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INTERVAL ESTIMATION OF NONCENTRALITY PARAMETER OF F-DISTRIBUTION

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ABSTRACT. Powers of statistical tests of hypotheses most often require noncentral χ^2, t or F-distributions. The noncentrality parameter (NCP) is a function of the sample size. In the process of designing experiments or, more specifically, planning a clinical trial, stating the proper sample size that would with specified power enable the detection of a desired difference, is an essential task. Ideally, a pilot study or other source of information on the value of test statistic in a similar setting is available. Hence estimation of the NCP becomes an important issue. Here we present a simple approach to construction of a confidence interval of an NCP of an F-distribution and compare the derived interval with a different type suggested in the literature.

1. Introduction

The noncentral χ^2, t or F-distributions occur the most often in testing statistical hypotheses. For example, in ANOVA setting, the commonly used test statistic follows a noncentral F-distribution with the noncentrality parameter (NCP) depending on the parameter in the alternative. The need to estimate the noncentrality parameter arises when the estimate of the effect size as a function of noncentrality parameter is presented, either as a point estimator or an interval estimator, in psychological applications. Another motivation comes from the use of the noncentrality parameter as a function of sample size in power analysis. The power of a test depends directly on the NCP of the corresponding distribution. The NCP is a function of the sample size. To determine the sample size based on the estimate of the effect size is an essential part of planning an experiment, e.g., a clinical trial.

Here we concentrate on the derivation of the confidence interval for the noncentrality parameter of the F-distribution, that in the balanced case may be adapted to derive a confidence interval for the effect size.

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There is a considerable amount of literature devoted to the topic of point estimation of NCP, addressing mainly the questions of unbiasedness, optimality, and admissibility of the estimators. See, e.g., [7], [1], [2], [4], [6], and for suggestions for interval estimation of NCP for noncentral t-distribution see, e.g., [3].

2. Maximum likelihood estimation of NCP

The density function $h(f|\lambda)$ for the noncentral F distribution with numerator degrees of freedom ν_1, denominator degrees of freedom ν_2, and noncentrality parameter λ is

$$h(f|\lambda) = e^{-\frac{\lambda}{2}} \sum_{r=0}^{\infty} \left(\frac{\lambda}{2}\right)^r g(f,r)\,, \tag{1}$$

where

$$g(f,r) = \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}+r} f^{\frac{\nu_1}{2}+r-1}}{r!\,\mathrm{B}\left(\frac{\nu_1}{2}+r, \frac{\nu_2}{2}\right)\left(1+\frac{\nu_1}{\nu_2}f\right)^{\frac{\nu_1+\nu_2}{2}+r}} \tag{2}$$

depends only on ν_1, ν_2, and f, and the Beta function

$$\mathrm{B}(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}\,.$$

Note that the noncentral F-distribution does not belong to the exponential family.

Assume that there is one observed value of f as a realization of the noncentral F variable. The likelihood function based on this observation is

$$L(\lambda|f) = e^{-\frac{\lambda}{2}} \sum_{r=0}^{\infty} \left(\frac{\lambda}{2}\right)^r g(f,r)\,. \tag{3}$$

THEOREM 1. *Let the random variable F follow noncentral F-distribution with an unknown noncentrality parameter λ and a given numerator and denominator degrees of freedom ν_1, ν_2, respectively. Let f be the observed value of F.*

The maximum likelihood estimate $\hat\lambda$ of λ is given by:

1. *for $f \le 1$, $\hat\lambda = 0$,*
2. *for $f > 1$, $\hat\lambda$ is the unique solution to the likelihood equation*

$$\sum_{r=0}^{\infty} \left(\frac{\lambda}{2}\right)^r g(f,r) \left(\frac{y}{(1+y)} \frac{(\nu_1+\nu_2+2r)}{(\nu_1+2r)} - 1\right) = 0\,, \tag{4}$$

where $y = \frac{\nu_1}{\nu_2} f$.

Note. Spruill in [7] considers a random sample of size n from noncentral F-distribution. The proof of Theorem 1 can follow the lines of the proof of Theorem 3.1. in his paper. For detailed proof of Theorem 1 the next two lemmas are needed. (We are grateful to Miloš Volauf for supplying us a proof of Lemma 1.)

Lemma 1. (Spruill [7] p. 223) *Suppose that* $\sum\limits_{i\geq 0}^{\infty} a_i t^i$ *and* $\sum\limits_{i\geq 0}^{\infty} b_i t^i$ *converge for all* $t > 0$, *and that for* i *sufficiently large* $b_i > 0$. *If*

$$\lim_{i\to\infty} \frac{a_i}{b_i} = \gamma$$

including $\pm\infty$, *then*

$$\lim_{t\to\infty} \frac{\sum\limits_{i\geqslant 0} a_i t^i}{\sum\limits_{i\geqslant 0} b_i t^i} = \gamma\,.$$

Lemma 2. (Lehmann [5] p. 313) *Let* $f(z) = \frac{\sum_{k=0}^{\infty} a_k z^k}{\sum_{k=0}^{\infty} b_k z^k}$, *where the constants* a_k, b_k *are positive, and* $\sum\limits_{k=0}^{\infty} a_k z^k$ *and* $\sum\limits_{k=0}^{\infty} b_k z^k$ *converge for all* $z > 0$. *Suppose that* $\frac{a_k}{b_k} < \frac{a_{k+1}}{b_{k+1}}$ *for all* k. *Then*

$$f'(z) = \frac{\sum\sum\limits_{k<n} (n-k)(a_n b_k - a_k b_n) z^{k+n-1}}{\left(\sum\limits_{k=0}^{\infty} b_k z^k\right)^2}$$

is positive and hence $f(z)$ *is an increasing function of* z.

Note. For the use of Lemma 2 we need its modified version. Suppose all the assumptions of Lemma 2 hold true except that $\frac{a_k}{b_k} > \frac{a_{k+1}}{b_{k+1}}$ for all k. Then $f(z)$ is decreasing.

Proof of Theorem 1. The derivative of the likelihood function given by (3) with respect to λ is

$$\begin{aligned}
\frac{\partial L(f,\lambda)}{\partial\lambda} &= \frac{1}{2} e^{-\frac{\lambda}{2}} \sum_{r=0}^{\infty} \left(\frac{\lambda}{2}\right)^r \left\{(r+1) g(f, r+1) - g(f,r)\right\} \\
&= \frac{1}{2} e^{-\frac{\lambda}{2}} \sum_{r=0}^{\infty} \left(\frac{\lambda}{2}\right)^r g(f,r) \left(\frac{y}{(1+y)} \frac{(\nu_1+\nu_2+2r)}{(\nu_1+2r)} - 1\right) \\
&= \frac{1}{2} e^{-\frac{\lambda}{2}} \sum_{r=0}^{\infty} \left(\frac{\lambda}{2}\right)^r g(f,r) \frac{\nu_1\nu_2(f-1) - 2r\nu_2}{(\nu_1+2r)(\nu_2+\nu_1 f)}\,.
\end{aligned}$$

Since both degrees of freedom are positive, it follows immediately that in case $f \leq 1$ each coefficient in the derivative is negative. Hence in this case the likelihood function (3) is monotone decreasing and its maximum is at $\hat{\lambda} = 0$.

The second part of the proof, for $f > 1$, is based on the following idea. The likelihood equation (4) can be expressed as

$$\frac{\sum\limits_{r=0}^{\infty} \left(\frac{\lambda}{2}\right)^r g(f,r) \left(\frac{y}{(1+y)} \frac{(\nu_1+\nu_2+2r)}{(\nu_1+2r)}\right)}{\sum\limits_{r=0}^{\infty} \left(\frac{\lambda}{2}\right)^r g(f,r)} = 1 . \tag{5}$$

At $\lambda = 0$, the left side of (5) takes the value $\frac{y}{(1+y)} \frac{(\nu_1+\nu_2)}{\nu_1}$, which for $f \geq 1$ is greater than 1. It follows from Lemma 1 that for $\lambda \to \infty$ the left side of (5) tends to $\frac{y}{(1+y)}$. Moreover, from Lemma 2 it follows that (5) is a monotone decreasing function in λ. Hence, (4) has a unique solution, completing the proof. □

3. Interval estimation of λ

Let f be the observed value of F. Let $\lambda(f, \nu_1, \nu_2, p)$ be the function that returns the noncentrality parameter λ for whom the observed value f is the upper pth quantile. Let the significance level be α. Denote

$$\lambda_L = \lambda\Big(f, \nu_1, \nu_2, 1 - \frac{\alpha}{2}\Big) ,$$
$$\lambda_U = \lambda\Big(f, \nu_1, \nu_2, \frac{\alpha}{2}\Big) .$$

Hedges and Olkin in [3] suggest the $100(1-\alpha)\,\%$ confidence interval on λ the interval $[\lambda_L, \lambda_U]$.

Note. Let F_α denote the upper α-quantile of central F-distribution with ν_1 and ν_2 the numerator and denominator degrees of freedom, respectively. Then, if $f < F_{1-\frac{\alpha}{2}}$, the function $\lambda\big(f, \nu_1, \nu_2, 1 - \frac{\alpha}{2}\big)$ is not defined, and similarly, if $f < F_{\frac{\alpha}{2}}$, the function $\lambda\big(f, \nu_1, \nu_2, \frac{\alpha}{2}\big)$ is not defined.

Our idea is to base a $100(1-\alpha)\,\%$ confidence interval on *likelihood ratio statistics*. Assume

$$H_0 : \quad \lambda = \lambda_0 \qquad \text{vs.} \qquad \lambda \neq \lambda_0 . \tag{6}$$

Define the likelihood ratio (LR) as

$$\rho(f, \lambda_0) = \frac{\sup\limits_{\lambda} h(f, \lambda)}{h(f, \lambda_0)} .$$

Defined in terms of LR, for a given value of λ_0, values of F *as extreme as* f are $E = \{F\colon \rho(F,\lambda_0) \ge \rho(f,\lambda_0)\}$. Denote the probability at λ_0 by P_{λ_0}. Then the corresponding p-value is

$$\pi(f,\lambda_0) = \mathrm{P}_{\lambda_0}(E).$$

For the LR test, the critical region is

$$R(\lambda_0) = \{F\colon \pi(F,\lambda_0) \le \alpha\}.$$

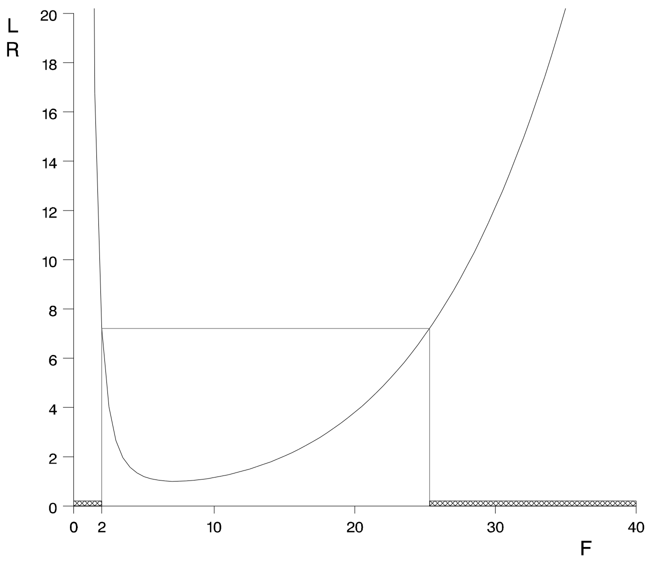

FIGURE 1. The set E for testing $H_0:\lambda = 20$ vs. $H_1:\lambda \ne 20$ where $f = 2$, $\nu_1 = 3$, and $\nu_2 = 10$.

Assume that $\rho(F,\lambda_0)$ is quasi-convex (we did not prove it). See Figure 1. Then the critical region $R(\lambda_0)$ is the complement of a finite interval with the endpoints $f_1(\lambda_0)$ and $f_2(\lambda_0)$. Moreover,

$$\begin{aligned}\alpha &= P_{\lambda_0}\{\rho(F,\lambda_0) \ge \alpha\}\\ &= P_{\lambda_0}\{F \le f_1(\lambda_0)\} + P_{\lambda_0}\{F \ge f_2(\lambda_0)\}\\ &= \alpha_1(\lambda_0) + \alpha_2(\lambda_0).\end{aligned}$$

In other words, the LR test allocates α to the two tails depending on λ_0; for small λ_0, most of α goes in the upper tail, with the proportion shifting back to the lower tail as λ_0 increases. The acceptance region is

$$A(\lambda_0) = \{F\colon \pi(F,\lambda_0) \ge \alpha\}.$$

For any λ, the probability $P_\lambda(A(\lambda)) \ge 1-\alpha$, which follows from the definition of $A(\lambda)$.

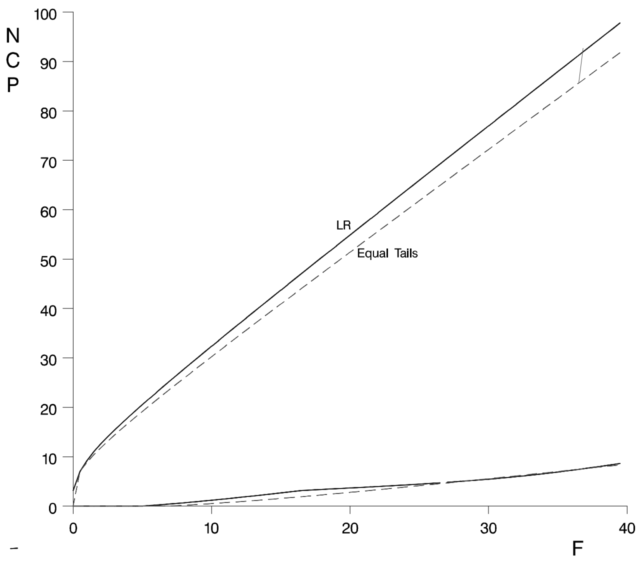

FIGURE 2. Graph of both confidence regions. The Equal Tail is based on equal probability $\frac{\alpha}{2}$ on both ends and the other one is the LR-based confidence interval. Here $\nu_1 = 1$ and $\nu_2 = 10$.

Define

$$CR = \{\lambda\colon F \in A(\lambda)\}\,.$$

Since

$$\lambda \in CR \iff F \in A(\lambda)\,,$$

CR represents a $100(1-\alpha)\,\%$ confidence region on λ. For illustration see Figure 2.

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