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UNIFIED APPROACH TO DETERMINING NONSENSITIVENESS REGIONS

Lubomír Kubáček — Ludmila Kubáčková

ABSTRACT. The aim of this paper is to give some survey of results obtained in researching the nonsensitiveness regions of a linear unbiasedly estimable function $h(\beta)$ of the first order parameters and its dispersion in models with constraints of the type I and II within the universal model, where no assumptions concerning the regularity of the model are taken into account.

1. Introduction, stimulation of the problem

In the framework of linear models, the variance and covariance parameters are often known only approximately (e.g., from certificates of the devices used) and, moreover, their values may change in time. Therefore the question arises how to determine such a neighbourhood of the actual values of the variance components that their changes in this neighbourhood do not cause any essential change of the estimator of the first order parameter and its variance, the confidence region and the power function of the test.

The situation is complicated by the fact that the covariance matrix of the observation vector is often not diagonal, even not sparse. This fact reflects in the remarkable behaviour of the estimator of the unknown parameter and its fundamental statistical characteristics. This may by convincingly illustrated on the simple case of the model

$$\left[\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\beta, \begin{pmatrix} 5, & \sqrt{5}\rho \\ \sqrt{5}\rho, & 1 \end{pmatrix}\right], \qquad \beta \in \mathbb{R}^1, \quad \rho \in [-1, 1],$$

see [1].

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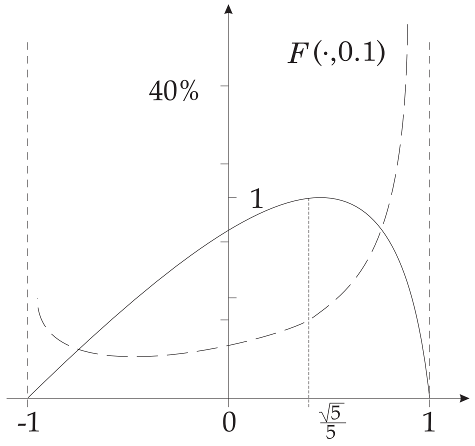

FIGURE 1. The dependence of the ratio of the standard deviations of the correction term (for $\delta\rho = 0.1$) and the ρ-LBLUE on the correlation coefficient ρ

The solid line on Fig. 1 expresses the dependence of the variance of the ρ-LBLUE on ρ and the dashed line then the course of the function

$$\sqrt{\frac{\operatorname{Var}\left\{\left[\partial\hat{\beta}(Y_1, Y_2, \rho)/\partial\rho\right]\delta\rho|\rho\right\}}{\operatorname{Var}\left[\hat{\beta}(Y_1, Y_2, \rho)|\rho\right]}} = \frac{4}{(6 - 2\sqrt{5}\rho)\sqrt{1-\rho^2}}\,\delta\rho$$

for $\delta\rho = 0.1$.

2. Preliminaries

In the following, the linear model of the form

$$\mathbf{Y} \sim_n \left(\mathbf{X}\beta,\ \operatorname{Var}(\mathbf{Y}) = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i\right)$$

will be considered. Here $\mathbf{Y}$ is an n-dimensional observation vector, $\mathbf{X}$ a known $n \times k$-dimensional design matrix, β a k-dimensional unknown vector of the first order parameters such that

$$\beta \in \begin{cases} \mathbb{R}^k, & \text{model without constraints}, \\ \mathcal{V}_I = \{\mathbf{u}\colon \mathbf{b} + \mathbf{B}\mathbf{u} = \mathbf{0}\}, & \text{model with constraints of the type I}. \end{cases}$$

($\mathbf{b} \in \mathcal{M}(\mathbf{B}) = \{\mathbf{B}\mathbf{u}\colon \mathbf{u} \in \mathbb{R}^k\}$ is a known vector and $\mathbf{B}$ a known $q \times k$-dimensional matrix). The model with constraints of the type I corresponds to the models of complete measurement with conditions. The model with constraints of the type II, i.e., the model of incomplete measurements with conditions, is written as follows

$$\mathbf{Y} \sim_n \left(\mathbf{X}\beta_1, \operatorname{Var}(\mathbf{Y}) = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i\right),$$

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \in \mathcal{V}_{II} = \left\{ \begin{pmatrix} \mathbf{v} \\ \mathbf{z} \end{pmatrix} : \mathbf{b} + (\mathbf{B}_1, \mathbf{B}_2) \begin{pmatrix} \mathbf{v} \\ \mathbf{z} \end{pmatrix} = \mathbf{0} \right\},$$

model with constraints of the type II

($\mathbf{b} \in \mathcal{M}(\mathbf{B}_1, \mathbf{B}_2)$ is a known vector, $\mathbf{B}_1$ and $\mathbf{B}_2$ known $q \times (k-l)$ and $q \times l$-dimensional matrices, respectively, $\vartheta = (\vartheta_1, \ldots \vartheta_p)'$ a vector of the second order parameters). As far as the covariance matrix of the observation vector is concerned, two forms of the model are distinguished:

(i) ϑ_i, $i = 1, \ldots, p$, are assumed to be only positive and the matrices $\mathbf{V}_i$, $i = 1, \ldots, p$, positive semidefinite. This model is called a model with variance components, the mixed linear model. This model corresponds to the simplest case of a measurement with various devices characterized by various standard deviations.

(ii) Generally $\vartheta \in \underline{\vartheta} \subset \mathbb{R}^p$, where the definition domain $\underline{\vartheta}$ of the second order parameters is an open set in $\mathbb{R}^p$ and $\mathbf{V}_i$, $i = 1, \ldots, p$, are symmetric matrices such that $\sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$ is a positive semidefinite matrix. This model is called model with covariance components.

This difference will play a substantial role in the following theory.

3. Sensitivity in models without constraints

Definition 3.1. Within a mixed linear model without constraints (see Section 2) the expression

$$\frac{\partial \widehat{\mathbf{h}'\beta}(\mathbf{Y}, \vartheta^*)}{\partial \vartheta_i}$$

is called the *sensitiveness* of a linear unbiasedly estimable function $\mathbf{h}'\beta$, $\mathbf{h} \in \mathcal{M}(\mathbf{X}')$, with respect to ϑ_i; here ϑ^* is the actual value of the parameter ϑ.

Definition 3.2. (an a priori nonsensitiveness region for a single function) Let $\varepsilon > 0$ be a small number chosen by the user. The *nonsensitiveness region* of the linear function $\mathbf{h}'\beta$ of the first order parameter is the region of the form

$$\left\{ \delta\vartheta : \frac{\operatorname{Var}\left[\widehat{\mathbf{h}'\beta}(\mathbf{Y}, \vartheta^* + \delta\vartheta) \,|\, \vartheta^*\right]}{\operatorname{Var}\left[\widehat{\mathbf{h}'\beta}(\mathbf{Y}, \vartheta^*) \,|\, \vartheta^*\right]} < 1 + \varepsilon^2 \right\}.$$

In the following let us denote $\mathbf{\Sigma}^* = \sum_{i=1}^{p} \vartheta_i^* \mathbf{V}_i$, where ϑ^* are the actual values of the second order parameters and, in contradiction to it, $\mathbf{\Sigma} = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$.

THEOREM 3.3. (an a priori nonsensitiveness region for a single function). *Let*

$$\mathbf{h} \in \mathcal{M}(\mathbf{X}'),$$
$$\{\mathbf{W}_h\}_{i,j} = \mathbf{L}_h'\mathbf{V}_i\mathbf{\Sigma}^{*-}\operatorname{Var}(\mathbf{v}^*\,|\,\vartheta^*)\mathbf{\Sigma}^{*-}\mathbf{V}_j\mathbf{L}_h\,,\ i,j = 1,\dots,p,$$
$$\mathbf{L}_h' = \begin{cases} \mathbf{h}'\mathbf{C}^{-1}\mathbf{X}'\mathbf{\Sigma}^{*-} & \text{if } r(\mathbf{X}) = k < n\,,\ \mathbf{\Sigma}^* \text{ is p.d.},\\ \mathbf{h}'\mathbf{C}^{-}\mathbf{X}'\mathbf{\Sigma}^{*-} & \text{if } \mathcal{M}(\mathbf{X}') \subset \mathcal{M}(\mathbf{\Sigma}^*),\\ \mathbf{h}'(\mathbf{X}'\mathbf{T}^{*-}\mathbf{X})^{-}\mathbf{X}'\mathbf{T}^{*-} & \text{otherwise}, \end{cases}$$
$$\mathbf{C} = \mathbf{X}'\mathbf{\Sigma}^{*-}\mathbf{X}\,,$$
$$\mathbf{T}^* = \mathbf{\Sigma}^* + \mathbf{X}\mathbf{X}',$$
$$\mathbf{v}^* = \Big\{\mathbf{I} - \mathbf{X}\big[(\mathbf{X}')^-_{m(\Sigma^*)}\big]'\Big\}\mathbf{Y}\,,$$
$$\operatorname{Var}(\mathbf{v}^*\,|\,\vartheta^*) = \mathbf{\Sigma}^* - \mathbf{X}\big[(\mathbf{X}')^-_{m(\Sigma^*)}\big]'\mathbf{\Sigma}^*\,,$$
$$\mathbf{M}_X = \mathbf{I} - \mathbf{X}\mathbf{X}^+\,.$$

Then

$$\vartheta \in \{\mathbf{u}\colon \mathbf{u}'\mathbf{W}_h\mathbf{u} \le \varepsilon_h\sigma_h\} \quad \text{nonsensitiveness region} \ \Rightarrow$$
$$\Rightarrow \ \frac{\sqrt{\operatorname{Var}\Big(\big[\widehat{\mathbf{h}'\beta}(\mathbf{Y}, \vartheta^* + \delta\vartheta\big]\,|\,\vartheta^*\Big)}}{\sqrt{\operatorname{Var}\Big(\big[\widehat{\mathbf{h}'\beta}(\mathbf{Y}, \vartheta^*\big]\,|\,\vartheta^*\Big)}} < \sqrt{1+\varepsilon^2}\,.$$

Here $\sigma_h = \sqrt{\operatorname{Var}\big[\widehat{\mathbf{h}'\beta}(\mathbf{Y},\vartheta^*)\,|\,\vartheta^*\big]} = \sqrt{\mathbf{h}\big[(\mathbf{X}')^-_{m(\Sigma^*)}\big]'\mathbf{\Sigma}^*(\mathbf{X}')^-_{m(\Sigma^*)}\mathbf{h}}$.

An a posteriori nonsensitiveness region $\mathcal{E}$ *for a single function is given by the relation* $\mathcal{E} = \big\{\delta\vartheta\colon \big|\widehat{\mathbf{h}'\beta}(\mathbf{y},\vartheta^*+\delta\vartheta) - \widehat{\mathbf{h}'\beta}(\mathbf{y},\vartheta^*)\big| < \varepsilon\sigma_h\big\}$, *where the number* $\varepsilon > 0$ *is chosen by a statistician.*

THEOREM 3.4. (an a posteriori nonsensitiveness region for a single function). *Let* $h(\beta) = \mathbf{h}'\beta$, $\beta \in \mathbb{R}^k$, *where* $\mathbf{h} \in \mathcal{M}(\mathbf{X}')$ *and*

$$\mathbf{p}_h' = \big(-\mathbf{L}_h'\mathbf{V}_1\mathbf{\Sigma}^{*-}\mathbf{v}^{**},\dots,-\mathbf{L}_h'\mathbf{V}_p\mathbf{\Sigma}^{*-}\mathbf{v}^{**}\big)\,,$$

where $\mathbf{v}^{**}$ *is a realization of the residual vector. Then the nonsenitiveness region for the function* $h(\cdot)$ *is*

$$\mathcal{E} = \big\{\delta\vartheta\colon\ |\mathbf{p}_h'\delta\vartheta| \le \varepsilon\sigma_h\big\}\,.$$

A nonsensitiveness region for the dispersion is defined as follows

$$\mathcal{D} = \Big\{\delta\vartheta\colon \Big|\operatorname{Var}\big[\widehat{\mathbf{h}'\beta}(\mathbf{Y},\vartheta^*+\delta\vartheta)\,\big|\,\vartheta^*+\delta\vartheta\big] - \operatorname{Var}\big[\widehat{\mathbf{h}'\beta}(\mathbf{Y},\vartheta^*)\,\big|\,\vartheta^*\big]\Big| \le \varepsilon^2\sigma_h^2\Big\}\,,$$

where the number $\varepsilon > 0$ *is chosen by a statistician.*

THEOREM 3.5. (*nonsensitiveness region for the dispersion*).

$$\mathcal{D} = \big\{\delta\vartheta\colon\ \big|(\mathbf{L}_h'\mathbf{V}_1\mathbf{L}_h,\dots,\mathbf{L}_h'\mathbf{V}_p\mathbf{L}_h)\delta\vartheta\big| \le \varepsilon^2\sigma_h^2\big\}\,.$$

4. Universal models with constraints of the type I

Definition 4.1. The universal model with constraints of the type I is

$$\mathbf{Y} \sim_n \left(\mathbf{X}\beta, \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i\right), \quad \beta \in \{\mathbf{u}\colon \mathbf{b} + \mathbf{B}\mathbf{u} = \mathbf{0}\} = \mathcal{V}_I\,, \ \vartheta \in \underline{\vartheta}\,,$$

where $\mathbf{B}$ is a given $q \times k$ matrix and $\mathbf{b} \in \mathcal{M}(\mathbf{B})$.

Lemma 4.2. *The* ϑ^*-LBLUE *of a function* $h(\beta) = \mathbf{h}'\beta$, $\beta \in \mathcal{V}_I$, *where* $\mathbf{h} \in \mathcal{M}(\mathbf{X}', \mathbf{B}')$, *is*

$$\widehat{\mathbf{h}'\beta}(\mathbf{Y}, -\mathbf{b}, \vartheta^*) = \mathbf{L}_h' \begin{pmatrix} \mathbf{Y} \\ -\mathbf{b} \end{pmatrix} = \left[(\mathbf{L}_h^{(1)})',\ (\mathbf{L}_h^{(2)})'\right] \begin{pmatrix} \mathbf{Y} \\ -\mathbf{b} \end{pmatrix},$$

where

$$\left[(\mathbf{L}_h^{(1)})',\ (\mathbf{L}_h^{(2)})'\right] =$$

$$= \begin{cases} \left(\mathbf{h}'\mathbf{P}^{C}_{\mathrm{Ker}(B)}\mathbf{C}^{+}\mathbf{X}'\mathbf{\Sigma}^{*-}, \mathbf{h}'\mathbf{C}^{+}\mathbf{B}'(\mathbf{B}\mathbf{C}^{+}\mathbf{B}')^{+}\right) \\ \qquad \textit{if} \ \ \mathcal{M}(\mathbf{X}\mathbf{M}_{B'}) \subset \mathcal{M}(\mathbf{\Sigma}^*) \quad \textit{and} \quad \mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{C}), \\ \left(\mathbf{h}'\mathbf{P}^{C+B'B}_{\mathrm{Ker}(B)}(\mathbf{C}+\mathbf{B}'\mathbf{B})^{+}\mathbf{X}'\mathbf{\Sigma}^{*-}, \mathbf{h}'(\mathbf{C}+\mathbf{B}'\mathbf{B})^{+}\mathbf{B}'\left[\mathbf{B}(\mathbf{C}+\mathbf{B}'\mathbf{B})^{+}\mathbf{B}'\right]^{+}\right) \\ \qquad \textit{if} \ \ \mathcal{M}(\mathbf{X}\mathbf{M}_{B'}) \subset \mathcal{M}(\mathbf{\Sigma}^*), \\ \left(\mathbf{h}'\mathbf{P}^{W}_{\mathrm{Ker}(B)}\mathbf{W}^{+}\mathbf{X}'(\mathbf{\Sigma}^* + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^{+}, \mathbf{h}'\mathbf{W}^{+}\mathbf{B}'(\mathbf{B}\mathbf{W}^{+}\mathbf{B}')^{+}\right) \ \textit{otherwise}. \end{cases}$$

Here $\mathrm{Ker}(\mathbf{B}) = \{\mathbf{u}\colon \mathbf{B}\mathbf{u} = \mathbf{0}\}$, $\mathbf{P}^{C}_{\mathrm{Ker}(B)} = \mathbf{I} - \mathbf{C}^{+}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-}\mathbf{B}')^{-}\mathbf{B}$ *and*

$$\mathbf{W} = \mathbf{X}'(\mathbf{\Sigma}^* + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^{+}\mathbf{X} + \mathbf{B}'\mathbf{B}.$$

Corollary 4.3. *One version of the minimum* $\begin{pmatrix} \mathbf{\Sigma}^*, & \mathbf{0} \\ \mathbf{0}, & \mathbf{0} \end{pmatrix}$ *g-inverse of the matrix* $(\mathbf{X}', \mathbf{B}')$ *is given by the relation*

$$\left((\mathbf{X}', \mathbf{B}')^{-}_{m\begin{pmatrix} \Sigma^*, & 0 \\ 0, & 0 \end{pmatrix}}\right)' =$$

$$= \begin{cases} \left(\mathbf{P}^{C}_{\mathrm{Ker}(B)}\mathbf{C}^{+}\mathbf{X}'\mathbf{\Sigma}^{*-}, \mathbf{C}^{+}\mathbf{B}'(\mathbf{B}\mathbf{C}^{+}\mathbf{B}')^{+}\right) \\ \qquad \textit{if} \ \ \mathcal{M}(\mathbf{X}\mathbf{M}_{B'}) \subset \mathcal{M}(\mathbf{\Sigma}^*) \quad \textit{and} \quad \mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{C}), \\ \left(\mathbf{P}^{C+B'B}_{\mathrm{Ker}(B)}(\mathbf{C}+\mathbf{B}'\mathbf{B})^{+}\mathbf{X}'\mathbf{\Sigma}^{*-}, (\mathbf{C}+\mathbf{B}'\mathbf{B})^{+}\mathbf{B}'\left[\mathbf{B}(\mathbf{C}+\mathbf{B}'\mathbf{B})^{+}\mathbf{B}'\right]^{+}\right) \\ \qquad \textit{if} \ \ \mathcal{M}(\mathbf{X}\mathbf{M}_{B'}) \subset \mathcal{M}(\mathbf{\Sigma}^*), \\ \left(\mathbf{P}^{W}_{\mathrm{Ker}(B)}\mathbf{W}^{+}\mathbf{X}'(\mathbf{\Sigma}^* + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')^{+}, \mathbf{W}^{+}\mathbf{B}'(\mathbf{B}\mathbf{W}^{+}\mathbf{B}')^{+}\right) \ \textit{otherwise}. \end{cases}$$

Lemma 4.4.

$$\mathbf{v}^* = \begin{pmatrix} \mathbf{Y} \\ -\mathbf{b} \end{pmatrix} - \begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix} \left[(\mathbf{X}', \mathbf{B}')^-_{m\begin{pmatrix} \Sigma^*, & 0 \\ 0, & 0 \end{pmatrix}} \right]' \left(\begin{pmatrix} \mathbf{Y} \\ -\mathbf{b} \end{pmatrix}\right) = \begin{pmatrix} \mathbf{v}_1^* \\ \mathbf{0} \end{pmatrix} .$$

Further

$$\mathrm{Var}(\mathbf{v}_1^*) = \begin{cases} \mathbf{\Sigma}^* - \mathbf{X}(\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^+\mathbf{X}' = \mathbf{\Sigma}^* - \mathbf{X}\big[\mathbf{C}^+ - \mathbf{C}^+\mathbf{B}'(\mathbf{B}\mathbf{C}^-\mathbf{B}')^-\mathbf{B}\mathbf{C}^+\big]\mathbf{X}' \\ \qquad \textit{if} \quad \mathcal{M}(\mathbf{X}\mathbf{M}_{B'}) \subset \mathcal{M}(\mathbf{\Sigma}^*) \quad \textit{and} \quad \mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{C}), \\ \mathbf{\Sigma}^* - \mathbf{X}(\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^+\mathbf{X}' = \\ \mathbf{\Sigma}^* - \mathbf{X}\big\{(\mathbf{C}+\mathbf{B}'\mathbf{B})^+ - (\mathbf{C}+\mathbf{B}'\mathbf{B})^+\mathbf{B}'\big[\mathbf{B}(\mathbf{C}+\mathbf{B}'\mathbf{B})^- \, B'\big]^- \times \\ \times \mathbf{B}(\mathbf{C}+\mathbf{B}'\mathbf{B})^+\big\}\mathbf{X}', \qquad \textit{if} \quad \mathcal{M}(\mathbf{X}\mathbf{M}_{B'}) \subset \mathcal{M}(\mathbf{\Sigma}^*), \\ \mathbf{\Sigma}^* - \mathbf{X}\big[\mathbf{W}^+ - \mathbf{W}^+\mathbf{B}'(\mathbf{B}\mathbf{W}^+\mathbf{B})^+\mathbf{B}'\mathbf{W}^+\big]\mathbf{X}' - \mathbf{X}\mathbf{M}_{B'}\mathbf{X}' \quad \textit{otherwise}. \end{cases}$$

Lemma 4.5. *Let* $\mathbf{h} \in \mathcal{M}(\mathbf{X}', \mathbf{B}')$*; then*

$$\mathrm{Var}\big[\widehat{\mathbf{h}'\beta}(\mathbf{Y}, \vartheta^*)\,|\,\vartheta^*\big] = \begin{cases} \mathbf{h}'(\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^+\mathbf{h} & \text{if} \quad \mathcal{M}(\mathbf{X}\mathbf{M}_{B'}) \subset \mathcal{M}(\mathbf{\Sigma}^*) \\ \qquad \text{and} \quad \mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{C}), \\ \mathbf{h}'\big[\mathbf{M}_{B'}(\mathbf{C}+\mathbf{B}'\mathbf{B})\mathbf{M}_{B'}\big]^+\mathbf{h} & \text{if} \quad \mathcal{M}(\mathbf{X}\mathbf{M}_{B'}) \subset \mathcal{M}(\mathbf{\Sigma}^*), \\ \mathbf{h}'(\mathbf{M}_{B'}\mathbf{W}\mathbf{B}_{B'})^+\mathbf{h} - \mathbf{h}'\mathbf{M}_{B'}\mathbf{h}, & \text{otherwise}. \end{cases}$$

Lemma 4.6. *Let* $\mathbf{h} \in \mathcal{M}(\mathbf{X}', \mathbf{B}')$*; then*

$$\partial\mathbf{h}' \left[(X', \mathbf{B}')^-_{m\begin{pmatrix} \Sigma, & 0 \\ 0, & 0 \end{pmatrix}} \right]' \begin{pmatrix} \mathbf{Y} \\ -\mathbf{b} \end{pmatrix} / \partial\vartheta_i \Bigg|_{\vartheta=\vartheta^*} = \big(\mathbf{L}_h^{(1)}\big)'\mathbf{V}_i\mathbf{\Sigma}^{*-}\mathbf{v}_1^* .$$

All quantities necessary for constructing the nonsensitiveness regions, i.e., $\mathbf{L}_h'$, $\mathrm{Var}(\mathbf{v}^*|\vartheta^*)$ and $\mathrm{Var}\big[\mathbf{h}'\beta(\mathbf{Y}, \vartheta^*)|\vartheta^*\big]$ are now known and thus the region of nonsensitiveness for the model with constraints of the type I can be determined using Theorem 3.3.

5. Universal models with constraints of the type II

Definition 5.1. The universal model with constraints of the type II is

$$\mathbf{Y} \sim_n \left(\mathbf{X}\beta_1, \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i \right) ,$$

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \in \left\{ \begin{pmatrix} \mathbf{u} \\ \mathbf{z} \end{pmatrix} : \mathbf{u} \in \mathbb{R}^k, \ \mathbf{z} \in \mathbb{R}^l, \ \mathbf{b} + \mathbf{B}_1\mathbf{u} + \mathbf{B}_2\mathbf{z} = \mathbf{0} \right\} = \mathcal{V}_{II}, \ \vartheta \in \underline{\vartheta}.$$

Theorem 5.2. *One version of the minimum* $\begin{pmatrix} \boldsymbol{\Sigma}^*, & \mathbf{0} \\ \mathbf{0}, & \mathbf{0} \end{pmatrix}$ *g-inverse of the matrix* $\begin{pmatrix} \mathbf{X}', & \mathbf{B}_1' \\ \mathbf{0}, & \mathbf{B}_2' \end{pmatrix}$) *is given by the relation*

$$\left[\begin{pmatrix} \mathbf{X}', & \mathbf{B}_1' \\ \mathbf{0}, & \mathbf{B}_2' \end{pmatrix}^-_{m\begin{pmatrix} \Sigma^*, & 0 \\ 0, & 0 \end{pmatrix}}\right]' = \begin{pmatrix} \boxed{1}, & \boxed{2} \\ \boxed{3}, & \boxed{4} \end{pmatrix},$$

where

$$\begin{aligned}
\boxed{1} &= \mathbf{M}_{B_1' M_{B_2}} \left[(\mathbf{M}_{B_1' M_{B_2}} \mathbf{X}')^-_{m(\Sigma^*)}\right]', \\
\boxed{2} &= \mathbf{W}^+ \mathbf{B}_1' (\mathbf{M}_{B_2} \mathbf{B}_1 \mathbf{W}^+ \mathbf{B}_1' \mathbf{M}_{B_2})^+, \\
\boxed{3} &= -\left[(\mathbf{B}_2')^-_{m(B_1 W^+ B_1')}\right]' \mathbf{B}_1 \mathbf{W}^+ \mathbf{X}' (\boldsymbol{\Sigma}^* + \mathbf{X}\mathbf{M}_{B_1' M_{B_2}} \mathbf{X}')^+, \\
\boxed{4} &= \left[(\mathbf{B}_2')^-_{m(B_1 W^+ B_1')}\right]'.
\end{aligned}$$

For any unbiasedly estimable function $h(\beta_1, \beta_2) = \mathbf{h}_1'\beta_1 + \mathbf{h}_2'\beta_2$, $\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \in \mathcal{V}_{II}$, the nonsensitiveness region can be determined by the help of the quantities $\{\mathbf{W}_h\}_{i,j}$, $i,j = 1,\dots,p$, $\mathbf{L}_h', \mathbf{v}^*, \mathrm{Var}(\mathbf{v}^*)$ and $\mathrm{Var}\left[\widehat{\mathbf{h}'\beta}(\mathbf{Y},\vartheta^*)\,|\,\vartheta^*\right]$ which are expressed in terms of the model with constraints of the type II. These expressions are

$$\begin{aligned}
\mathbf{L}_h' &= \left((\mathbf{L}_h^{(1)})', (\mathbf{L}_h^{(2)})'\right), \\
(\mathbf{L}_h^{(1)})' &= \mathbf{h}_1'\boxed{1} + \mathbf{h}_2'\boxed{3}, \\
(\mathbf{L}_h^{(2)})' &= \mathbf{h}_1'\boxed{2} + \mathbf{h}_2'\boxed{4}, \\
\mathbf{v}^* &= \begin{pmatrix} \mathbf{Y} \\ -\mathbf{b} \end{pmatrix} - \begin{pmatrix} \mathbf{X}, & \mathbf{0} \\ \mathbf{B}_1, & \mathbf{B}_2 \end{pmatrix} \begin{pmatrix} \boxed{1}, & \boxed{2} \\ \boxed{3}, & \boxed{4} \end{pmatrix}, \begin{pmatrix} \mathbf{Y} \\ -\mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1^* \\ \mathbf{0} \end{pmatrix}, \\
\{\mathbf{W}_h\}_{i,j} &= (\mathbf{L}_h^{(1)})' \mathbf{V}_i \boldsymbol{\Sigma}^{*-} \mathrm{Var}({\mathbf{v}^*}_1) \boldsymbol{\Sigma}^{*-} \mathbf{V}_j \mathbf{L}_h^{(1)}, \\
{\mathbf{v}^*}_1 &= \mathbf{Y} - \mathbf{X}\boxed{1}\mathbf{Y} - \mathbf{X}\boxed{2}(-\mathbf{b}), \\
\mathrm{Var}({\mathbf{v}^*}_1) &= \left(\mathbf{I} - \mathbf{X}\boxed{1}\right)\boldsymbol{\Sigma}^*\left(\mathbf{I} - \mathbf{X}\boxed{1}\right)', \\
\mathrm{Var}\left[\widehat{\mathbf{h}'\beta}(\mathbf{Y},\vartheta^*)\,|\,\vartheta^*\right] &= (\mathbf{h}_1', \mathbf{h}_2') \begin{pmatrix} \boxed{1}, & \boxed{2} \\ \boxed{3}, & \boxed{4} \end{pmatrix} \begin{pmatrix} \boldsymbol{\Sigma}^*, & \mathbf{0} \\ \mathbf{0}, & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boxed{1}', & \boxed{3}' \\ \boxed{2}', & \boxed{4}' \end{pmatrix} \begin{pmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \end{pmatrix}.
\end{aligned}$$

It is obvious how to proceed in the case of the other mentioned kinds of nonsensitiveness regions.

All given assertions are proved in detail in [5].

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Department of Mathematical Analysis
and Applied Mathematics
Faculty of Science
Palacký University
Tomkova 40
CZ–779 00 Olomouc
CZECH REPUBLIC

E-mail: kubacekl@risc.upol.cz