct $y, y' \in Y$, their distance $d_D(y, y')$ in D is at least $m+1 \geq 2$; we may suppose that the numbering of the y_i's is chosen such that $d_D(y_1, y_\alpha)$ is minimum among these distances. Then there exists a shortest (directed) path $(y_1, x_1, \ldots, x_r, y_\alpha)$ in D from y_1 to y_α of length $r+1 := d_D(y_1, y_\alpha) \geq m+1$, and it follows that $x_i \notin Y$ for $i = 1, \ldots, r$, and that D does not contain any arc of the set E_0 formed by the following pairs:

(y_i, x_j) for $i = 1, \ldots, \alpha-1$ and $j = 2, \ldots, r$;
(x_i, y_j) for $i = 1, \ldots, r-1$ and $j = 2, \ldots, \alpha$;
(x_i, x_j) for $j \geq i+2$, $i, j = 1, \ldots, r$; (y_i, y_j) for $i \neq j$, $i, j = 1, \ldots, \alpha$.

Denote by s the number of vertices of D which are different from $y_1, \ldots, y_\alpha$, $x_1, \ldots, x_r$. Then we get $\alpha + r + s = n = 2m+2$, $s \geq 0$, thus $2m+2 \geq (m+1) + m + s$ and therefore, $0 \leq s \leq 1$. Since the assumption $\alpha \geq m+3$ would imply the contradiction $\alpha + r + s \geq m + 3 + m + 0 > 2m+2$, we have $\alpha \leq m+2$ and therefore, either $\alpha = m+2$ or $\alpha = m+1$.

Case A: $s = 0$. Then $V = \{x_1, \ldots, x_r, y_1, \ldots, y_\alpha\}$ and $\alpha + r = 2m+2$, and this implies that either

(a) $\alpha = m+2$, $r = m$ or

(b) $\alpha = m+1$, $r = m+1$.

Since D is strongly connected and $A \cap E_0 = \emptyset$, it follows that (y_i, x_1) for $i = 1, \ldots, \alpha-1$, (y_α, x_{j_0}) for some $j_0 \in \{1, \ldots, r\}$, (x_r, y_j) for $j = 2, \ldots, \alpha$, and (x_{i_0}, y_1) for some $i_0 \in \{1, \ldots, r\}$ are arcs in D.

If $\alpha \geq 3$, the assumption $j_0 \geq 2$ results in the contradiction $d_D(y_\alpha, y_2) \leq 1 + d_D(x_{j_0}, x_r) + 1 = r - j_0 + 2 \leq r = d_D(y_1, y_\alpha) - 1$, and analogously the assumption $i_0 \leq r-1$ yields the contradiction $d_D(y_2, y_1) \leq 1 + d_D(x_1, x_{i_0}) + 1 = i_0 + 1 \leq r = d_D(y_1, y_\alpha) - 1$. Hence it follows that $j_0 = 1$, $i_0 = r$, and that $(y_\alpha, x_1), (x_r, y_1) \in A$ but (y_α, x_j) for $j = 2, \ldots, r$ and (x_i, y_1) for $i = 1, \ldots, r-1$ are not arcs in D. Now it is obvious that $D \in \mathcal{F}(r, \alpha)$, i.e., $D \in \mathcal{F}(m, m+2)$

in subcase (a) or $D \in \mathcal{F}(m+1, m+1)$ in subcase (b), which is a contradiction to our assumption on D.

If $\alpha \leq 2$, then because of $m \geq 1$, subcase (a) is not possible; thus $\alpha = r = m+1$ and $m = 1$. If we assume that $j_0 = i_0$, then $d_D(y_2, y_1) = d_D(y_\alpha, y_1) \leq d_D(y_\alpha, x_{j_o}) + d_D(x_{i_0}, y_1) \leq 2 = r = d_D(y_1, y_\alpha) - 1$, a contradiction to the minimum property of $d_D(y_1, y_\alpha)$. Thus we obtain that only either $j_0 = 1$, $i_0 = 2$ or $j_0 = 2$, $i_0 = 1$ can occur, i.e., in the first case $(y_2, x_1), (x_2, y_1) \in A$ and $(y_2, x_2), (x_1, y_1) \notin A$, in the second case $(y_2, x_2), (x_1, y_1) \in A$ and $(y_2, x_1), (x_2, y_1) \notin A$. The first case implies $D \in \mathcal{F}(2,2) = \mathcal{F}(m+1, m+1)$, the second case $(x_2, x_1) \in A$ (because D is strongly connected) and therefore, $D \in \mathcal{F}'(1,1) = \mathcal{F}'(m,m)$. In both cases we have a contradiction.

Case B: $s = 1$. Then $V = \{x_1, \dots, x_r, y_1, \dots, y_\alpha, z\}$ and $\alpha + r = 2m+1$. Since $r \geq m$, $\alpha \geq m+1$ it follows $r = m$, $\alpha = m+1$.

(**a**) Let $(y_{m+1}, x_p) \in A$ for some $p \in \{2, \dots, m\}$. It follows $m \geq 2$, and because $Y = \{y_1, \dots, y_{m+1}\}$ is independent in D^m, we get $(x_m, y_j) \notin A$ for $j = 1, \dots, m$. This and $A \cap E_0 = \emptyset$ imply $(x_i, y_j) \notin A$ for $i = 1, \dots, m$ and $j = 2, \dots, m$.

(**a1**) Let $(x_q, y_1) \in A$ for some $q \in \{1, \dots, m\}$. If $q \geq p$ then $d_D(y_{m+1}, y_1) \leq 1 + d_D(x_p, x_q) + 1 = 2 + (q-p) \leq q \leq m$, a contradiction since Y is independent in D^m. Thus $q < p$, and (because the assumption that $(y_i, x_1) \in A$ for some $i \in \{2, \dots, m+1\}$ would imply the contradiction $d_D(y_i, y_1) \leq 1 + d_D(x_1, x_q) + 1 = 2 + (q-1) = q+1 \leq p \leq m$) we obtain $(y_i, x_1) \notin A$ for $i = 2, \dots, m+1$. Together with $A \cap E_0 = \emptyset$ it follows $(y_i, x_j) \notin A$ for $i = 2, \dots, m$ and $j = 1, \dots, m$. So we have shown that $(v, y_k), (y_k, v) \notin A$ for any $v \in V - \{y_k, z\}$ and $k = 2, \dots, m$. Because D is strongly connected, we obtain $(z, y_k), (y_k, z) \in A$ for $k = 2, \dots, m$, and, moreover, there must exist two arcs $(u, z), (z, u') \in A$ with some $u, u' \in \{x_1, \dots, x_m, y_1, y_{m+1}\}$. If $m \geq 3$ we get $d_D(y_2, y_3) \leq d_D(y_2, z) + d_D(z, y_3) = 1 + 1 = 2 < m$, a contradiction. Thus $m = 2$. The assumptions $(z, y_3) \in A$, $(z, y_1) \in A$, $(y_3, z) \in A$, $(y_1, z) \in A$ would imply, respectively, $d_D(y_2, y_3) \leq d_D(y_2, z) + d_D(z, y_3) \leq 2 = m$, $d_D(y_2, y_1) \leq d_D(y_2, z) + d_D(z, y_1) \leq 2 = m$, $d_D(y_3, y_2) \leq d_D(y_3, z) + d_D(z, y_2) \leq 2 = m$, $d_D(y_1, y_2) \leq d_D(y_1, z) + d_D(z, y_2) \leq 2 = m$, which is a contradiction in every case. Hence it follows that $u, u' \in \{x_1, x_2\}$. Because of $1 \leq q < p \leq m = 2$ we get $q = 1$, $p = 2$ and therefore, $(x_1, y_1) \in A$ and $(y_3, x_2) \in A$. Furthermore, we have $(x_2, y_2), (y_2, x_1) \notin A$, $(x_2, y_1) = (x_m, y_1) \notin A$, $(z, y_1) \notin A, (y_3, x_1) = (y_{m+1}, x_1) \notin A$, $(y_3, z) \notin A$, and $Y = \{y_1, y_2, y_3\}$ is independent in D^m. Since D is strongly connected, there must exist an arc $e \in A$ from $\{x_2, y_3\}$ to $\{x_1, y_1, y_2, z\}$ and also an arc $e' \in A$ from $\{y_2, y_3, x_2, z\}$ to $\{x_1, y_1\}$. If $(x_2, x_1) \notin A$, only $e = (x_2, z)$ and

$e' = (z, x_1)$ is possible; in this case, however, $D^2 = D^m$ has the hamiltonian cycle $(z, y_1, x_1, y_3, x_2, y_2, z)$ which is a contradiction. Therefore, $(x_2, x_1) \in A$. If $u \neq u'$, say $u = x_2$, $u' = x_1$, we get the hamiltonian cycle of D^2 as before (analogously for $u = x_1$, $u' = x_2$), and if $u = u'$, say $u = u' = x_2$, we get the hamiltonian cycle $(z, y_3, x_1, y_1, x_2, y_2, z)$ in D^2 (analogously for $u = u' = x_1$), which is always a contradiction. Thus subcase (**a1**) cannot occur, and we have

(**a2**) $(x_i, y_1) \notin A$ for $i = 1, \dots, m$. Then it follows $(x_i, y_j) \notin A$ for $i, j = 1, \dots, m$, and because Y is independent in D^m and D is strongly connected, $(z, y_j) \in A$ for $j = 1, \dots, m$. Assume $(y_k, z) \in A$ for some $k \in \{1, \dots, m+1\}$. Since $m \geq 2$, there is a $j \neq k$, $j \in \{1, \dots, m\}$, and we get $d_D(y_k, y_j) \leq d_D(y_k, z) + d_D(z, y_j) = 2 \leq m$, a contradiction. Therefore, $(y_i, z) \notin A$ for $i = 1, \dots, m+1$. Since $(y_i, x_j) \notin A$ for $i = 1, \dots, m$ and $j = 2, \dots, m$ (these pairs belong to E_0) it follows $(y_i, x_1) \in A$ for $i = 1, \dots, m$, and further there exists an arc $(x_q, z) \in A$ for some $q \in \{1, \dots, m\}$. If $q \leq m-2$ we would get $d_D(y_1, y_m) \leq d_D(y_1, x_1) + d_D(x_1, x_q) + d_D(x_q, z) + d_D(z, y_m) = 1 + (q-1) + 1 + 1 = q + 2 \leq m$, a contradiction; thus $q \in \{m-1, m\}$ and therefore, $(x_i, z) \notin A$ for $i = 1, \dots, m-2$. For $q = m-1$ and $p \leq q$ we would obtain $d_D(y_{m+1}, y_1) \leq d_D(y_{m+1}, x_p) + d_D(x_p, x_q) + d_D(x_q, z) + d_D(z, y_1) = 1 + (q-p) + 1 + 1 = 3 + q - p \leq 3 + (m-1) - 2 = m$, and for $q = m$ and $p \geq 3$ analogously $d_D(y_{m+1}, y_1) \leq 3 + q - p \leq 3 + m - 3 = m$, which is a contradiction in either case. Thus in case that $(x_{m-1}, z) \in A$ we have $p > m-1$, i.e., $(y_{m+1}, x_j) \notin A$ for $j = 2, \dots, m-1$ and $(y_{m+1}, x_m) \in A$, and in case that $(x_m, z) \in A$ it follows $p = 2$, i.e., $(y_{m+1}, x_2) \in A$. In the second case (considering $(y_i, x_1), (z, y_i) \in A$ for $i = 1, \dots, m$, $(x_m, y_{m+1}) \in A$, and $m \geq 2$), $(y_{m+1}, z, y_m, x_m, y_{m-1}, x_{m-1}, \dots, y_2, x_2, y_1, x_1, y_{m+1})$ would be a hamiltonian cycle in D^m, a contradiction. Hence it follows that $(x_m, z) \in A$ is not possible and only the first case $(x_{m-1}, z) \in A$ can occur. Now we have $(y_{m+1}, x_m) \in A$, $(y_{m+1}, z), (x_m, z), (y_{m+1}, x_i), (y_{m+1}, y_j), (x_m, y_j) \notin A$ for $i = 2, \dots, m-1$ and $j = 1, \dots, m$; since the strongly connected digraph D has an arc from $\{y_{m+1}, x_m\}$ to $\{x_1, \dots, x_{m-1}, y_1, \dots, y_m, z\}$ we get $(y_{m+1}, x_1) \in A$ or $(x_m, x_k) \in A$ for some $k \in \{1, \dots, m-1\}$. If $(y_{m+1}, x_1) \in A$ then $(y_{m+1}, z, y_1, x_1, y_2, x_2, \dots, y_{m-1}, x_{m-1}, y_m, x_m, y_{m+1})$ would be a hamiltonian cycle in D^m; this contradiction implies that $(x_m, x_k) \in A$ for some $k \in \{1, \dots, m-1\}$. If $k \geq 3$ we get $d_D(y_{m+1}, y_1) \leq d_D(y_{m+1}, x_m) + d_D(x_m, x_k) + d_D(x_k, x_{m-1}) + d_D(x_{m-1}, z) + d_D(z, y_1) = 1 + 1 + (m-1-k) + 1 + 1 = m - k + 3 \leq m$, a contradiction since Y is independent in D^m. Therefore, $(x_m, x_1) \in A$ or $(x_m, x_2) \in A$. In the first case $(y_{m+1}, x_1, y_1, z, y_2, x_2, \dots, y_{m-1}, x_{m-1}, y_m, x_m, y_{m+1})$, and in the second case — which implies $m \geq 3$ — $(y_{m+1}, z, y_1, x_1,$

$y_2, x_2, \ldots, y_{m-1}, x_{m-1}, y_m, x_m, y_{m+1})$ would be a hamiltonian cycle in D^m, a contradiction. So we have proved that subcase (**a**) is not possible, and we can suppose that $(y_{m+1}, x_j) \notin A$ for $j = 2, \ldots, m$.

(**b**) Let $(x_p, y_1) \in A$ for some $p \in \{1, \ldots, m-1\}$. By analogous arguments as at the beginning of subcase (**a**) we get $m \geq 2$, $(y_i, x_1) \notin A$ for $j = 2, \ldots, m+1$, and $(y_i, x_j) \notin A$ for $i = 2, \ldots, m$ and $j = 1, \ldots, m$. Furthermore, we have $(y_{m+1}, x_j) \notin A$ for $j = 2, \ldots, m$, and because the assumption $(y_{m+1}, x_1) \in A$ yields the contradiction $d_D(y_{m+1}, y_1) \leq d_D(y_{m+1}, x_1) + d_D(x_1, x_p) + d_D(x_p, y_1) = 1 + (p-1) + 1 = p + 1 \leq m$ it follows $(y_{m+1}, x_j) \notin A$ for $j = 1, \ldots, m$. Now, the situation is completely analogous to that in subcase (**a2**); this can be easily seen by considering $\overleftarrow{D}$ (i.e., by replacing the direction of each arc in D by its opposite direction, which is automatically transmitted to the powers of D and implies $\overleftarrow{D} \in \mathcal{F}(n)$) and by renumbering the vertices $x_1, \ldots, x_m, y_1, \ldots, y_{m+1}$ according to $x'_i := x_{m-i+1}$, $i = 1 \ldots, m$, and $y'_j := y_{m-j+2}$, $j = 1, \ldots, m+1$. In the same way as in subcase (**a2**) we get a contradiction; this proves that subcase (**b**) is not possible. Hence it follows that $(x_i, y_1) \notin A$ for $i = 1, \ldots, m-1$.

(**c**) In **Case B** until now we have proved that D cannot contain any arc of the set E_1 consisting of the pairs belonging to E_0, and the pairs (y_{m+1}, x_j), $j = 2, \ldots, m$, and (x_i, y_1), $i = 1, \ldots, m-1$; i.e., E_1 consists of the following pairs: (y_i, x_j) for $i = 1, \ldots, m+1$ and $j = 2, \ldots, m$; (x_i, y_j) for $i = 1, \ldots, m-1$ and $j = 1, \ldots, m+1$; (x_i, x_j) for $j \geq i+2$, $i, j = 1, \ldots, m$; (y_i, y_j) for $i \neq j$, $i, j = 1, \ldots, m+1$. Furthermore, $V = \{x_1, \ldots, x_m, y_1, \ldots, y_{m+1}, z\}$, and D contains the path $(y_1, x_1, x_2, \ldots, x_m, y_{m+1})$.

(**c1**) Let $(y_i, z), (z, y_j) \in A$ for some $i, j \in \{1, \ldots, m+1\}$.

(**c.1.1**) Suppose that $(y_i, z), (z, y_j) \in A$ for some $i, j \in \{1, \ldots, m+1\}$ with $i \neq j$. If $m \geq 2$ then we get $d_D(y_i, y_j) \leq d_D(y_i, z) + d_D(z, y_j) = 2 \leq m$ which is a contradiction since Y is independent in D^m. This implies $m = 1$. If $i = 2$, $j = 1$ we get the hamiltonian cycle (y_2, z, y_1, x_1, y_2) in $D = D^m$, a contradiction. Therefore, $i = 1$, $j = 2$, i.e., $(y_1, z), (z, y_2) \in A$ but at least one of (y_2, z) or (z, y_1) is not in A. Further (y_1, x_1, y_2) is a path in D. Since D is strongly connected, there exist arcs in D from y_2 to $\{x_1, y_1, z\}$ and from $\{x_1, y_2, z\}$ to y_1 and therefore, from y_2 to $\{x_1, z\}$ and from $\{x_1, z\}$ to y_1. If $(y_2, x_1) \in A$ then the assumption $(x_1, y_1) \in A$ would yield the hamiltonian cycle (y_2, x_1, y_1, z, y_2) in D, which is a contradiction, and if $(y_2, z) \in A$ then $(z, y_1) \notin A$. Hence it follows that only the following cases can occur:

(**c1.1.1**) $(y_2, x_1) \in A$, $(z, y_1) \in A$, $(x_1, y_1) \notin A$, (**c1.1.2**) $(y_2, z) \in A$, $(x_1, y_1) \in A$, $(z, y_1) \notin A$, $(y_2, x_1) \notin A$.

In the first case we get $(y_2, z) \notin A$, and since there is an arc from $\{x_1, y_2\}$ to $\{y_1, z\}$ in D, it is necessarily $(x_1, z) \in A$. Thus the arc set of D is either $A = \{(y_1, x_1), (y_1, z), (y_2, x_1), (x_1, y_2), (x_1, z), (z, y_1), (z, y_2)\}$ or $A = \{(y_1, x_1), (y_1, z), (y_2, x_1), (x_1, y_2), (x_1, z), (z, y_1), (z, y_2), (z, x_1)\}$.

In the second case there is an arc from $\{y_2, z\}$ to $\{x_1, y_1\}$ in D and thus necessarily $(z, x_1) \in A$. Now we get two possibilities for the arc set A of D, and commuting the denotations of the vertices z and x_1 we get the same possible digraphs as in the first case. Therefore, in case (**c1.1**), D necessarily belongs to the class $\mathcal{F}'''(1,1) = \mathcal{F}'''(m,m)$, which contradicts to the assumption on D.

(**c1.2**) It is not possible that $(y_i, z), (z, y_j) \in A$ for some $i \neq j$, $i, j \in \{1, \dots, m+1\}$. It follows that there is exactly one $k \in \{1, \dots, m+1\}$ such that $(y_k, z), (z, y_k) \in A$ and $(y_i, z), (z, y_i) \notin A$ for every $i \neq k$, $i \in \{1, \dots, m+1\}$. Furthermore, we have $(y_i, x_1), (x_m, y_i) \in A$ for every $i \neq k$, $i = 1, \dots, m+1$.

(**c1.2.1**) Let $(y_k, x_1) \notin A$. Then $k \geq 2$, and since D is strongly connected, there exists an $l \in \{1, \dots, m\}$ with $(z, x_l) \in A$. If $l \geq 3$ then $d_D(y_k, y_1) \leq d_D(y_k, z) + d_D(z, x_l) + d_D(x_l, x_m) + d_D(x_m, y_1) = 1 + 1 + (m-l) + 1 = m - l + 3 \leq m$, a contradiction. Therefore, $l \in \{1, 2\}$ and $(z, x_j) \notin A$ for $j = 3, \dots, m$.

Let $l = 2$. Then $m \geq 2$. If $(x_m, y_k) \in A$ then $(y_k, z, y_1, x_1, \dots, y_{k-1}, x_{k-1}, y_{k+1}, x_k, \dots, y_{m+1}, x_m, y_k)$ is a hamiltonian cycle in D^m, a contradiction. Thus $(x_m, y_k) \notin A$. Then there exists a $t \in \{1, \dots, m\}$ with $(x_t, z) \in A$. If $t \leq m-2$ then $d_D(y_1, y_k) \leq d_D(y_1, x_t) + d_D(x_t, z) + d_D(z, y_k) = t + 1 + 1 \leq m$, a contradiction. Thus $t \in \{m-1, m\}$. If $t = m-1$ then $(y_1, z, y_k, x_m, y_{k+1}, x_{m-1}, y_{k+2}, x_{m-2}, \dots, y_{m+1}, x_{k-1}, y_{k-1}, x_{k-2}, y_{k-2}, x_{k-3}, \dots, y_2, x_1, y_1)$ is a hamiltonian cycle in D^m, a contradiction. Thus $t = m$; but then $(y_k, z, y_1, x_1, y_2, x_2, \dots, y_{k-1}, x_{k-1}, y_{m+1}, x_k, y_m, x_{k+1}, \dots, y_{k+1}, x_m, y_k)$ is a hamiltonian cycle in D^m, which is a contradiction. Hence $l = 2$ is impossible, and we have $l = 1$ and therefore, $(z, x_j) \notin A$ for $j = 2, \dots, m$.

(**c1.2.1.1**) Let $(x_p, z) \in A$ for some $p \in \{1, \dots, m\}$. If $p \leq m-1$ then $m \geq 2$ and $(y_1, z, y_k, x_1, y_2, x_2, \dots, y_{k-1}, x_{k-1}, y_{k+1}, x_k, \dots, y_{m+1}, x_m, y_1)$ is a hamiltonian cycle in D^m, a contradiction. Thus $p = m$, i.e., $(x_m, z) \in A$, $(x_i, z) \notin A$ for $i = 1, \dots, m-1$. Hence it follows that $D \in \mathcal{F}'(m,m)$, which is a contradiction.

(**c1.2.1.2.**) Now we can suppose $(x_i, z) \notin A$ for $i = 1, \dots, m$. Because there exists an arc from $\{x_1, \dots, x_m, y_1, \dots, y_{k-1}, y_{k+1}, \dots, y_{m+1}\}$ to $\{y_k, z\}$, it follows $(x_m, y_k) \in A$. Therefore, we have $D \in \mathcal{F}''(m,m)$, a contradiction.

(**c.1.2.2**) Now we have to suppose $(y_k, x_1) \in A$. There exists an arc from $\{y_1, \dots, y_{k-1}, y_{k+1}, \dots, y_{m+1}, x_1, \dots, x_m\}$ to (y_k, z); this implies $(x_m, y_k) \in A$ or $(x_l, z) \in A$ for some $l \in \{1, \dots, m\}$.

(**c1.2.2.1**) Let $(x_m, y_k) \in A$. Then we renumber $Y = \{y_1, \dots, y_{m+1}\}$ by $y'_1, \dots, y'_{m+1}$ such that $y'_{m+1} = y_k$.

(**c1.2.2.1.1**). If $(z, x_q) \in A$ for some $q \in \{1, \dots, m\}$, then for $q \geq 2$, $(y'_{m+1}, z, y'_1, x_1, \dots, y'_m, x_m, y'_{m+1})$ would be a hamiltonian cycle in D^m, a contradiction. Therefore, $q = 1$ and $(z, x_j) \notin A$ for $j = 2, \dots, m$. If $(x_l, z) \in A$ for some $l \in \{1, \dots, m\}$, then for $l \leq m-1$, $(y'_1, z, y'_{m+1}, x_m, y'_m, x_{m-1}, \dots, y'_2, x_1, y'_1)$ would be a hamiltonian cycle in D^m, a contradiction. Thus $l = m$ and $(x_i, z) \notin A$ for $i = 1, \dots, m-1$. Therefore, we have $(x_i, z) \notin A$ for $i = 1, \dots, m-1$, and either $(x_m, z) \notin A$ or $(x_m, z) \in A$. In both cases it follows $D \in \mathcal{F}'(m, m)$, a contradiction.

(**c.1.2.2.1.2**) Now we have $(z, x_j) \notin A$ for $j = 1, \dots, m$. If $(x_l, z) \in A$ for some $l \in \{1, \dots, m\}$, then for $l \leq m-1$, again $(y'_1, z, y'_{m+1}, x_m, y'_m, x_{m-1}, \dots, y'_2, x_1, y'_1)$ would be a hamiltonian cycle in D^m, a contradiction. Thus $l = m$ and $(x_i, z) \notin A$ for $i = 1, \dots, m-1$. Therefore, we have $(x_i, z) \notin A$ for $i = 1, \dots, m-1$, and either $(x_m, z) \notin A$ or $(x_m, z) \in A$. In both cases it follows $D \in \mathcal{F}'(m, m)$, a contradiction.

(**c1.2.2.2**) Now let $(x_m, y_k) \notin A$; then $k \leq m$ and $(x_l, z) \in A$ for some $l \in \{1, \dots, m\}$. If $l \leq m-1$ then $(y_{m+1}, z, y_k, x_1, y_1.x_2, \dots, y_{k-1}, x_k, y_{k+1}, x_{k+1}, \dots, y_m, x_m, y_{m+1})$ would be a hamiltonian cycle in D^m, a contradiction. Thus $l = m$ and $(x_i, z) \notin A$ for $i = 1, \dots, m-1$. If $(z, x_p) \in A$ for some $p \in \{1, \dots, m\}$ then if $p \geq 2$ it follows $m \geq 2$ and $(y_{m+1}, x_m, y_k, z, y_1, x_1, y_2, x_2, \dots, y_{k-1}, x_{k-1}, y_{k+1}, x_k, y_{k+2}, x_{k+1}, \dots, y_m, x_{m-1}, y_{m+1})$ would be a hamiltonian cycle in D^m, a contradiction; therefore, $p = 1$ and $(z, x_j) \notin A$ for $j = 2, \dots, m$. Hence it follows that $(z, x_j) \notin A$ for $j = 2, \dots, m$, and either $(z, x_1) \notin A$ or $(z, x_1) \in A$. In the first case necessarily $D \in \mathcal{F}''(m, m)$, in the second case $D \in \mathcal{F}'(m, m)$, a contradiction in both cases.

(**c.2**) Now we have to suppose that there do not exist any $i, j \in \{1, \dots, m+1\}$ such that both $(y_i, z), (z, y_j) \in A$.

(**c2.1**) Let $(y_k, z) \in A$ for some $k \in \{1, \dots, m+1\}$. Then $(z, y_j) \notin A$ for $j = 1, \dots, m+1$. Thus $(z, x_l) \in A$ for some $l \in \{1, \dots, m\}$ and furthermore, for every $j \in \{1, \dots, m+1\}$, there is an arc from $\{x_1, \dots, x_m, y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_{m+1}, z\}$ to y_j in D, i.e., $(x_m, y_j) \in A$ for $j = 1, \dots, m+1$. If $l \geq 3$ then for any $p \neq k$, $p \in \{1, \dots, m+1\}$, $d_D(y_k, y_p) \leq d_D(y_k, z) + d_D(z, x_l) + d_D(x_l, x_m) +$

$d_D(x_m, y_p) = 1 + 1 + (m - l) + 1 = m - l + 3 \le m$, a contradiction. Therefore, $l \in \{1, 2\}$.

With $Q := \{y_i \colon i \in \{1, \dots, m+1\} \wedge (y_i, z) \in A\}$ we have $y_k \in Q$ and $(y, z) \notin A$ for every $y \in Y - Q$; but for each $i = 1, \dots, m+1$ there is an arc from y_i to $V - \{y_i\}$ in D, this implies $(y_i, x_1) \in A$ for each $y_i \notin Q$, $i = 1, \dots, m+1$. Now we can renumber the vertices of Y in the following way: Let $Q = \{y'_1, \dots, y'_q\}$, and let $\overline{Q} = \{y'_{m+2-t}, \dots, y'_{m+1}\}$ consist of all vertices $y_j \in Y$ with $(y_j, x_1) \in A$. Clearly, $Q \ne \emptyset$, $y_1 \in \overline{Q}$, $Y - Q \subseteq \overline{Q}$, $Q \cup \overline{Q} = Y$, therefore, $1 \le q \le m+1$, $1 \le t \le m+1$, $q + t \ge m+1$. If we denote $x_0 := z$, it is proved:

$A \cap E_1 = \emptyset$; $V = Y \cup \{x_0, x_1, \dots, x_m\}$; $(x_m, y'_j) \in A$ for $j = 1, \dots, m+1$; $(x_0, x_l) \notin A$ for $l = 3, \dots, m$; $(y'_i, x_0) \in A$ for $i = 1, \dots, q$; $(y'_i, x_0) \notin A$ for $i = q+1, \dots, m+1$; $(y'_i, x_1) \notin A$ for $i = 1, \dots, m+1-t$; $(y'_i, x_1) \in A$ for $i = m+2-t, \dots, m+1$; D contains the path $(x_1, \dots, x_m)$ and at least one of the arcs (x_0, x_1) or (x_0, x_2). If $(x_0, x_2) \in A$ then $(y'_1, x_0, y'_2, x_2, y'_3, x_3, \dots, y'_m, x_m, y'_{m+1}, x_1, y'_1)$ would be a hamiltonian cycle in D^m, a contradiction. Thus $(x_0, x_2) \notin A$ and therefore, $(x_0, x_1) \in A$.

Hence is follows that for some $q, t \in \{1, \dots, m+1\}$ with $q + t \ge m+1$, $D \in \mathcal{F}_1(m, m+1, q, t)$ which is a contradiction.

(c2.2) Let $(z, y_k) \in A$ for some $k \in \{1, \dots, m+1\}$. Then $(y_i, z) \notin A$ for $i = 1, \dots, m+1$. Thus $(x_l, z) \in A$ for some $l \in \{1, \dots, m\}$ and furthermore, for every $i \in \{1, \dots, m+1\}$ there is an arc from y_i to $\{x_1, \dots, x_m, y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_{m+1}, z\}$, i.e., $(y_i, x_1) \in A$ for $i = 1 \dots, m+1$. If $l \le m-2$ then for any $p \ne k$, $p \in \{1, \dots, m+1\}$, $d_D(y_p, y_k) \le d_D(y_p, x_1) + d_D(x_1, x_l) + d_D(x_l, z) + d_D(z, y_k) = 1 + (l-1) + 1 + 1 = l + 2 \le m$, a contradiction. Therefore, $l \in \{m-1, m\}$, and $(x_i, z) \notin A$ for $i = 1, \dots, m-2$.

With $Q := \{y_j \colon j \in \{1, \dots, m+1\} \wedge (z, y_j) \in A\}$ we have $y_k \in Q$ and $(z, y) \notin A$ for every $y \in Y - Q$; but for each $j = 1, \dots, m+1$, there is an arc from $V - \{y_j\}$ to y_j in D which implies $(x_m, y_j) \in A$ for each $y_j \notin Q$, $j = 1, \dots, m+1$. Now we can renumber the vertices of Y in the following way: Let $Q = \{y'_1, \dots, y'_q\}$, and let $\overline{Q} = \{y'_{m+2-t}, \dots, y'_{m+1}\}$ consist of all vertices $y_i \in Y$ with $(x_m, y_i) \in A$. Obviously, $Q \ne \emptyset$, $y_{m+1} \in \overline{Q}$, $Y - Q \subseteq \overline{Q}$, $Q \cup \overline{Q} = Y$ and therefore, $1 \le q \le m+1$, $1 \le t \le m+1$, $q + t \ge m+1$. If we denote $x_0 := z$, it is proved: $A \cap E_1 = \emptyset$, $V = Y \cup \{x_1, \dots, x_m, x_0\}$; $(y'_i, x_1) \in A$ for $i = 1, \dots, m+1$; $(x_l, x_0) \notin A$ for $l = 1, \dots, m-2$; $(x_0, y'_j) \in A$ for $j = 1, \dots, q$; $(x_0, y'_j) \notin A$ for $j = q+1, \dots, m+1$; $(x_m, y'_j) \in A$ for $j = m+2-t, \dots, m+1$; $(x_m, y'_j) \notin A$ for $j = 1, \dots, m+1-t$; D contains the path $(x_1, \dots, x_m)$ and at least one of the arcs (x_m, x_0) or (x_{m-1}, x_0). If $(x_{m-1}, x_0) \in$

A then $(x_0, y_1', x_1, y_2', x_2, \dots, y_m', x_m, y_{m+1}', x_0)$ would be a hamiltonian cycle in D^m, a contradiction. Thus $(x_{m-1}, x_0) \notin A$ and therefore, $(x_m, x_0) \in A$. Hence it follows that for some $q, t \in \{1, \dots, m+1\}$ with $q + t \geq m+1$, $D \in \mathcal{F}_1^*(m, m+1, q, t)$, a contradiction.

(**c2.3**) Now we have to suppose that $(y_i, z), (z, y_i) \notin A$ for every $i = 1, \dots$ $\dots, m+1$. Then obviously, $(y_i, x_1) \in A$ for $i = 1, \dots, m+1$, $(x_m, y_j) \in A$ for $j = 1, \dots, m+1$, and there exist some $k, l \in \{1, \dots, m\}$ such that $(x_k, z), (z, x_l) \in A$. If $k < l$ then $k \leq m-1$, $l \geq 2$, $l - k \leq m-1$ and $(y_1, z, y_2, x_1, y_3, x_2, \dots, y_{m+1}, x_m, y_1)$ would be a hamiltonian cycle in D^m, a contradiction. Therefore, $k \geq l$.

If $2 \leq l \leq k \leq m-1$ then we get the same contradiction. Hence it follows:

If $(x_k, z) \in A$ for some $k \in \{1, \dots, m-1\}$, then $(z, x_j) \notin A$ for $j = 2, \dots, m$, and $(z, x_1) \in A$;

if $(z, x_l) \in A$ for some $l \in \{2, \dots, m\}$, then $(x_i, z) \notin A$ for $i = 1, \dots, m-1$, and $(x_m, z) \in A$;

if $(x_i, z) \notin A$ for $i = 1, \dots, m-1$ and $(z, x_j) \notin A$ for $j = 2, \dots, m$, then $(x_m, z) \in A$ and $(z, x_1) \in A$.

(**c2.3.1**) Let $(x_k, z) \in A$ for some $k \in \{1, \dots, m-1\}$. Put $t := \max \{i \in \{1, \dots, m-1\} : (x_i, z) \in A\}$. Then $(z, x_j) \notin A$, $j = 2, \dots, m$, $(z, x_1) \in A$, $(x_i, z) \notin A$ for $i = t+1, \dots, m$. Obviously, $D \in \mathcal{F}^*(m, m+1, t)$ and $t \leq m-1$, which is a contradiction.

(**c2.3.2**) Let $(z, x_l) \in A$ for some $l \in \{2, \dots, m\}$. Denote $t := \min\{j \in \{2, \dots, m\} : (z, x_j) \in A\}$. Then $(x_i, z) \notin A$ for $i = 1, \dots, m-1$, $(x_m, z) \in A$, $(z, x_j) \notin A$ for $j = 1, \dots, t-1$. Thus we get $D \in \mathcal{F}(m, m+1, t)$ and $t \geq 2$ which is a contradiction.

(**c2.3.3**) Let $(x_i, z) \notin A$ for $i = 1, \dots, m-1$ and $(z, x_j) \notin A$ for $j = 2, \dots, m$. Then $(x_m, z), (z, x_1) \in A$, and $d_D(y_i, z) = m+1$, $d_D(z, y_i) = m+1$ for $i = 1, \dots, m+1$. Therefore, $Y \cup \{z\}$ would be an independent vertex subset in D^m, which is a contradiction, since $\alpha(D^m) = \alpha = m+1 = |Y|$.

Thus our assumption on D has resulted in a contradiction in all cases, which completes the proof of Theorem 3.

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Antoni Marczyk
University of Mining and Metallurgy
Faculty of Mathematics
Al. Mickiewicza 30
PL–30-059 Kraków
POLAND

E-mail: marczyk@uci.agh.edu.pl

Günter Schaar
Freiberg University of Mining and Technology
Institute of Theoretical Mathematics
D–09596 Freiberg
GERMANY

E-mail: schaar@mathe.tu-freiberg.de