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# EFFICIENCY OF CROP ROTATION DESIGNS WITH UNEQUAL NUMBERS OF REPLICATIONS

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ABSTRACT. A mild modification of the notion of overall efficiency factor and its application to the crop rotation designs are considered in the paper. The practical considerations are restricted to the supplemented designs comprising four-field cropping systems.

## 1. Introduction

We are interested in comparing the efficiency of different crop rotation experiments. Let us have a closer look into the nature of crop rotation experiments. *Crop rotation* or *cropping system* is the fixed sequence of species of plants which appear on the same plot in the course of many years. We start an experiment with all plants included into the sequence. Two examples of four-field cropping systems are given below.

| year | plot no. | | | |
|---|---|---|---|---|
| no. | 1 | 2 | 3 | 4 |
| 1 | **wheat** | beans | **wheat** | **wheat** |
| 2 | **wheat** | **wheat** | beans | **wheat** |
| 3 | **wheat** | **wheat** | **wheat** | beans |
| 4 | beans | **wheat** | **wheat** | **wheat** |

| year | plot no. | | | |
|---|---|---|---|---|
| no. | 1 | 2 | 3 | 4 |
| 1 | **wheat** | rape | beans | **wheat** |
| 2 | **wheat** | **wheat** | rape | beans |
| 3 | beans | **wheat** | **wheat** | rape |
| 4 | rape | beans | **wheat** | **wheat** |

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We assume that different sequences of plants create various levels of soil fertility in the plots. Such additional effects of soil fertility accumulated during the full rotation (four years) we treat as the *objects*. From the agricultural point of view, the usefulness of rotation should be estimated on the basis of the yields of all species. But such the yields are incomparable because of the character of the plant products (for example a field of corn and seeds of rape). Sometimes the same species appears in the rotation more than once, therefore we can treat such a plant as the *test crop* and we only take the yields of this species for the statistical analysis. In the crop rotations considered, wheat is treated as the test crop. The incidence matrix for the first crop rotation has the form

$$\mathbf{N} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} .$$

The rows are related to the plots, the columns correspond to the years, 1 indicates that the wheat was on the plot, 0 refers to the other species. Crop rotation designs can be formulated as cyclic designs, i.e., their incidence matrices are circulant. The initial row provides a concise representation of the cropping system. For example, the second design is represented by $[1,1,0,0]$. In this paper we are interested in crop rotation experiments described by the following initial rows: $\mathbf{r}_1 = [1,1,1,0]$, $\mathbf{r}_2 = [1,1,0,0]$, $\mathbf{r}_3 = [1,0,1,0]$. Besides, we consider supplemented designs, i.e., designs with incidence matrices

$$\mathbf{N}_1' = \left(\mathbf{N}_{\mathbf{r}_1}' \vdots \mathbf{N}_{\mathbf{r}_2}'\right), \quad \mathbf{N}_2' = \left(\mathbf{N}_{\mathbf{r}_1}' \vdots \mathbf{N}_{\mathbf{r}_3}'\right).$$

For the sake of convenience, we assign the symbol $\Delta_{\mathbf{r}}$ to the design described by the incidence matrix $\mathbf{N}_{\mathbf{r}}$ with the initial row $\mathbf{r}$. Consequently, for supplemented designs we employ the symbols $\Delta^*_{\mathbf{r}_2}$ and $\Delta^*_{\mathbf{r}_3}$. Moreover, we will use the following notation. The number of objects is $v$ (equal to the number of plots in rotation), $k$ stands for the number of plots in one year with the test crop, $b$ denotes the number of years. The number of 1's in each row is treated as the number of replications $r_i$. It should be pointed out that cyclic designs $\Delta_{\mathbf{r}_i}$ are equireplicated designs, but $\Delta^*_{\mathbf{r}_j}$ are not.

## 2. Linear models and the estimation of parameters

The crop rotation experiment can be described in accordance with the modified model for the two-factor split-plot design (see, e.g., Przybysz (1982)). The complete description for any observation is as follows:

$$y_{ijk} = \mu + \rho_i + \alpha_j + e_{ij} + \beta_k + \varphi_{jk} + \varepsilon_{ijk} ,$$

where $\mu$ is a general level effect, $\rho_i, \alpha_j, \beta_k$ denote the effects of $i$th replication ($i = 1, \ldots, r$), $j$th plot ($j = 1, \ldots, v$) and the $k$th year ($k = 1, \ldots, b$). Besides, $\varphi_{jk}$ stands for interaction effect of respective factors. Here $e_{ij}$ and $\varepsilon_{ijk}$ are random errors of experimental units and random technical errors, respectively.

The model of the experiment can be written in matrix notation as

$$\mathbf{y} = \mathbf{1}\mu + \mathbf{R}'\boldsymbol{\rho} + \boldsymbol{\Delta}'\boldsymbol{\alpha} + \mathbf{e} + \mathbf{D}'\boldsymbol{\beta} + \mathbf{G}'\boldsymbol{\varphi} + \boldsymbol{\varepsilon}\,.$$

For the application, our original model *comprising only one replication* can be stated as that for the incomplete block design

$$\widetilde{\mathbf{y}} = \mathbf{1}\mu + \widetilde{\boldsymbol{\Delta}}'\boldsymbol{\alpha} + \widetilde{\mathbf{D}}'\boldsymbol{\beta} + \tilde{\mathbf{e}}\,,$$

where $\widetilde{\mathbf{y}}$ is $n \times 1$ vector of observations, $n = bk$, $\widetilde{\boldsymbol{\Delta}}'$, $\widetilde{\mathbf{D}}'$ are $n \times v$, $n \times b$ design matrices for treatments and years, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are $v \times 1$, $b \times 1$ vectors of treatment parameters and year effects, $\tilde{\mathbf{e}}$ is $n \times 1$ random vector of errors. We adopt this form solely for the purpose of evaluating efficiency factors. The approach undertaken is both in accord with the proper model and sufficient for our purpose.

We are interested in drawing inferences about the treatment effects $\alpha_j$ only. After elimination of the parameter $\boldsymbol{\beta}$ we get the so-called reduced normal equations $\mathbf{C}\widehat{\boldsymbol{\alpha}} = \mathbf{q}$, where $\mathbf{C} = \mathbf{r}^{\delta} - \mathbf{N}\mathbf{k}^{-\delta}\mathbf{N}'$ is the *information matrix* for the treatment effects, $\widehat{\boldsymbol{\alpha}}$ is the estimator of the vector of object effects, $\mathbf{q}$ denotes the vector of adjusted treatment totals and $\mathbf{r}^{\delta} = \operatorname{diag}(r_1, r_2, \ldots, r_v)$. The solutions of the reduced normal equations can be written by means of the Moore-Penrose $g$-inverse matrix $\mathbf{C}^{-}$ whose elements will be denoted by $c^{ij}$.

The unbiased estimator of the estimable function $\mathbf{c}'\boldsymbol{\alpha}$ is $\mathbf{c}'\widehat{\boldsymbol{\alpha}}$ and the variance of this estimator is given by

$$\operatorname{var}(\mathbf{c}'\widehat{\boldsymbol{\alpha}}) = \mathbf{c}'\mathbf{C}^{-}\mathbf{c}\sigma^2\,. \tag{1}$$

In view of the above, the pairwise comparisons are estimable with estimators $\mathbf{c}'_{ij}\widehat{\boldsymbol{\alpha}}$ and from (1)

$$\operatorname{var}\ \mathbf{c}'_{ij}\widehat{\boldsymbol{\alpha}} = \mathbf{c}'_{ij}\,\mathbf{C}^{-}\,\mathbf{c}_{ij}\,\sigma^2 = \left(c^{ii} + c^{jj} - 2c^{ij}\right)\sigma^2 = v_{ij}\sigma^2\,. \tag{2}$$

The average variance of all pairwise comparisons is

$$\overline{\operatorname{var}} = \frac{2\sigma^2}{v(v-1)} \sum_{i>j}\sum v_{ij}\,.$$

For the orthogonal block design, the variance (2) is of the form

$$\operatorname{var}_o\ \mathbf{c}'_{ij}\widehat{\boldsymbol{\alpha}} = \mathbf{c}'_{ij}\mathbf{C}^{-}\,\mathbf{c}_{ij}\,\sigma^2 = \mathbf{c}'_{ij}\mathbf{r}^{-\delta}\mathbf{c}_{ij}\,\sigma^2 = \sigma^2\left(\tfrac{1}{r_i} + \tfrac{1}{r_j}\right).$$

## 3. Efficiency factors

The initial considerations of this section follow J o h n (1987). They are concerned with the equireplicated case and are rather familiar.

The *coefficient of efficiency* is defined as a ratio of variances calculated for the same contrast in the complete and investigated design, respectively, under the assumption that the error variance is the same in both designs. So, in the case of the pairwise comparison $\boldsymbol{c}_{ij}'\widehat{\boldsymbol{\alpha}} = \widehat{\alpha}_i - \widehat{\alpha}_j$, we obtain the following formula

$$E_{ij} = \frac{2\sigma^2/r}{\sigma^2 v_{ij}} = \frac{2}{r\, v_{ij}}\,. \tag{3}$$

Incomplete block designs are judged mainly by the average variance of estimated pairwise comparisons. This suggests defining the average efficiency factor using average variances in the complete and investigated design, respectively

$$E = \frac{2\sigma^2/r}{\overline{\mathrm{var}}} = \frac{v(v-1)\sigma^2}{r\sum\limits_{i>j}\sum \mathrm{var}\ \boldsymbol{c}_{ij}'\widehat{\boldsymbol{\alpha}}}\,.$$

To incorporate the notion of efficiency for cropping system designs, the concept of the average efficiency factor should be generalized. Extending formula (3) to the case of unequal replications, the average coefficient $E$ can be obtained as the ratio of average variances in the orthogonal and supplemented designs, respectively, as

$$E = \frac{\overline{\mathrm{var}}_o}{\overline{\mathrm{var}}} = \frac{\sum\limits_{i>j}\sum\left(\frac{1}{r_i}+\frac{1}{r_j}\right)}{\sum\limits_{i>j}\sum v_{ij}}\,.$$

A significant remark should be given here. We measure the efficiency when the replication vector is regarded as fixed. This means that we assume the replication vector to be the same in the orthogonal and supplemented design.

Due to the comparative character of crop rotation experiments, we recognize *inter-contrasts* to be of main interest (cf. P r z y b y s z *et al.* 1998). So, an overall average efficiency factor $E$ does not give an exact information relating to the usefulness of the design. To supply the appropriate criterion for comparing the supplemented designs, we restate the efficiency factor (3) in the following way

$$E^* = \frac{\overline{\mathrm{var}}_o^*}{\overline{\mathrm{var}}^*} = \frac{\sum\limits_{1\le i\le v_A,\ v_A+1\le j\le v}\left(\frac{1}{r_i}+\frac{1}{r_j}\right)\sigma^2}{\sum\limits_{\mathcal{C}_{AB}} \mathrm{var}\ \boldsymbol{c}_{ij}'\widehat{\boldsymbol{\alpha}}}\,,$$

where $v_A, v_B$ denote the number of treatments pertaining to the competitive cropping systems, $v = v_A + v_B$, and the sum in the denominator extends over the set $\mathcal{C}_{AB}$ of all inter-contrasts. To sum up, the efficiency factor $E^*$ should provide a useful summary of the properties of the supplemented designs.

# 4. Applications

## 4.1 The efficiency of cyclic designs.

First we address the problem of estimating the coefficients of efficiency for the design $\Delta_{\mathbf{r}_1}$. In this case we have the circulant information matrix of the form

$$\mathbf{C} = 1/3 \begin{pmatrix} 6 & -2 & -2 & -2 \\ & \dots & \dots & \end{pmatrix}.$$

For this information matrix $\mathbf{C}$, one can obtain the following eigenvalues and corresponding eigenvectors:

$$\mathbf{1} \leftrightarrow \lambda_1 = 0\,, \quad \{[-1,1,0,0],\,[-1,0,1,0],\,[-1,0,0,1]\} \leftrightarrow \lambda_{2,3,4} = 8/3\,.$$

Examination of this result reveals the fact that all pairwise comparisons belong to the space spanned by the foregoing vectors, i.e., the remaining comparisons are obtained as follows: $\boldsymbol{c}_{23} = \boldsymbol{g}_1 - \boldsymbol{g}_2$ and $\boldsymbol{c}_{34} = \boldsymbol{g}_2 - \boldsymbol{g}_3$, where $\boldsymbol{g}_i$ denote eigenvectors in the appropriate sequence. Hence we can "identify" pairwise comparisons $\boldsymbol{c}_{ij}$ with the *basic contrasts* $\boldsymbol{p}_k$ (normalized eigenvectors of the information matrix). Besides, one can readily establish the following formula

$$\operatorname{var}\ \boldsymbol{p}_i'\widehat{\boldsymbol{\alpha}} = \boldsymbol{p}_i'\,\mathbf{C}^-\boldsymbol{p}_i\,\sigma^2 = \sigma^2\,/\,\lambda_i\,.$$

The efficiency coefficient of the *basic contrast* is of the form

$$e_k = \lambda_k\,/r\,.$$

Therefore we can draw the conclusion that every pairwise efficiency factor is equal to the canonical efficiency factor $E_{ij} = e = 0.889$. The average efficiency factor $E$ is the harmonic mean of the canonical efficiency factors, $E = \mathbf{0.889}$.

Let us now look at the block design $\Delta_{\mathbf{r}_2}$. The eigenvalues and corresponding eigenvectors are of the form: $\mathbf{1} \leftrightarrow \lambda_1 = 0$, $\{[0,-1,0,1]\,,[-1,0,1,0]\} \leftrightarrow \lambda_{2,3} = 1$, $\{[-1,1,-1,1]\} \leftrightarrow \lambda_4 = 2$. $\mathbf{C}^-$ and $\mathbf{V}$ are circulant matrices with the initial rows $1/8\,[5,-1,-3,-1]$ and $[0,3/2,2,3/2]$, respectively. The average efficiency is equal to $E = \mathbf{0.6}$.

For design $\Delta_{\mathbf{r}_3}$ we obtain a circulant information matrix $\mathbf{C}$ with initial row $[1,0,-1,0]$. It is readily apparent that $\operatorname{rank}(\mathbf{C}) = 2$, thereby the investigated design is disconnected.

## 4.2 Efficiency of supplemented designs.

We now proceed to estimating efficiency coefficients for supplemented designs. Let us focus on the case $\Delta^*_{\mathbf{r}_2}$. Some details pertaining to matrices $\mathbf{C}$ and $\mathbf{C}^-$

will be omitted. The method described earlier leads us to the following variance coefficients matrix

$$\mathbf{V} = \sigma_e^2 \left( \begin{array}{cccc|cccc} 0 & 0.721 & 0.727 & 0.721 & 0.877 & 0.877 & 0.968 & 0.968 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ - & -- & -- & -- & -- & -- & -- & -- \\ & & & & 0 & 1.136 & 1.273 & 1.136 \\ & & & & \dots & \dots & \dots & \dots \end{array} \right).$$

For the sake of simplicity, we have distinguished three submatrices. Each of them is a circulant matrix (the other rows are obtained from the first one by a cyclical rotation). Thus the upper left and the bottom right submatrices are relevant to the comparisons made among treatments of the same cyclic design (intra-contrasts), i.e., $\Delta_{\mathbf{r}_1}$ and $\Delta_{\mathbf{r}_2}$, respectively. The remaining submatrix refers to variances of inter-contrasts. It is noteworthy that within each of the submatrices mentioned above we obtained two levels of precision. The existence of distinct coefficients is a consequence of the different number of times that respective pairs of treatments were applied to the same block. It is convenient to present the computed results in the following table:

Table 1. Efficiency of comparisons within respective cropping systems

| pairwise comparison | cropping system design | |
|---|---|---|
| | cyclic | supplemented |
| 12 14 23 34 | 0.889 | 0.925 |
| 13 24 | 0.889 | 0.017 |
| 56 58 67 78 | 0.667 | 0.880 |
| 57 68 | 0.500 | 0.786 |

It is noteworthy that in both cases the efficiency coefficients obtained within the supplemented design exceed the coefficients for designs examined separately.

Let us consider the efficiency of design $\Delta^*_{\mathbf{r}_3}$. It is worth to recall that $\Delta_{\mathbf{r}_3}$ fails to be connected, but this fact does not violate the connectedness of the supplemented design. By virtue of (2) the variance coefficient matrix $\mathbf{V}$ is of the form

$$\mathbf{V} = \sigma_e^2 \left( \begin{array}{cccc|cccc} 0 & 0.732 & 0.714 & 0.732 & 0.887 & 1.012 & 0.887 & 1.012 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ - & -- & -- & -- & -- & -- & -- & -- \\ & & & & 0 & 1.375 & 1.000 & 1.375 \\ & & & & \dots & \dots & \dots & \dots \end{array} \right).$$

It is of interest to compare the coefficients $E_{ij}$ related to $\Delta_{\mathbf{r}_1}$ with the coefficients corresponding to the same treatments but obtained within the design $\Delta^*_{\mathbf{r}_2}$. Such a comparison is made in the table below.

TABLE 2. Efficiency of comparisons within cropping system $\Delta_{r_1}$

| intra-contrast | supplemented design | |
|---|---|---|
| | $\Delta^*_{r_2}$ | $\Delta^*_{r_3}$ |
| 12 14 23 34 | 0.925 | 0.911 |
| 13 24 | 0.917 | 0.933 |

The comparison of the average parameters of the supplemented designs is arranged in Table 3.

## 5. Final remarks

Summing up, the supplemented designs are aimed at comparing the cropping systems they comprise, so in these experiments the elementary contrasts are not of equal interest. The appropriate criterion for comparing the supplemented designs should be based on the efficiency factor $E^*$. The overall factor $E^*$ dominates the coefficient $E$ in the case of all supplemented designs examined. For more details and examples, we refer the reader to Przybysz *et al.* (1998).

TABLE 3. Efficiency of supplemented designs

| supplemented design | average variance | | efficiency | |
|---|---|---|---|---|
| | $\overline{\mathrm{var}}$ | $\overline{\mathrm{var}}^*$ | E | $E^*$ |
| $\Delta^*_{r_2}$ | 0.935 | **0.923** | 0.891 | **0.903** |
| $\Delta^*_{r_3}$ | 0.966 | **0.949** | 0.863 | **0.878** |

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