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EQUIVARIANT ESTIMATORS AND THEIR ASYMPTOTIC REPRESENTATIONS

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ABSTRACT. We apply the results of Jurečková and Milhaud (1999) on the characterization of (generally nonnormal) distributions in invariant models to location and regression equivariant estimators and arrive at a special form of the convolution theorem. In the location model, we derive an interesting characterization of the distribution of a general sample statistic.

1. Introduction

Let $X_1, \dots, X_n$ be *i.i.d.* observations with joint density $f(x-\theta)$, $\theta \in \mathbb{R}^1$. Consider the problem of estimation of θ under the condition that the loss incurred when estimating θ by t is equal to $L(\theta, t) = (t-\theta)^2$. The loss is invariant under the transformations $X_i' = X_i + c$, $\theta' = \theta + c$, $i = 1, \dots, n$, $c \in \mathbb{R}^1$; this invariance leads to the class of *translation-equivariant estimators:* The estimator T_n is called *translation equivariant* if

$$T_n(X_1 + c, \dots, X_n + c) = T_n(X_1, \dots, X_n) + c, \qquad \forall c \in \mathbb{R}^1.$$

Let $\mathbf{Y}$ be a maximal invariant with respect to the group of translations, e.g., $\mathbf{Y} = (X_2 - X_1, \dots, X_n - X_1)$ or $\mathbf{Y} = (X_1 - \bar{X}, \dots, X_n - \bar{X})$. Every equivariant estimator T_n with finite risk satisfies the inequalities

$$\mathbb{E}_\theta\big(T_n - \theta - \mathbb{E}_0(T_n|\mathbf{Y})\big)^2 \le \mathbb{E}_\theta(T_n - \theta)^2$$

and hence *an equivariant estimator T_n has minimal risk if and only if*

$$\mathbb{E}_0(T_n|\mathbf{Y}) = 0. \tag{1.1}$$

The minimum risk equivariant estimator (*MRE, Pitman estimator*) θ_n^* of θ can be expressed as $\theta_n^* = T_n - \mathbb{E}_0(T_n|\mathbf{Y})$ with T_n being any equivariant estimator

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with a finite risk. In the terminology of Kagan, Linnik and Rao (1973) terminology, (1.1) means that T_n has constant [zero] regression with respect to $\mathbf{Y}$. In the special case $T_n = \bar{X}_n$, Kagan, Linnik, Rao (1967) proved that, provided $\mathbb{E}(X_1) = 0$, the equation

$$\mathbb{E}(\bar{X}|X_1 - \bar{X}, \ldots, X_n - \bar{X}) = 0$$

can be satisfied if and only if F is normal.

Assume that T_n is a translation equivariant estimator which is *asymptotically linear* in the sense that it admits an asymptotic representation

$$T_n = \theta + \frac{1}{n}\sum_{i=1}^{n} \psi(X_i - \theta) + R_n, \quad R_n = o_p(n^{-1/2}) \tag{1.2}$$

with an appropriate function ψ, for $\theta \in \mathbb{R}^1$. In the special case $T_n = \bar{X}_n$ is $R_n \equiv 0$, the approximation holds as an identity. The relation (1.1) then characterizes the normal distribution. In all other cases, the function ψ is nonlinear and $R_n \neq 0$. Jurečková and Milhaud (1999) proved that the constant regression of the leading term on the right-hand-side of (1.2) with respect to $\mathbf{Y}$ provides a characterization of the underlying distribution even in the non-normal case; this characterization is continuous with respect to a certain distance in the space of ψ-functions.

Theorem 1.1. (Jurečková and Milhaud (1999)) *Let f be an absolutely continuous density with positive and finite Fisher information and with characteristic function $\varphi(t) \neq 0$, $t \in \mathbb{R}^1$. Let $\psi : \mathbb{R}^1 \mapsto \mathbb{R}^1$ be a function satisfying*

$$0 < \int_{\mathbb{R}^1} \psi^2(x) f(x) dx < \infty \qquad \text{and} \qquad \int_{\mathbb{R}^1} \psi(x) f(x) dx = 0. \tag{1.3}$$

Then, provided $n \geq 4$,

$$\mathbb{E}_0\left(\sum_{i=1}^{n} \psi(X_i)|\mathbf{Y}\right) = C \quad \text{a.s. } [F] \tag{1.4}$$

holds for some constant C if and only if $\psi(x) = A\psi_f(x)$ for almost all $x \in \mathbb{R}^1$ *for some constant $A \neq 0$, where*

$$\psi_f(x) = -\frac{f'(x)}{f(x)}, \qquad x \in \mathbb{R}^1. \tag{1.5}$$

This result illustrates that the score function ψ_f plays a central role in statistical inference; in fact, $\sum_{i=1}^{n} \psi_f(X_i)$ is the component of the asymptotic representation (1.2), orthogonal to $\mathbf{Y}$. In Section 2 we shall illustrate one of many possible applications of these results, leading to a special form of the convolution theorem.

More generally, consider the linear regression model

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{E} \tag{1.6}$$

with the vector of observations $\mathbf{Y} = (Y_1, \ldots, Y_n)'$, matrix $\mathbf{X} = \mathbf{X}_n$ of order $n \times p$ of the full rank p, parameter $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)'$ and with the vector $\mathbf{E} = (E_1, \ldots, E_n)'$ of i.i.d. errors, $\mathbb{E}(E_i) = 0$, $i = 1, \ldots, n$, distributed according to the density f. Assume that f is absolutely continuous and has a non-vanishing characteristic function φ. We may assume, without loss of generality, that $\mathbf{X}$ takes the form

$$\mathbf{X} = \begin{bmatrix} \mathbf{I}_p \\ \mathbf{X}_1 \end{bmatrix} \tag{1.7}$$

with $\mathbf{X}_1$ of order $(n-p) \times p$ (if this is not the case, we may reorder the observations to get $\mathbf{X} = \begin{bmatrix} \mathbf{X}_2 \\ \mathbf{X}_3 \end{bmatrix}$ with a regular $p \times p$ matrix $\mathbf{X}_2$, replace $\boldsymbol{\beta}$ by $\mathbf{X}_2\boldsymbol{\beta}$ and put $\mathbf{X}_1 = \mathbf{X}_3\mathbf{X}_2^{-1}$). Consider the problem of estimating $\boldsymbol{\beta}$ under the condition that the loss function incurred when estimating $\boldsymbol{\beta}$ by $\mathbf{b}$ is

$$L(\mathbf{b}, \boldsymbol{\beta}) = \|\mathbf{b} - \boldsymbol{\beta}\|^2 . \tag{1.8}$$

The problem is invariant under the group of transformations $\mathcal{G} = \{\mathbf{Y} + \mathbf{X}\mathbf{b}, \mathbf{b} \in \mathbb{R}^p\}$, and the vector

$$\mathbf{Z} = \mathbf{Y} - (\mathbf{Y}_1', \mathbf{Y}_1'\mathbf{X}_1')', \qquad \mathbf{Y}_1 = (Y_1, \ldots, Y_p)', \tag{1.9}$$

is one of the maximal invariants for $\mathcal{G}$. By Section 7.7 of Kagan, Linnik and Rao (1973), the admissibility of the least squares estimator with respect to the loss (1.8) implies that f is normal.

An estimator $\mathbf{T}_n(\mathbf{Y})$ of $\boldsymbol{\beta}$ is called *regression equivariant* if $\mathbf{T}_n(\mathbf{Y} + \mathbf{X}\mathbf{b}) = \mathbf{T}_n(\mathbf{Y}) + \mathbf{b}$ $\forall \mathbf{b} \in \mathbb{R}^p$. A broad class of estimators $\mathbf{T}_n$ of $\boldsymbol{\beta}$ admits the asymptotic representation

$$\begin{aligned} (\mathbf{X}_n'\mathbf{X}_n)^{1/2}(\mathbf{T}_n(\mathbf{Y}) - \boldsymbol{\beta}) &= (\mathbf{X}_n'\mathbf{X}_n)^{-1/2}\mathbf{X}_n' \sum_{i=1}^n \mathbf{e}_i \psi(Y_i - \mathbf{e}_i'\mathbf{X}_n\boldsymbol{\beta}) + \mathbf{R}_n \\ &= \mathbf{M}_n(\mathbf{Y}, \boldsymbol{\beta}) + \mathbf{R}_n \quad \text{(say)}, \\ \|\mathbf{R}_n\| &= o_p(1) \end{aligned} \tag{1.10}$$

as $n \to \infty$, where $\mathbf{e}_i = (e_{i1}, \ldots, e_{in})'$ with $e_{ij} = \delta_{ij}$, $1 \le i, j \le n$. Sufficient conditions for (1.10) can be found in Jurečková and Sen (1996). The representation holds as an identity if $\mathbf{T}_n$ is the least squares estimator $\hat{\boldsymbol{\beta}}_n$ and ψ is linear.

The leading term $\mathbf{M}_n(\mathbf{Y},\boldsymbol{\beta})$ on the right-hand side of (1.10) is not regression equivariant unless ψ is linear. However, it is *locally uniformly asymptotically equivariant* under general conditions in the sense that

$$\mathbf{M}_n(\mathbf{Y}+\mathbf{X}_n(\mathbf{X}_n'\mathbf{X}_n)^{-1/2}\mathbf{t},\boldsymbol{\beta}) = \mathbf{M}_n(\mathbf{Y},\boldsymbol{\beta}) + \mathbf{t} + o_p(1) \tag{1.11}$$

uniformly for $\mathbf{t}\in\mathbb{R}^p$, $\|\mathbf{t}\|\le C$ for any fixed C, $0<C<\infty$. The following theorem shows that, similarly to the case of the location model, the fact that the conditional expectation of the leading term on the right hand side of (1.10) vanishes characterizes the distribution of the errors in the model (1.6).

THEOREM 1.2. (Jurečková and Milhaud (1999)) *Consider the modle (1.6) satisfying the condition (1.7) and with density f possessing positive and finite Fisher information and nowhere zero characteristic function φ. Then*

(*i*) $$\mathbf{X}'\mathbb{E}_{\mathbf{0}}(\sum_{i=1}^{n}\mathbf{e}_i\psi_f(Y_i)|\mathbf{Z}) = \mathbf{0}.$$

(*ii*) *Let $\psi:\mathbb{R}^1\mapsto\mathbb{R}^1$ satisfy* (1.3) *and*

$$\mathbf{X}'\mathbb{E}_{\mathbf{0}}(\sum_{i=1}^{n}\mathbf{e}_i\psi(Y_i)|\mathbf{Z}) = \mathbf{0}. \tag{1.12}$$

Provided $n\ge 3+p$, then

$$\psi(x) = A\psi_f(x) \quad \text{a.e. } x\in\mathbb{R}^1, \quad A=\text{const}, \quad A\neq 0 \tag{1.13}$$

where $\mathbf{Z}$ is the maximal invariant defined in (1.9).

Hallin, Jurečková and Milhaud (1998) further extended these results to the simple linear regression model with autoregressive errors.

2. Convolution theorem

Theorems 1.1 and 1.2 lead to a special form of the Hájek (1971) convolution theorem, valid for equivariant estimators in the location and linear regression models.

THEOREM 2.1.

(i) *Let $X_1,\dots,X_n$ be independent observations, identically distributed according to the probability density $f(x-\theta)$, $\theta\in\mathbb{R}_1$, where f has positive and finite Fisher information $I(f)$ and nowhere zero characteristic function φ. Let $T_n=T_n(X_1,\dots,X_n)$ be an equivariant estimator of θ with finite quadratic risk and such that*

$$n^{1/2}(T_n-\theta) = O_p(1) \tag{2.14}$$

as $n \to \infty$. Let θ_n^* be the minimum risk estimator of θ, which admits the asymptotic representation (1.2) *with $\psi \equiv \psi_f / I(f)$. Then*

$$n^{1/2}(T_n - \theta) = \delta_n^*(\theta) + \Delta_n + \tilde{R}_n \tag{2.15}$$

where $\delta_n^*(\theta)$ and $\Delta_n(\theta)$ are uncorrelated random variables, Δ_n is location invariant $\forall n \geq 3$ and, as $n \to \infty$,

$$\begin{aligned} &\delta_n^*(\theta) \xrightarrow{\mathcal{D}} \mathcal{N}\Big(0, \big(I(f)\big)^{-1}\Big), \\ &\Delta_n = \begin{cases} O_p(1) & \text{and} \\ o_p(1) & \text{if } T_n \text{ is asymptotically efficient}, \end{cases} \\ &\text{and} \qquad \tilde{R}_n = o_p(1). \end{aligned} \tag{2.16}$$

(ii) *Let the observations $Y_1, \ldots, Y_n$ follow the model* (1.6) *satisfying the condition* (1.7) *and*

$$\lim_{n\to\infty} \max_{1 \leq i \leq n} \big(\mathbf{x}_i'(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_i\big) = 0 \tag{2.17}$$

with $\mathbf{x}_i'$ being the i-th row of $\mathbf{X}_n$, $i = 1, \ldots, n$. Assume that the density f of the errors satisfies the conditions stated in (i). Let $\mathbf{T}_n = \mathbf{T}_n(\mathbf{Y})$ be a regression equivariant estimator of $\boldsymbol{\beta}$ with finite quadratic risk and such that

$$(\mathbf{X}_n\mathbf{X}_n')^{1/2}(\mathbf{T}_n - \boldsymbol{\beta}) = O_p(1) \qquad \text{as} \quad n \to \infty. \tag{2.18}$$

Let $\mathbf{T}_n^*$ be the minimum risk estimator of $\boldsymbol{\beta}$, which admits the asymptotic representation (1.10) *with $\psi \equiv \psi_f / I(f)$. Then*

$$(\mathbf{X}_n'\mathbf{X}_n)^{1/2}(\mathbf{T}_n - \boldsymbol{\beta}) = \boldsymbol{\delta}_n^*(\boldsymbol{\beta}) + \boldsymbol{\Delta}_n + \tilde{\mathbf{R}}_n \tag{1.19}$$

where $\boldsymbol{\delta}_n^*(\boldsymbol{\beta})$ and $\boldsymbol{\Delta}_n$ are random vectors, uncorrelated for $n \geq p + 3$, $\boldsymbol{\Delta}_n$ is regression invariant and, as $n \to \infty$,

$$\begin{aligned} &\boldsymbol{\delta}_n^*(\boldsymbol{\beta}) \xrightarrow{\mathcal{D}} \mathcal{N}_p\Big(\mathbf{0}\big(I(f)\big)^{-1}\mathbf{I}_p\Big), \\ &\|\boldsymbol{\Delta}_n\| = \begin{cases} O_p(1) & \text{and} \\ o_p(1) & \text{if } \mathbf{T}_n \text{ is asymptotically efficient}, \end{cases} \\ &\big\|\tilde{R}_n(\boldsymbol{\beta})\big\| = o_p(1). \end{aligned} \tag{2.20}$$

P r o o f. We shall prove only part (ii), which then implies (i) as a special case. Denote $\mathbf{Q}_n = \mathbf{X}_n'\mathbf{X}_n$, for simplicity. Put

$$\boldsymbol{\delta}_n^*(\boldsymbol{\beta}) = \mathbf{Q}_n^{1/2}\big\{\mathbf{T}_n - \boldsymbol{\beta} - \mathbb{E}_{\mathbf{0}}\left[\mathbf{T}_n|\mathbf{Z}_n\right]\big\} \tag{2.21}$$

and

$$\boldsymbol{\Delta}_n = \mathbf{Q}_n^{1/2} \mathbb{E}_{\mathbf{0}} \left[\mathbf{T}_n | \mathbf{Z}_n \right]. \tag{2.22}$$

Then $\boldsymbol{\delta}_n^*(\boldsymbol{\beta}) = \mathbf{Q}_n^{1/2}(\mathbf{T}_n^* - \boldsymbol{\beta})$ and, by (1.10),

$$\mathbf{Q}_n^{1/2}(\mathbf{T}_n^* - \boldsymbol{\beta}) = \left(I(f)\right)^{-1} \mathbf{Q}_n^{-1/2} \mathbf{X}_n' \sum_{i=1}^{n} \mathbf{e}_i \psi_f(\mathbf{Y}_i - \mathbf{e}_i' \mathbf{X}_n \boldsymbol{\beta}) + \mathbf{R}_n^*, \ \|\mathbf{R}_n^*\| = o_p(1); \tag{2.23}$$

hence, the asymptotic normality of $\boldsymbol{\delta}_n^*$ follows under (2.17). Moreover, by (2.21), (2.22) and (2.23),

$$\boldsymbol{\Delta}_n = \mathbf{Q}_n^{1/2}(\mathbf{T}_n - \mathbf{T}_n^*) = O_p(1) \tag{2.24}$$

and it is $o_p(1)$ if $\mathbf{T}_n$ is asymptotically efficient. Further, by (2.23) and by Theorem 2.1,

$$\begin{aligned}
&\mathbb{E}_{\boldsymbol{\beta}} \left(\boldsymbol{\delta}_n^*(\boldsymbol{\beta}) \cdot \boldsymbol{\Delta}_n' \right) \\
&\quad = \left(I(f)\right)^{-1} \mathbf{Q}_n^{-1/2} \mathbb{E}_{\boldsymbol{\beta}} \left[\mathbf{X}_n' \sum_{i=1}^{n} \mathbf{e}_i \psi_f(\mathbf{Y}_i - \mathbf{e}_i' \mathbf{X}_n \boldsymbol{\beta}) \left(\mathbb{E}_{\mathbf{0}}(\mathbf{T}_n | \mathbf{Z}) \right)' \right] \mathbf{Q}_n^{1/2} \\
&\quad = \left(I(f)\right)^{-1} \mathbf{Q}_n^{-1/2} \mathbb{E}_{\mathbf{0}} \left\{ \mathbb{E}_{\mathbf{0}} \left[\mathbf{X}_n' \sum_{j=1}^{n} \mathbf{e}_j \psi_f(\mathbf{Y}_j) | \mathbf{Z} \right] \left(\mathbb{E}_0 [\mathbf{T}_n | \mathbf{Z}] \right)' \right\} \mathbf{Q}_n^{1/2} \\
&\quad = \mathbf{0} = \mathbb{E}_{\boldsymbol{\beta}} \left(\boldsymbol{\delta}_n^*(\boldsymbol{\beta}) \right) \cdot \mathbb{E}_{\boldsymbol{\beta}} (\boldsymbol{\Delta}_n'),
\end{aligned} \tag{2.15}$$

which implies that $\boldsymbol{\delta}_n^*(\boldsymbol{\beta}_n^*)$ and $\boldsymbol{\Delta}_n$ are uncorrelated. On the other hand, because (1.12) characterizes ψ_f, any other choice $\psi \neq \psi_f$ leads to correlated vectors unless $\mathbf{T}_n$ is the minimum risk equivariant estimator. $\square$

3. Score function

The score function φ_f plays a basic role in statistical inference; for example in the maximum likelihood estimation, score tests, locally or asymptotically optimal rank tests, among others. Theorem 2.1 shows, roughly speaking, that the role of $\psi_f(X)/I(f)$ under the density f is similar to that of X under normal f. Another illustration of the role of ψ_f is provided by the result of Jurečková and Milhaud (1994), concerning the multivariate location model, showing that the quadratic risk of the random vector $\boldsymbol{\theta} + \boldsymbol{\Psi}_{f,n}(\mathbf{X} - \boldsymbol{\theta})$, where

$$\boldsymbol{\Psi}_{f,n}(\mathbf{X} - \boldsymbol{\theta}) = \left(nI(f)\right)^{-1} \left(\sum_{j=1}^{n} \psi_f(X_{1j} - \theta_1), \ldots, \sum_{j=1}^{n} \psi_f(X_{pj} - \theta_p) \right)'$$

can be shrunken similarly, analogously as the risk of the multivariate mean $\bar{\mathbf{X}}$ under the normal density. On the other hand, $\boldsymbol{\theta} + \boldsymbol{\Psi}_{f,n}(\mathbf{X} - \boldsymbol{\theta})$ is generally not a statistic; it is independent of $\boldsymbol{\theta}$ only if ψ_f is linear and hence only if the underlying distribution is normal.

Because ψ_f uniquely characterizes the distribution f and ψ_f corresponding to normal f is linear, the knowledge of the logarithmic derivative of the density of an estimator or of another statistic would be of great interest; its curvature would show the distance of the distribution from the normal. This problem was successfully solved by Klaassen (1980) for location equivariant estimators. The following theorem is an extension of his result to a more general setup and it gives an interesting characterization of the finite-sample distribution of general sample statistics. These results again demonstrate the central role of the score function in the statistical inference.

Theorem 3. *Let $X_1, \dots, X_n$ be i.i.d. observations from a distribution with the distribution function $F(x-\theta)$ where F has an absolutely continuous density f with positive and finite Fisher information. Let $S_n = S_n(X_1, \dots, X_n)$ be a statistic; let G_θ and g_θ denote its distribution function and density, respectively, under θ. Then*

$$\frac{\partial \log\ g_\theta(y)}{\partial \theta} = \mathbb{E}_\theta \left[-\sum_{i=1}^{n} \frac{f'(X_i-\theta)}{f(X_i-\theta)} \middle| S(X_1, \dots, X_n) = y \right]. \tag{3.1}$$

Proof. For $\delta > 0$,

$$\begin{aligned} \frac{1}{\delta}&\left[P_{\theta+\delta}(S \le y) - P_\theta(S \le y)\right] \\ &= \frac{1}{\delta} \int_{S \le y} \cdots \int \left[\prod_{i=1}^{n} f(x_i - \theta - \delta) - \prod_{i=1}^{n} f(x_i - \theta) \right] dx_1 \dots dx_n\,, \end{aligned} \tag{3.2}$$

hence

$$\begin{aligned} \frac{\partial G_\theta(y)}{\partial \theta} &= \int_{S(x_1,\dots,x_n) \le y} \cdots \int \sum_{i=1}^{n} \left(-\frac{f'(x_i-\theta)}{f(x_i-\theta)} \right) \prod_{i=1}^{n} f(x_i-\theta)\, dx_1 \dots dx_n \\ &= \mathbb{E}_\theta \left[\sum_{i=1}^{n} \left(-\frac{f'(x_i-\theta)}{f(x_i-\theta)} \right) \cdot I[S(X_1,\dots,X_n) \le y] \right], \end{aligned} \tag{3.3}$$

thus

$$\begin{aligned} \frac{\partial^2 G_\theta(y)}{\partial \theta \partial y} &= \frac{\partial g_\theta(y)}{\partial \theta} \\ &= \mathbb{E}_\theta \left[-\sum_{i=1}^{n} \frac{f'(X_i-\theta)}{f(X_i-\theta)} \middle| S(X_1,\dots,X_n) = y \right] g_\theta(y)\,, \end{aligned} \tag{3.4}$$

which implies 3.1. □

COROLLARY 3.1. *Let* $S(X_1, \dots, X_n) = -\sum\limits_{i=1}^{n} \frac{f'(X_i)}{f(X_i)}$. *Then*

$$\frac{\partial \log\, g_\theta(y)}{\partial\theta} = \mathbb{E}_\theta \left[S(X_1 - \theta, \dots, X_n - \theta) \,|\, S(X_1, \dots, X_n) = y \right]. \tag{3.5}$$

Specifically,

$$\left.\frac{\partial \log\, g_\theta(y)}{\partial\theta}\right|_{\theta=0} = y\,, \qquad y \in \mathbb{R}^1. \tag{3.6}$$

Remark. *Unlike in the case of normal* f, where $\frac{\partial \log\, g_\theta(y)}{\partial\theta} \equiv y$, *for general* f *the function* $\frac{\partial \log g_\theta(y)}{\partial\theta}$ *is linear in* y *only locally in a neighbourhood of* $\theta = 0$.

COROLLARY 3.2. (Klaassen 1980.) *Let* $S(X)$ *be equivariant under to the shift in location, i.e.,*

$$S(X_1 + c, \dots, X_n + c) = S(X_1, \dots, X_n) + c \qquad \forall c \in \mathbb{R}^1.$$

Then

$$-\frac{g'(y)}{g(y)} = \mathbb{E}_0 \left[\sum_{i=1}^{n} \frac{f'(X_i)}{f(X_i)} \Big| S(\mathbf{X}) = y \right] \tag{3.7}$$

where g *is the density of* $(S(\mathbf{X}) - \theta)$ *under* θ.

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