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A GOODNESS-OF-FIT TEST FOR CAUCHY DISTRIBUTION

František Rublík

ABSTRACT. An explicit formula for the quantile test statistic for testing goodness-of-fit for the Cauchy distribution is presented. The proof of convergence in distribution includes also an asymptotic representation of a trigonometric L-estimator of scale, which is shown to hold for the Cauchy distribution. Critical values for the test statistic are given for the range $n = 16 - 55$.

1. Introduction and the main results

Suppose that $x_1, \dots, x_n$ is a random sample and $\{F(\cdot, \theta) \colon \theta \in \Theta\}$ is a family of distribution functions. One of the most popular statistics for testing goodness of fit is the Anderson–Darling test statistic

$$A^2 = n \int\limits_{-\infty}^{+\infty} \frac{\big(F_n(t) - \hat F(t)\big)^2}{\hat F(t)\big(1 - \hat F(t)\big)} d\hat F(t) = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \log\Big(Z^{(i)}\big(1 - Z^{(n+1-i)}\big)\Big) , \tag{1.1}$$

where

$$F_n(t) = \frac{1}{n} \sum_{i=1}^{n} \chi_{(-\infty,t\rangle}(x_i) , \tag{1.2}$$

$$\hat F(t) = F(t, \hat\theta) , \quad Z_i = F(x_i, \hat\theta) , \tag{1.3}$$

with $\hat\theta = \hat\theta_n(x_1, \dots, x_n)$ standing for an estimator of the parameter and $Z^{(i)}$ denoting the ith order statistic computed from $Z_1, \dots, Z_n$. In what follows we deal with the Cauchy distributions. Their parameter set is $\Theta = \{(\mu, \sigma)' \in$

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$\mathbb{R}^2\colon \sigma > 0\bigr\}$ and for $\theta = (\mu, \sigma)' \in \Theta$ the Cauchy distribution $C(\mu, \sigma)$ is defined by means of the density

$$f(x, \theta) = \frac{1}{\sigma} f\Bigl(\frac{x-\mu}{\sigma}\Bigr), \qquad f(x) = \frac{1}{\pi(1+x^2)}. \tag{1.4}$$

The distribution function and the pth quantile of this distribution are given by the formulas

$$F(x, \theta) = F\Bigl(\frac{x-\mu}{\sigma}\Bigr), \qquad F(t) = \frac{1}{2} + \frac{1}{\pi}\arctan(t), \tag{1.5}$$

$$\xi(p, \theta) = F^{-1}(p, \theta) = \sigma\xi_p + \mu, \qquad \xi_p = \tan\bigl((p-0.5)\pi\bigr). \tag{1.6}$$

Since an explicit formula for the maximum likelihood estimator is known only for $n = 3$ and $n = 4$ (cf. [2]) and the use of numerical methods for the computation of the MLE is difficult because of the presence of several local maxima of the likelihood function (cf. p. 311 of [3]), it is convenient to use some another estimator for constructing goodness-of-fit test statistic. The estimators used in constructing A^2 on p. 164 of [1] are

$$\begin{gathered} \hat{\mu}_n = \sum_{i=1}^{n} g_{ni} x_n^{(i)}, \qquad g_{ni} = \frac{1}{n} J\Bigl(\frac{i}{n+1}\Bigr), \\ J(u) = \frac{\sin\bigl(4\pi(u-0.5)\bigr)}{\tan\bigl(\pi(u-0.5)\bigr)} = -4\sin^2(\pi u)\cos(2\pi u), \end{gathered} \tag{1.7}$$

and

$$\hat{\sigma}_n = \sum_{i=1}^{n} c_{ni} x_n^{(i)}, \tag{1.8}$$

where $x_n^{(i)}$ denotes the ith order statistic and

$$c_{ni} = \frac{1}{n} J\Bigl(\frac{i}{n+1}\Bigr), \quad J(u) = \frac{8\tan\bigl(\pi(u-0.5)\bigr)}{\sec^4\bigl(\pi(u-0.5)\bigr)} = -8\cos(\pi u)\sin^3(\pi u). \tag{1.9}$$

However, for the constants determined by (1.7) the equality $\sum_i g_{ni} = 1$ does not hold. Hence $\hat{\mu}_n(ay_1 + b, \dots, ay_n + b) \neq a\hat{\mu}_n(y_1, \dots, y_n) + b$ which causes the distribution of A^2 to depend on the parameter of the sampled Cauchy distribution. Estimates of the probability of the rejection of the null hypothesis that the sample is taken from a Cauchy distribution, based on $N = 5000$ simulations of samples from the Cauchy $C(\mu, 1)$ distribution, for some selected values of μ and n are given in the following tables; the critical constants are taken from the table 4.26 on p. 163 of [1], $c(0.1, n) = 1.40$, 1.94, 1.41, $c(0.05, n) = 1.77$, 3.57, 2.14, $c(0.025, n) = 2.00$, 6.91, 3.37, and all values are related to $n = 5$, $n = 25$ and $n = 50$, respectively.

	$n=5$					$n=25$				
μ	0	5	10	15	20	0	5	10	15	20
$P(A^2 > c(0.1, n))$	0.1	0.35	0.62	0.75	0.82	0.1	0.12	0.23	0.44	0.65
$P(A^2 > c(0.05, n))$	0.05	0.24	0.52	0.68	0.77	0.05	0.05	0.08	0.17	0.34
$P(A^2 > c(0.025, n))$	0.025	0.20	0.46	0.64	0.73	0.025	0.026	0.029	0.04	0.079

	$n=50$				
μ	0	5	10	15	20
$P(A^2 > c(0.1, n))$	0.1	0.12	0.20	0.36	0.57
$P(A^2 > c(0.05, n))$	0.05	0.06	0.09	0.18	0.35
$P(A^2 > c(0.025, n))$	0.025	0.028	0.038	0.066	0.138

These values show that the statistic A^2 based on (1.7) does not yield good results for testing the qoodness-of-fit for Cauchy distributions for moderately large sample sizes $n \leq 50$. In deriving properties of the quantile test statistics we shall use the following properties of the scale estimator from (1.8) and (1.9).

Theorem 1.1.

For the constants determined by (1.9) *the equalities*

$$c_{n,j} = -c_{n,n+1-j}\,, \qquad \sum_{j=1}^{n} c_{nj} = 0 \tag{1.10}$$

hold. Consequently, if $x_1 = \cdots = x_n$ *does not hold and* $n > 1$, *then for the estimator* (1.8)

$$\hat{\sigma}_n > 0\,, \quad \hat{\sigma}_n(ax_1 + b, \ldots, ax_n + b) = |a|\,\hat{\sigma}_n(x_1, \ldots, x_n)\,. \tag{1.11}$$

(II) *Suppose that* $x_1, \ldots, x_n$ *is a random sample from the Cauchy* $C(0,1)$ *distribution and*

$$L_n = \sum_{i=1}^{n} d_{ni} x_n^{(i)}\,, \qquad d_{ni} = \int_{\frac{i-1}{n}}^{\frac{i}{n}} J(u)\,du\,, \tag{1.12}$$

where J *is defined in* (1.9). *Put*

$$\psi(x) = -2\cos\bigl(2\arctan(x)\bigr)\,. \tag{1.13}$$

Then

$$L_n - 1 = \frac{1}{n}\sum_{i=1}^{n} \psi(x_i) + \mathcal{O}_P(n^{-1})\,. \tag{1.14}$$

(III) *If* $x_1, \ldots, x_n$ *is a random sample from the Cauchy* $C(0,1)$ *distribution, then for the estimators described by* (1.8), (1.9) *and* (1.12) *the relation*

$$L_n - \hat{\sigma}_n = \mathcal{O}_P(n^{-1}) \tag{1.15}$$

holds. Hence

$$\hat{\sigma}_n - 1 = \frac{1}{n}\sum_{i=1}^{n} \psi(x_i) + \mathcal{O}_P(n^{-1})\,. \tag{1.16}$$

The sample distribution function is defined in (1.2). Further, let

$$\hat{\xi}_{p,n} = \inf\{t\colon F_n(t) \geq p\}\,, \tag{1.17}$$

$$\hat{\mu}_n = \begin{cases} \frac{x^{(k)}+x^{(k+1)}}{2}\,, & n = 2k\,, \\ x^{(k+1)}\,, & n = 2k+1\,, \end{cases} \tag{1.18}$$

denote the pth sample quantile and the sample median, respectively.

THEOREM 1.2. *Let an integer $k \geq 1$ and real numbers $0 < p_1 < \cdots < p_k < 1$ be given. Suppose that $x_1, \dots, x_n$ is a random sample from the Cauchy $C(\mu,\sigma)$ distribution and put*

$$\mathbf{\Delta}_n = \begin{pmatrix} F(\hat{\xi}_{p_1,n}, \hat{\theta}_n) - p_1 \\ F(\hat{\xi}_{p_2,n}, \hat{\theta}_n) - p_2 \\ \vdots \\ F(\hat{\xi}_{p_k,n}, \hat{\theta}_n) - p_k \end{pmatrix}, \tag{1.19}$$

where $F(\cdot,\theta)$ is the function (1.5) and $\hat{\theta}_n = (\hat{\mu}_n, \hat{\sigma}_n)'$ is composed of the estimators defined by (1.18) and (1.8), (1.9).

(I) *The vector of differences $\mathbf{\Delta}_n$ has an exact null distribution, i.e.,*

$$\mathcal{L}\big(\mathbf{\Delta}_n \,|\, C(\mu,\sigma)\big) = \mathcal{L}\big(\mathbf{\Delta}_n \,|\, C(0,1)\big) \tag{1.20}$$

for all $(\mu,\sigma) \in \Theta$ and all $n > 1$.

(II) *As $n \to \infty$,*

$$\mathcal{L}\big(\sqrt{\mathbf{n}}\mathbf{\Delta}_\mathbf{n} | C(\mu,\sigma)\big) \longrightarrow N(\mathbf{0},\mathbf{\Sigma})\,. \tag{1.21}$$

Here

$$\mathbf{\Sigma} = \mathbf{A} + \mathbf{G}\,, \tag{1.22}$$

$\mathbf{A}$ *is the symmetric $k \times k$ matrix with the elements $a_{ij} = p_i(1-p_j)$ for all $i \leq j$, and*

$$\begin{aligned} G_{i,j} = {} & \frac{\sin^2(p_i\pi)\sin^2(p_j\pi)}{4} - \frac{\sin^2(p_i\pi)}{2}\min\{p_j\,, 1-p_j\} \\ & - \frac{\sin^2(p_j\pi)}{2}\min\{p_i\,, 1-p_i\} - \frac{\sin(2p_i\pi)\sin(2p_j\pi)}{2\pi^2}\,. \end{aligned} \tag{1.23}$$

If $p_i \neq \frac{1}{2}$ for $i = 1, \dots, k$, then the matrix (1.22) is regular and for

$$T_n = n\mathbf{\Delta}_n'\mathbf{\Sigma}^{-1}\mathbf{\Delta}_n \tag{1.24}$$

the weak convergence to the chi-square distribution with k degrees of freedom

$$\mathcal{L}\big(T_n | C(\mu,\sigma)\big) \longrightarrow \chi^2_k \tag{1.25}$$

holds as $n \to \infty$.

In accordance with (1.20) and (1.24), the null hypothesis that the sample comes from a Cauchy distribution is rejected, whenever $T_n > q(\alpha, n, p_1, \dots, p_k)$, where α denotes the level of significance. Some values of $q(\alpha, n) = q(\alpha, n, 0.1, 0.9)$, obtained from $N = 50000$ simulations for each sample size, are given in the following table.

n	16	17	18	19	20	21	22	23
$q(0.1,n)$	9.369	8.475	7.408	6.633	9.839	12.424	11.107	10.237
$q(0.05,n)$	13.270	12.271	10.875	9.842	13.526	16.600	14.803	13.937
n	24	25	26	27	28	29	30	31
$q(0.1,n)$	9.216	8.396	7.558	6.881	6.251	5.731	8.057	9.924
$q(0.05,n)$	12.538	11.682	10.576	9.749	8.980	8.299	11.072	13.149
n	32	33	34	35	36	37	38	39
$q(0.1,n)$	9.178	8.435	7.702	7.190	6.578	6.098	5.607	5.289
$q(0.05,n)$	12.200	11.426	10.544	9.852	9.158	8.555	7.861	7.531
n	40	41	42	43	44	45	46	47
$q(0.1,n)$	6.983	8.590	8.025	7.451	6.982	6.455	6.085	5.661
$q(0.05,n)$	9.504	11.455	10.752	10.055	9.484	8.894	8.296	7.905
n	48	49	50	51	52	53	54	55
$q(0.1,n)$	5.313	5.053	6.441	7.820	7.340	6.902	6.492	6.028
$q(0.05,n)$	7.446	7.024	8.828	10.330	9.772	9.368	8.739	8.218

Values of the probability $P\big(T_n > q(0.05, n)\big)$ of rejection by the statistics (1.24) with $k = 2$, $p_1 = 0.1$, $p_2 = 0.9$, presented in the following table, are based on $N = 2000$ simulations for each alternative and sample size n considered.

n	21	25	30	35	40	45
chi-square $k = 1$	0.52	0.79	0.9	0.97	0.99	0.99
chi-square $k = 2$ *(exponential)*	0.20	0.41	0.54	0.71	0.79	0.89
chi-square $k = 5$	0.06	0.18	0.24	0.39	0.47	0.60
uniform	0.09	0.28	0.41	0.57	0.68	0.78
normal	0.03	0.08	0.11	0.21	0.27	0.36
Poisson $\lambda = 1$	0.25	0.32	0.50	0.53	0.63	0.64
Poisson $\lambda = 5$	0.12	0.16	0.21	0.31	0.36	0.46

It can be seen from the table that this test begins to exert its power for sample sizes of at least 30. The simulations suggest that within alternative classes of probabilities which are not location and scale parameter families, the sensitivity of this test increases with the growth of excess and asymmetry, described by the coefficients

$$b_1 = \frac{E\big((X-\mu)^3\big)}{\sigma^3}\,, \qquad b_2 = \frac{E\big((X-\mu)^4\big)}{\sigma^4}\,,$$

but, in general, the magnitude of these quantities does not imply greater (or smaller) sensitivity, because for the uniform distribution $b_1 = 0$, $b_2 = 1.8$, while for the χ^2_5 distribution $b_1 = 1.26$, $b_2 = 5.4$.

2. Proofs

Proof of Theorem 1.1. (I) The validity of (1.10) is obvious. Hence $\hat\sigma_n = \sum\limits_i c_{ni}(x_n^{(i)} - \hat\mu_n)$, where $\hat\mu_n$ denotes the sample median, and since $\operatorname{sign}\big(c_{ni}(x_n^{(i)} - \hat\mu_n)\big) \ge 0$, the rest of the proof is evident.

(II) Put

$$\Phi(s) = -\int_s^1 J(u)\,\mathrm{d}u\,. \tag{2.1}$$

Then $\Phi'(s) = J(s)$ and (cf. (1.5))

$$\begin{aligned} 1 &= \int_0^1 2\sin^2(2\pi u)\,\mathrm{d}u = \int_0^1 J(u)F^{-1}(u)\,\mathrm{d}u \\ &= \int_{\mathbb{R}} xJ\big(F(x)\big)\,\mathrm{d}F(x) = \int_{\mathbb{R}} x\,\mathrm{d}\Phi\big(F(x)\big)\,. \end{aligned}$$

Thus, making use of the integration by parts (cf. [5], p. 157), and proceeding as in the proof of (4.3.10)–(4.3.13) of [4], one obtains that

$$L_n - 1 - \frac{1}{n}\sum_{i=1}^n \psi(x_i) = \int_0^1 V_n(z)\,\mathrm{d}F^{-1}(z)\,, \tag{2.2}$$

$$\big|V_n(z)\big| = \big|\Phi\big(U_n(z)\big) - \Phi(z) - \Phi'(z)\big(U_n(z) - z\big)\big|\,, \tag{2.3}$$

$$U_n(z) = \frac{1}{n}\sum_{i=1}^n \chi_{\langle 0,z\rangle}(Y_i)\,, \tag{2.4}$$

where $Y_1, \dots, Y_n$ is a random sample from the uniform distribution on the interval $(0,1)$. As observed on p. 144 of [4] (the proof can also be found on p. 43 of [6]), for every fixed $\eta \in (0, 1/2)$

$$\sup\left\{\sqrt{n}\left|U_n(z) - z\right| / \big(z(1-z)\big)^{1/2-\eta} \colon 0 < z < 1\right\} = \mathcal{O}_P(1)\,.$$

Thus, given $\varepsilon > 0$, $\eta \in (0, 1/8)$ there exists a constant M_1 such that for the sets

$$A_n = \left\{\sqrt{n}\left|U_n(z) - z\right| \le M_1\big(z(1-z)\big)^{1/2-\eta} \quad \text{for all} \quad z \in (0,1)\right\} \tag{2.5}$$

the inequality

$$\limsup_n \big(1 - P(A_n)\big) < \varepsilon \tag{2.6}$$

holds. Further,

$$\left|\Phi''(u)\right| = \left|J'(u)\right| \le M_2 \sin^2(\pi u) \tag{2.7}$$

and by using the Taylor formula we obtain that on A_n

$$\begin{aligned} \left|V_n(z)\right| &= \left|\Phi''\big(z + \alpha(U_n(z) - z)\big)\right| \big(U_n(z) - z\big)^2/2 \\ &\le M_2 \sin^2\big(\pi\big(z + \alpha(U_n(z) - z)\big)\big) \big(U_n(z) - z\big)^2 \\ &\le \frac{M_3}{n} \sin^2\big(\pi\big(z + \alpha(U_n(z) - z)\big)\big) z^{1-2\eta}\,. \end{aligned} \tag{2.8}$$

There exists $\delta_1 \in (0,1)$ such that $\left|\sin(\pi x)/x\right| < 7$ for all $x \in (0, \delta_1)$. Put $\delta = \delta_1/2$. Then for all sufficiently large n and all $z \in (0, \delta)$ on A_n

$$\begin{gathered} 0 \le z + \alpha\big(U_n(z) - z\big) \le z + M_1/\sqrt{n} < \delta_1\,, \\ \left|\sin\big(\pi\big(z + \alpha(U_n(z) - z)\big)\big)\right| \le 7(z + z^{1/2-\eta}) \le 14 z^{1/2-\eta} \end{gathered}$$

and from (2.8) we obtain that

$$\left|V_n(z)\right| \le \frac{M_4}{n} z^{2(1-2\eta)}$$

for all $0 < z < \delta$. This, together with (1.6) means that for all sufficiently large n on A_n

$$\int_0^\delta \left|V_n(z)\right| \mathrm{d}F^{-1}(z) \le \frac{M_4}{n} \int_0^\delta z^{2(1-2\eta)} \frac{\pi}{\cos^2\big((z-1/2)\pi\big)} \,\mathrm{d}z = \frac{M_5}{n}\,, \tag{2.9}$$

because for some $\alpha \in (0,1)$

$$\lim_{z \to 0} \frac{z^{2(1-2\eta)}}{\cos^2\big((z-1/2)\pi\big) z^{-\alpha}} = 0\,. \tag{2.10}$$

Similarly on A_n for some $\delta \in (0, 1/2)$

$$\int_{1-\delta}^{1} |V_n(z)| \, \mathrm{d}F^{-1}(z) \le \frac{M_6}{n}, \tag{2.11}$$

and by means of (2.8)

$$\int_{\delta}^{1-\delta} |V_n(z)| \, \mathrm{d}F^{-1}(z) \le \frac{M_7}{n}. \tag{2.12}$$

Obviously, the relations (2.2), (2.6), (2.9), (2.11) and (2.12) imply the validity of (1.14).

(II) Put

$$J_n(u) = J\Big(\frac{i}{n+1}\Big) \qquad \text{if} \quad \frac{i-1}{n} < u \le \frac{i}{n}, \ i = 1, \dots, n,$$

$$\Phi_n(s) = -\int_s^1 J_n(u) \, \mathrm{d}u .$$

Employing integration by parts we obtain that (cf. (1.12), (2.1), (1.2), (2.4))

$$\begin{aligned} |\hat{\sigma}_n - L_n| &= \left| \int x \, \mathrm{d}\Big[\Phi_n\big(F_n(x)\big) - \Phi\big(F_n(x)\big)\Big] \right| \\ &= \left| - \int \Big[\Phi_n\big(F_n(x)\big) - \Phi\big(F_n(x)\big)\Big] \, \mathrm{d}x \right| \\ &= \left| \int_0^1 \Big[\Phi\big(U_n(z)\big) - \Phi_n\big(U_n(z)\big)\Big] \, \mathrm{d}F^{-1}(z) \right| \le I_1 + I_2 , \end{aligned} \tag{2.13}$$

where, because $\Phi_n(0) = \Phi(0) = 0$,

$$\begin{aligned} I_1 &= \left| \int_0^{1/2} \Bigg[\int_0^{U_n(z)} \big(J(t) - J_n(t)\big) \, \mathrm{d}t \Bigg] \, \mathrm{d}F^{-1}(z) \right| , \\ I_2 &= \left| \int_{1/2}^{1} \Bigg[\int_{U_n(z)}^{1} \big(J(t) - J_n(t)\big) \mathrm{d}t \Bigg] \, \mathrm{d}F^{-1}(z) \right| . \end{aligned} \tag{2.14}$$

Fix $\varepsilon > 0$, choose $\eta \in (0, 1/8)$ and put

$$\begin{aligned} &d_n = d_n(\varepsilon) = -\frac{\log(-\varepsilon/2)}{n}, \quad D_n = D_n(\varepsilon) = (1-\varepsilon/2)^{1/n}, \\ &B_n = A_n \cap \big\{ Y_n^{(1)} > d_n , \ Y_n^{(n)} < D_n \big\}, \end{aligned} \tag{2.15}$$

where A_n is the set (2.5) and Y_i are the variables from (2.4). Obviously,

$$\limsup_n \big(1 - P(B_n)\big) < 2\varepsilon \tag{2.16}$$

and on B_n the inequality

$$I_1 \le \int_{d_n}^{1/2} \left[\int_0^{1/n} \big|J(t) - J_n(t)\big| \,\mathrm{d}t + \int_{1/n}^{U_n(z)} \big|J(t) - J_n(t)\big| \,\mathrm{d}t \right] \mathrm{d}F^{-1}(z) \tag{2.17}$$

holds. But (cf. (2.7))

$$\int_{d_n}^{1/2} \left[\int_0^{1/n} \big|J(t) - J_n(t)\big| \,\mathrm{d}t \right] \mathrm{d}F^{-1}(z) \le \frac{M_2}{n^2} \big[F^{-1}(1/2) - F^{-1}(d_n)\big] = \frac{M_2}{n} O(1)\,, \tag{2.18}$$

where $O(1)$ converges to a positive real number as $n \to \infty$. Further, if $\frac{i-1}{n} < t \le \frac{i}{n}$, $2 \le i \le n$, then (cf. (2.7))

$$\big|J(t) - J_n(t)\big| \le \frac{M_2}{n} \sin^2\Big(\pi\big(t + \alpha(\frac{i}{n+1} - t)\big)\Big) \le \frac{M_8}{n} t^2\,,$$

because

$$0 < \frac{t + \alpha\big(\frac{i}{n+1} - t\big)}{t} \le \frac{\frac{i}{n}}{\frac{i-1}{n}} \le 2\,.$$

Thus on B_n

$$\int_{1/n}^{U_n(z)} \big|J(t) - J_n(t)\big| \,\mathrm{d}t \le \frac{M_8}{n} U_n(z)^3 \le \frac{M_9}{n} z^{3(1/2-\eta)}\,,$$

which, together with (1.6), means that on B_n

$$\int_{d_n}^{1/2} \left[\int_{1/n}^{U_n(z)} \big|J(t) - J_n(t)\big| \,\mathrm{d}t \right] \mathrm{d}F^{-1}(z)$$

$$\le \frac{M_9}{n} \int_0^{1/2} z^{3(1/2-\eta)} \frac{\pi}{\cos^2\big((z - 1/2)\pi\big)} \,\mathrm{d}z = \frac{M_{10}}{n}\,, \tag{2.19}$$

because, since $\eta \in (0, 1/8)$, the equality (2.10) holds for some $\alpha \in (0,1)$. The relations (2.16) – (2.19) imply that

$$I_1 = \mathcal{O}_P(n^{-1})\,.$$

The validity of

$$I_2 = \mathcal{O}_P(n^{-1})$$

can be proved similarly, and using (2.13) we obtain that the relation (1.15) is true. Now (1.16) follows from (1.14). □

Proof of Theorem 1.2. Let $x_1, \dots, x_n$ be a random sample from the Cauchy $C(0,1)$ distribution. An application of Theorem 4.2.1 of [4] and of (1.16) yields the relations

$$\hat{\theta}_n - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{n} \sum_{j=1}^{n} \mathbf{l}(x_j) + o_P(n^{-1/2}), \quad \mathbf{l}(x) = \begin{pmatrix} \frac{\pi}{2}\,\mathrm{sign}(x) \\ -2\cos\big(2\arctan(x)\big) \end{pmatrix}.$$

Thus

$$E\big(\mathbf{l}(x)\big) = \mathbf{0}, \qquad \mathbf{L} = \mathrm{Var}\big(\mathbf{l}(x)\big) = \begin{pmatrix} \frac{\pi^2}{4}, & 0 \\ 0, & 2 \end{pmatrix}. \tag{2.20}$$

Since (1.11) holds and $\hat{\mu}_n(az_1 + b, \dots, az_n + b) = a\hat{\mu}_n(z_1, \dots, z_n) + b$ if $a > 0$, it is obvious that the assumptions of Theorem 2.2 of [7] are satisfied. This means that both (1.20) and (1.21) hold,

$$\mathbf{\Sigma} = \mathbf{A} + \mathbf{D}\mathbf{\Psi}\mathbf{C}' + \mathbf{C}\mathbf{\Psi}'\mathbf{D} + \mathbf{D}\mathbf{\Psi}\mathbf{L}\mathbf{\Psi}'\mathbf{D}, \tag{2.21}$$

where (cf. (1.6))

$$\mathbf{D} = \mathrm{diag}\big(f(\xi_1), \dots, f(\xi_k)\big), \quad f(x) = \frac{1}{\pi(1+x^2)}, \quad \xi_i = \xi_{p_i},$$

$$\mathbf{\Psi} = \begin{pmatrix} 1, & \xi_1 \\ \vdots & \\ 1, & \xi_k \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} \mathrm{cov}\big(\chi_{(-\infty,\xi_1)}(x), \mathbf{l}(x,\theta)\big) \\ \vdots \\ \mathrm{cov}\big(\chi_{(-\infty,\xi_k)}(x), \mathbf{l}(x,\theta)\big) \end{pmatrix} \tag{2.22}$$

and the jth row of the matrix $\mathbf{C}$ equals $\Big(-\frac{\pi}{2}\min\{p_j, 1-p_j\}, \ \frac{\sin(2p_j\pi)}{\pi}\Big)$. Combining (2.20) – (2.22) we easily obtain the validity of (1.22) and (1.23). The regularity of the matrix (1.22) follows from Theorem 2.2 and from Theorem 2.1(III) of [7]. The rest of the proof is obvious. □

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Institute of Measurement Science
Slovak Academy of Sciences
Dúbravská cesta 9
SK–84219 Bratislava
SLOVAKIA

E-mail: umerrubl@savba.savba.sk