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CHEMICAL BALANCE WEIGHING DESIGNS UNDER THE RESTRICTION ON THE NUMBER OF OBJECTS PLACED ON THE PANS

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ABSTRACT. The paper studies the problem of estimation of the weights of p objects in n weighings using a chemical balance weighing design under the restriction $k_1 + k_2 < p$, where k_1 and k_2 represent the number of objects placed on the right and left pans, respectively. The incidence matrices of balanced bipartite block designs are used to construct chemical balance weighing designs. The condition for estimates of weights to be uncorrelated is given.

1. Introduction

The results of n weighing operations to determine the individual weights of p objects with a balance that is corrected for bias will fit into the linear model

$$\mathbf{y} = \mathbf{X}\omega + e\,, \tag{1.1}$$

where $\mathbf{y}$ is an $n \times 1$ random observed vector of the recorded results of the weights, $\mathbf{X} = (x_{ij})$, $i = 1, \dots, n$, $j = 1, \dots, p$, is an $n \times p$ matrix of known elements with $x_{ij} = -1, 1, 0$ if the jth object is kept on the right pan, left pan, or is not included in the ith weighing operation, respectively, ω is the $p \times 1$ vector representing the unknown weights of objects and $\mathbf{e}$ is an $n \times 1$ random vector of errors such that $E(\mathbf{e}) = \mathbf{0}_n$ and $E(\mathbf{ee}') = \sigma^2 \mathbf{I}_n$, where $\mathbf{0}_n$ is the $n \times 1$ vector with zero elements everywhere, $\mathbf{I}_n$ is the $n \times n$ identity matrix, "E" stands for the expectation and $\mathbf{e}'$ is used for transpose of $\mathbf{e}$. The matrix $\mathbf{X}$ is the design matrix and we refer to the chemical balance weighing design $\mathbf{X}$ with the covariance matrix $\sigma^2 \mathbf{I}_n$.

If the matrix $\mathbf{X}$ is of full column rank, the least squares estimates of the true weights are given by

$$\widehat{\omega} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'y\,, \tag{1.2}$$

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and the variance-covariance matrix of $\widehat{\omega}$ is

$$\mathrm{Var}(\widehat{\omega}) = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}\,. \tag{1.3}$$

The problem is to choose the matrix $\mathbf{X}$ in such a way that variance factors are minimized. A detailed account of weighing designs can be obtained from Raghavarao [9] and Banerjee [1]. Several methods of constructing optimal chemical balance weighing designs without any restriction on the number of objects placed on either pan are available in the literature. Dey [6], Saha [10], Kageyama and Saha [8] and others have shown how optimum chemical balance weighing designs can be constructed from the incidence matrices of balanced incomplete block designs for $p = v$ objects. Ceranka and Katulska [4] have constructed optimum chemical balance weighing designs from two incidence matrices of balanced incomplete block designs. Saha and Kageyama [11] have constructed optimum chemical balance weighing designs for $v+1$ objects in $4(r-\lambda)$ weighings from incidence matrices of balanced incomplete block designs for v treatments. In the same case, Ceranka and Katulska [5] study another method of constructing. Some results of constructing chemical balance weighing designs under the restriction on the number of objects placed on either pan was given by Swamy [12].

In the present paper we study another method of constructing the design matrix $\mathbf{X}$ for a chemical balance weighing design under the restriction on the number of objects placed on either pan. It is based on incidence matrices of balanced bipartite block designs.

2. Balanced bipartite block designs

A balanced bipartite block design is an arrangement of v treatments into b blocks such that

(i) Each block containing k distinct treatments is divided into 2 subblocks containing k_1 and k_2 treatments, respectively, $k = k_1 + k_2$;

(ii) Each treatment appears in r blocks;

(iii) Every pair of treatments with different subblocks appears together in λ_1 blocks;

(iv) Every pair of treatments with the same subblock appears together in λ_2 blocks.

The integers $v, b, r, k_1, k_2, \lambda_1, \lambda_2$ are called the parameters of the balanced bipartite block design. Let $\mathbf{N}$ be the incidence matrix of this design. The parameters are not all independent and they are related by the following identities, see: Huang [7],

$$b = \frac{\lambda_1 v(v-1)}{2k_1k_2},$$
$$\lambda_2 = \frac{\lambda_1\left[k_1(k_1-1)+k_2(k_2-1)\right]}{2k_1k_2}, \qquad (2.1)$$
$$r = \frac{\lambda_1 k(v-1)}{2k_1k_2}.$$

In particular case, when $k_1 = k_2$, (2.1) reduces to

$$b = \frac{\lambda_1 v(v-1)}{2k_1^2},$$
$$\lambda_2 = \frac{\lambda_1(k_1-1)}{k_1}, \qquad (2.2)$$
$$r = \frac{\lambda_1(v-1)}{k_1}.$$

It is clear that in this case

$$\lambda_1 - \lambda_2 = \frac{\lambda_1}{k_1} \geq 1.$$

When $k_1 = 1$ then $\lambda_2 = 0$ and the balanced bipartite block design reduces to the balanced incomplete block design with the parameters $v = v$, $b = \lambda_1 v(v-1)/2$, $r = \lambda_1(v-1)$, $k = 2$, $\lambda = \lambda_1$. Balanced bipartite block designs with $k_1 = k_2$ are called tournament designs and have been studied by Bose and Cameron [2], [3].

3. The incidence matrix of the balanced bipartite block design as the matrix of the chemical balance weighing design

Let $\mathbf{N}$ be the incidence matrix of the balanced bipartite block design with the parameters $v, b, r, k_1, k_2, \lambda_1, \lambda_2$. From $\mathbf{N}'$ we obtain another matrix $\mathbf{X}$ by replacing the k_1 unities of each row which correspond to the elements belonging to the first subblock by -1. Thus, each row of $\mathbf{X}$ will contain k_1 elements equal to -1, k_2 elements equal to 1 and $v-k_1-k_2$ elements equal to 0. The matrix $\mathbf{X}$ is the design matrix of the chemical balance weighing design. In this design we have $n = b$ and $p = v$. Clearly, such a design implies that in each weighing, exactly $k = k_1 + k_2$ objects are weighed, k_1 of them on the right pan and k_2 on the left. Also, each object is weighed r times in the b weighing operations.

Lemma 3.1. *The chemical balance weighing design* $\mathbf{X}$ *based on the incidence matrix of the balanced bipartite block design is nonsingular if and only if* $k_1 \neq k_2$.

P r o o f. For the design matrix $\mathbf{X}$ given above, we have

$$\mathbf{X}'\mathbf{X} = \left[r - (\lambda_2 - \lambda_1)\right]\mathbf{I}_v + (\lambda_2 - \lambda_1)\boldsymbol{1}_v\boldsymbol{1}_v',$$

where $\boldsymbol{1}_v$ is the $v \times 1$ vector with the unit elements everywhere.

It is easy to see that

$$\det(\mathbf{X}'\mathbf{X}) = \left[r - (\lambda_2 - \lambda_1)\right]^{v-1}\left[r + (v-1)(\lambda_2 - \lambda_1)\right]. \tag{3.1}$$

The determinant (3.1) is positive if and only if

$$r \neq \lambda_2 - \lambda_1 \tag{3.2}$$

and

$$r \neq -(v-1)(\lambda_2 - \lambda_1). \tag{3.3}$$

Using (2.1) it can be shown that (3.2) implies

$$kv \neq (k_1 - k_2)^2,$$

which is always satisfied since $v \geq k = k_1 + k_2$. Again, using (2.1) it can be established that (3.3) implies

$$(k_1 - k_2)^2 \neq 0.$$

The last expression is positive if and only if $k_1 \neq k_2$. So the lemma is proved. □

If the design $\mathbf{X}$ given above is nonsingular then

$$(\mathbf{X}'\mathbf{X})^{-1} = \frac{1}{r - (\lambda_2 - \lambda_1)}\mathbf{I}_v - \frac{\lambda_2 - \lambda_1}{\left[r - (\lambda_2 - \lambda_1)\right]\left[r + (v-1)(\lambda_2 - \lambda_1)\right]}\boldsymbol{1}_v\boldsymbol{1}_v'.$$

From above it follows that

$$\operatorname{Var}(\widehat{\omega}_i) = \frac{\sigma^2}{r - (\lambda_2 - \lambda_1)}\left[1 - \frac{\lambda_2 - \lambda_1}{r + (v-1)(\lambda_2 - \lambda_1)}\right], \quad i = 1, \dots, v, \tag{3.4}$$

and

$$\operatorname{Cov}(\widehat{\omega}_i, \widehat{\omega}_{i'}) = -\frac{\sigma^2(\lambda_2 - \lambda_1)}{\left[r - (\lambda_2 - \lambda_1)\right]\left[r + (v-1)(\lambda_2 - \lambda_1)\right]}, \tag{3.5}$$

$$i \neq i', \quad i' = 1, \dots, v.$$

Theorem 3.1. *For a nonsingular chemical balance weighing design with* $\mathbf{X}$ *given above the estimated weights are uncorrelated if and only if*

$$k_1 - k_2 = \sqrt{k_1 + k_2}\,. \tag{3.6}$$

P r o o f . From (3.5) we have that $\mathrm{Cov}(\widehat{\omega}_i, \widehat{\omega}_{i'}) = 0$, $i, i' = 1, \ldots, v$, $i \neq i'$, if and only if $\lambda_2 = \lambda_1$. But from (2.1) $\lambda_2 = \{\lambda_1[k_1(k_1-1) + k_2(k_2-1)]\}/(2k_1k_2)$, then $k_1 + k_2 = (k_1 - k_2)^2$. From this we have (3.6). Hence the theorem is proved. □

When (3.6) is satisfied, obviously, $k_1 \neq k_2$. Note that $k_1 + k_2$ being a perfect square is a necessary condition for the estimates to be uncorrelated.

From Theorem 3.1 we have

Corollary 3.1. *Let* k_1 *and* k_2 *are positive integers. The condition*

$$k_1 - k_2 = \sqrt{k_1 + k_2}$$

is satisfied if and only if

$$k_1 = \frac{q(q+1)}{2} \qquad \text{and} \qquad k_2 = \frac{q(q-1)}{2}\,,$$

where q *is a positive integer greater than 1.*

From Corollary 3.1 and the relations between parameters of the balanced bipartite block designs (2.1) we have the following corollary.

Corollary 3.2. *The existence of the bipartite block design with parameters*

$$v,\, b = \frac{2\lambda_1 v(v-1)}{q^2(q^2-1)}\,,\ r = \frac{2\lambda_1(v-1)}{q^2-1}\,,\ k_1 = \frac{q(q+1)}{2}\,,\ k_2 = \frac{q(q-1)}{2}\,,\ \lambda_1 = \lambda_2 \tag{3.7}$$

implies the existence of the chemical balance weighing design for which the estimated weights are uncorrelated.

Corollary 3.3. *If the estimated weights are uncorrelated then*

$$\mathrm{Var}(\widehat{\omega}_i) = \frac{\sigma^2}{r}\,, \qquad i = 1, \ldots, v\,.$$

4. The balance bipartite block designs leading to uncorrelated estimated weights

We have seen in Corollary 3.2 that if parameters of balanced bipartite block designs are defined by (3.7), then a chemical balance weighing design with $\mathbf{X}$ gives uncorrelated estimated weights. Under this condition we have the following theorems.

Theorem 4.1. *A chemical balance weighing design with* $\mathbf{X}$ *based on the balanced bipartite block design with parameters*

(i) $v = 3s + 1,\ b = st(3s + 1)/2,\ r = 2st,$
where s *and* t *are positive integers,* $s \geq 2$*, and* t *is not divided by* 3*,*

(ii) $v = s,\ b = st(s - 1)/2,\ r = 2t(s - 1),$
where s *and* t *are positive integers,* $s \geq 5$*,*

$k_1 = 3,\ k_2 = 1,\ \lambda_1 = \lambda_2 = t$ *gives uncorrelated estimated weights.*

Theorem 4.2. *A chemical balance weighing design with* $\mathbf{X}$ *based on the balanced bipartite block design with the parameters*

(i) $v = 36s + 1,\ b = st(36s + 1),\ r = 9st,$

(i') $v = 9(4s + 1),\ b = t(4s + 1)(9s + 2),\ r = t(9s + 2),$
where s *is a positive integer and* $t = 1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35 \pmod{36}$*,*

(ii) $v = 18s + 1,\ b = st(18s + 1)/2,\ r = 9st/2,$

(ii') $v = 9(2s + 1),\ b = t(2s + 1)(9s + 4)/2,\ r = t(9s + 4)/2,$
where s *is a positive integer and* $t = 2, 10, 14, 22, 26, 34 \pmod{36}$*,*

(iii) $v = 12s + 1,\ b = st(12s + 1)/3,\ r = 3st,$

(iii') $v = 3(4s + 3),\ b = t(4s + 3)(3s + 2)/3,\ r = t(3s + 2),$
where s *is a positive integer and* $t = 3, 15, 21, 33 \pmod{36}$*,*

(iv) $v = 9s + 1,\ b = st(9s + 1)/4,\ r = 9st/4,$

(iv') $v = 9u,\ b = ut(9u - 1)/4,\ r = t(9u - 1)/4,$
where s *and* u *are positive integers* $u \geq 2$ *and* $t = 4, 8, 16, 20, 28, 32 \pmod{36}$*,*

(v) $v = 6s + 1,\ b = st(6s + 1)/6,\ r = 3st/2,$

(v') $v = 3(2s + 1),\ b = t(2s + 1)(3s + 1)/6,\ r = t(3s + 1)/2,$
where s *is a positive integer,* $s \geq 2$*, and* $t = 6, 30 \pmod{36}$*,*

(vi) $v = 4s + 1,\ b = st(4s + 1)/9,\ r = st,$
where s *is a positive integer* $s \geq 3$*, and* $t = 9 \pmod{36}$*,*

(vii) $v = 3s + 1,\ b = st(3s + 1)/12,\ r = 3st/4,$

(vii') $v = 3u,\ b = tu(3u - 1)/12,\ r = t(3u - 1)/4,$
where s *and* u *are positive integers,* $s \geq 3$*,* $u \geq 4$*, and* $t = 12, 24 \pmod{36}$*,*

(viii) $v = 2s + 1,\ b = st(2s + 1)/18,\ r = st/2,$
where s *is a positive integer,* $s \geq 5$*,* $t = 18 \pmod{36}$*,*

$k_1 = 6,\ k_2 = 3,\ \lambda_1 = \lambda_2 = t$ *gives uncorrelated estimated weights.*

In the same way we can formulated the theorem for $k_1 = 10$ and $k_2 = 6$.

TABLE 1. Parameters of balanced bipartite block designs

v	b	r	k_1	k_2	λ_1	λ_2	v	b	r	k_1	k_2	λ_1	λ_2
5	10	8	3	1	3	3	16	40	10	3	1	1	1
5	20	16	3	1	6	6	16	80	20	3	1	2	2
6	15	10	3	1	3	3	18	34	17	6	3	4	4
6	30	20	3	1	6	6	19	19	9	6	3	2	2
7	7	4	3	1	1	1	19	38	18	6	3	4	4
7	14	8	3	1	2	2	19	57	12	3	1	1	1
7	21	12	3	1	3	3	21	21	16	10	6	6	6
7	28	16	3	1	4	4	21	35	15	6	3	3	3
7	35	20	3	1	5	5	22	77	14	3	1	1	1
8	28	14	3	1	3	3	25	25	16	10	6	5	5
9	36	16	3	1	3	3	25	50	18	6	3	3	3
10	10	9	6	3	4	4	25	100	16	3	1	1	1
10	15	6	3	1	1	1	27	39	13	6	3	2	2
10	20	18	6	3	8	8	31	31	16	10	6	4	4
10	30	12	3	1	2	2	37	37	9	6	3	1	1
10	45	18	3	1	3	3	37	74	18	6	3	2	2
11	55	20	3	1	3	3	41	41	16	10	6	3	3
13	13	9	6	3	3	3	45	55	11	6	3	1	1
13	26	8	3	1	1	1	61	61	16	10	6	2	2
13	26	18	6	3	6	6	76	95	20	10	6	2	2
13	52	16	3	1	2	2							

Based on Corollary 3.2 and Theorems 4.1, 4.2 we present the list of all parameters combinations of the range $v, b \leq 100$, $r \leq 20$, $k_1 + k_2 = 4, 9, 16$, which give uncorrelated estimated weights.

REFERENCES

[1] BANERJEE, K. S.: *Weighing Designs for Chemistry, Medicine Economics*, Operations Research, Statistics, Marcel Dekker Inc., New York, 1975.

[2] BOSE, R. C.—CAMERON, J. M.: *The bridge tournament problem and calibration designs for comparing pairs of objects*, J. Res. NBS (Math. and Math. Phys.) **69B** (1965), 323–332.

[3] BOSE, R. C.—CAMERON, J. M.: *Calibration designs based on solution to the tournament problem*, J. Res. NBS (Math. and Math. Phys.) **71B** (1967), 149–160.

[4] CERANKA, B.—KATULSKA, K.: *On construction of optimum chemical balance weighing designs*, J. Indian Statist. Assoc. **26** (1988), 27–30.

[5] CERANKA, B.—KATULSKA, K.: *On some optimum chemical balance weighing designs for $v+1$ objects*, J. Japan Statist. Soc. **18** (1988), 47–50.

[6] DEY, A.: *On some chemical balance weighing designs*, Austral. J. Statist. **13** (1971), 137–141.

[7] HUANG, C.: *Balanced bipartite weighing designs*, J. Combin. Theory **21** (1976), 20–34.

[8] KAGEYAMA, S.—SAHA, G. M.: *Note on the construction of optimum chemical balance weighing designs*, Ann. Inst. Statist. Math. **35A** (1983), 447–452.

[9] RAGHAVARAO, D.: *Constructions and Combinatorial Problems in Designs of Experiments*, John Wiley, New York, 1971.

[10] SAHA, G. M.: *A note on relations between incomplete block and weighing designs*, Ann. Inst. Statist. Math. **27** (1975), 387–390.

[11] SAHA, G. M.—KAGEYAMA, S.: *Balanced arrays and weighing designs*, Austral. J. Statist. **26** (1984), 119–124.

[12] SWAMY, M. N.: *Use of balanced bipartite weighing designs as chemical balance designs*, Comm. Statist. Theory Methods **11** (1982), 769–785.

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