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A CHARACTERIZATION OF THE GEOMETRIC DISTRIBUTION

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ABSTRACT. Let X and Y have discrete distributions on the nonnegative integers with the property that

$$\Pr[Y = y] = \frac{1}{\nu^*} \sum_{x \geq y+1} \Pr[X = x]\,,$$

where $0 < \nu^* < \infty$ is the mean value of X. Then X and Y are identically distributed if and only if X has geometric distribution. This is a (new) characterization of the geometric distribution.

Introduction

Let X be a discrete random variable (r.v.) (called *basic* r.v.) defined on the nonnegative integers with probability mass function (pmf)

$$\{P_0^*, P_1^*, \dots\} \tag{1}$$

and probability generating function (pgf)

$$G^*(t) = \sum_{x \geqq 0} P_x^* t^x \tag{2}$$

and finite mean $\nu^* > 0$. From this basic r.v. we derive a new r.v. The distribution of this r.v. is characterized by probabilities proportional to the complement of the basic cumulative distribution function. The pmf of this partial-sums distribution is

$$P_x = \frac{1}{\nu^*} \sum_{j \geqq x+1} P_j^*\,, \qquad x = 0, 1, 2, \dots \tag{3}$$

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(see, e.g., [1], p.448) and its pgf is

$$G(t) = \frac{1 - G^*(t)}{(1-t)\nu^*} \tag{4}$$

(which can be easy verified).

Some results

Theorem 1. *The only basic r.v. identical with its partial-sums distribution obtained by the summation process (3) has geometric distribution.*

Proof. With respect to the previous notation let $G(t) = G^*(t)$, i.e., (according to (4))

$$G(t)(1-t) = \frac{1}{\nu} - \frac{G(t)}{\nu}, \tag{5}$$

where ν is the mean value of the required basic (and also partial-sums) r.v. Differentiating (5) we obtain

$$G'(t)(1-t) - G(t) = -\frac{1}{\nu}G'(t),$$

or, equivalently,

$$G'(t)\left(\frac{1+\nu}{\nu} - t\right) = G(t). \tag{6}$$

Substituting

$$G(t) = \sum_{i \geqq 0} P_i t^i$$

and

$$G'(t) = \sum_{i \geqq 1} i P_i t^{i-1}$$

into (6) we have

$$\begin{aligned} P_1\frac{1+\nu}{\nu} + 2P_2\frac{1+\nu}{\nu}t + 3P_3\frac{1+\nu}{\nu}t^2 \cdots - P_1 t - 2P_2 t^2 - 3P_3 t^3 \dots \\ = P_0 + P_1 t + P_2 t^2 + \dots, \end{aligned}$$

which gives the system of equations

$$\begin{aligned} P_0 &= P_1\frac{1+\nu}{\nu}, \\ P_1 &= P_2\frac{1+\nu}{\nu}, \\ P_2 &= P_3\frac{1+\nu}{\nu}, \\ &\vdots \end{aligned}$$

which, in turn defines the unique pmf

$$P_x = \frac{1}{1+\nu}\Big(\frac{\nu}{1+\nu}\Big)^x, \qquad x = 0, 1, \dots,$$

of the geometric distribution with mean value ν. The appropriate pgf is

$$G(t) = \frac{1}{1+\nu}\Big(1 - \frac{\nu t}{1+\nu}\Big)^{-1}. \tag{7}$$

Now let X have geometric distribution with mean value ν^*. Its pgf is

$$G^*(t) = \frac{1}{1+\nu^*}\Big(1 - \frac{\nu^* t}{1+\nu^*}\Big)^{-1}. \tag{8}$$

Substituting (8) into (4) we easy obtain $G(t) = G^*(t)$. □

Theorem 2. *Let X be the basic r.v. with pmf (1), pgf (2) and finite mean ν^*. Suppose then Y has partial-sums distribution with pmf (3), $k \in \mathbb{N} = \{1, 2, \dots\}$, $0 < \varrho < 1$. X has "modified" geometric (ϱ) distribution with pmf*

$$\begin{aligned} &P_0^*\,, \\ &P_x^* = (1 - P_0^*)(1-\varrho)\varrho^{x-k}\,, \qquad x = k, k+1, \dots\,, \end{aligned} \tag{9}$$

$0 \leqq P_0^* < 1$, *if and only if Y has pmf*

$$P_x = \begin{cases} \frac{1-\varrho}{k(1-\varrho)+\varrho}\,, & x = 0, 1, \dots, k-1\,, \\ \frac{(1-\varrho)\varrho^{x-k+1}}{k(1-\varrho)+\varrho}\,, & x = k, k+1, \dots\,. \end{cases} \tag{10}$$

Proof. *Necessity*: Since (9) holds,

$$\nu^* = \mathcal{E}(X) = \sum_{j=k}^{\infty}(1-P_0^*)j(1-\varrho)\varrho^{j-k} = \frac{k(1-\varrho)+\varrho}{1-\varrho}(1-P_0^*)\,,$$

and so the pmf of Y is (by (3))

$$P_0 = \frac{1-\varrho}{k(1-\varrho)+\varrho} \sum_{j=k}^{\infty} (1-\varrho)\varrho^{j-k} = \frac{1-\varrho}{k(1-\varrho)+\varrho}\,,$$

$$\vdots$$

$$P_{k-1} = \frac{1-\varrho}{k(1-\varrho)+\varrho} \sum_{j=k}^{\infty} (1-\varrho)\varrho^{j-k} = \frac{1-\varrho}{k(1-\varrho)+\varrho}\,,$$

$$P_k = \frac{1-\varrho}{k(1-\varrho)+\varrho} \sum_{j=k+1}^{\infty} (1-\varrho)\varrho^{j-k} = \frac{(1-\varrho)\varrho}{k(1-\varrho)+\varrho}\,,$$

$$P_{k+1} = \frac{(1-\varrho)\varrho^2}{k(1-\varrho)+\varrho}\,,$$

$$P_{k+2} = \frac{(1-\varrho)\varrho^3}{k(1-\varrho)+\varrho}\,,$$

$$\vdots$$

The necessity is proved.

Sufficiency: Let the pmf of Y be (10), $P_x = \frac{1}{\nu^*} \sum\limits_{j \geqq x+1} P_j^*$ and let ν^* be the mean value of X. It is obvious that

$$P_i - P_{i-1} = 0\,, \qquad i = 1, 2, \dots, k-1$$

(if $k \geqq 2$), i.e.,

$$P_1^* = P_2^* = \dots = P_{k-1}^* = 0\,. \tag{11}$$

Further from (3) and (10)

$$\begin{aligned} P_{k+j}^* = \nu^*(P_{k+j-1} - P_{k+j}) &= \nu^* \Big[\frac{(1-\varrho)\varrho^j}{k(1-\varrho)+\varrho} - \frac{(1-\varrho)\varrho^{j+1}}{k(1-\varrho)+\varrho}\Big] \\ &= \nu^* \frac{(1-\varrho)^2\varrho^j}{k(1-\varrho)+\varrho}\,, \qquad j = 0, 1, \dots\,. \end{aligned} \tag{12}$$

As $\sum\limits_{i=0}^{\infty} P_i^* = 1$, we have (using (11) and (12))

$$1 = \sum_{i=0}^{\infty} P_i^* = P_0^* + \sum_{i=0}^{\infty} \nu^* \frac{(1-\varrho)^2\varrho^i}{k(1-\varrho)+\varrho} = P_0^* + \nu^* \frac{(1-\varrho)}{k(1-\varrho)+\varrho}\,,$$

i.e.,

$$\nu^* \frac{(1-\varrho)}{k(1-\varrho)+\varrho} = 1 - P_0^*\,. \tag{13}$$

Substituting (13) into (12) and taking into account (11) we easily obtain (9). □

Corollary 1. *According to Theorem 1 we can formulate a characterization of the geometric distribution given in the Abstract.*

REFERENCES

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