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COMBINED TWO-SAMPLE TESTS FOR RANDOMLY CENSORED DATA

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ABSTRACT. Assume that we are given a randomly censored sample and used the uncensored observations to define an empirical process and the censored observations to define another empirical process. The main result of this article is that the latter two empirical processes can be jointly approximated by two independent Brownian bridges. This result is used in the two-sample problem with randomly censored data to obtain the asymptotic distributions of a number of combined two-sample tests.

1. Introduction

Albers and Akritas (1987) introduced and studied combined rank tests for the two-sample problem when the data is randomly censored form the right. Their main idea is the treatment of the uncensored observations and the censored observations as two separate groups. For each group they proposed a suitable two-sample rank statistic and the resulting rank statistics are combined into a single test statistic for the whole data. Albers and Akritas (1987) were motivated by Gill's (1980) comment which expresses the need for developing simple rank tests in the random censorship model analogous to ordinary rank tests in the uncensored case. The formula of the variance of the asymptotic normal distribution of the combined test statistic of Theorem 1 (Equation 3.3) of Albers and Akritas (1987) suggests that the rank statistic based on the uncensored observations is asymptotically independent of the rank statistic based on the censored observations. The main purpose of this article is to pinpoint the source of this asymptotic independence. This will allow us to extend the idea of combining to other two-sample tests.

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We will adopt here the general scheme of random censorship of Burke *et al.* (1981) as defined next. Let X be a real-valued random variable with a continuous distribution function (DF) $F(t) = \Pr\{X < t\}$, $t \in \mathbb{R}$. For a fixed integer l let $A_1^1,\ A_1^2, \dots, A_1^l$ be pairwise disjoint events. Let $\{X_j,\ A_{1j}^1, A_{1j}^2, \dots, A_{1j}^l \colon j = 1, 2, \dots, m\}$ be independent replicas of $\{X, A_1^1,\ A_1^2, \dots, A_1^l\}$ defined on some probability space $(\Omega, \mathcal{A}, P)$. For convenience, we will assume that $A_1^1, A_1^2, \dots, A_1^l$ form a partition of Ω and $\tau_i = \Pr\{A_1^i\} > 0$, $i = 1, 2, \dots, l$.

For $i = 1, 2, \dots, l$, define

(i) the sub-DF
$$F^i(t) = \Pr\left\{X < t\,,\ A_1^i\right\}, \tag{1.1}$$

(ii) the conditional DF
$$F_i(t) = \Pr\left\{X < t \,|\, A_1^i\right\}. \tag{1.2}$$

Note that $F_i(t) = F^i(t)/F^i(\infty)$.

(iii) The empirical sub-DF
$$F_m^i(t) = \frac{1}{m} \sum_{j=1}^{m} I(X_j < t \text{ and } A_{1j}^i \text{ occurs})\,, \tag{1.3}$$

and the corresponding empirical process
$$\alpha_m^i(t) = m^{\frac{1}{2}} \left(F_m^i(t) - F^i(t)\right), \qquad t \in \mathbb{R}\,. \tag{1.4}$$

(iv) The empirical (conditional) DF
$$F_{im}(t) = \frac{1}{M_i} \sum_{j=1}^{M_i} I(X_j^i < t) = \frac{F_m^i(t)}{F_m^i(\infty)}\,, \tag{1.5}$$

the corresponding empirical process
$$\alpha_{im}(t) = M_i^{\frac{1}{2}} \left(F_{im}(t) - F_i(t)\right), \qquad t \in \mathbb{R}\,, \tag{1.6}$$

the corresponding uniform empirical process
$$\xi_{im}(t) = M_i^{\frac{1}{2}} \left(E_{im}(t) - t\right), \qquad 0 \le t \le 1\,, \tag{1.7}$$

and the corresponding uniform quantile process
$$u_{im}(t) = M_i^{\frac{1}{2}} \left(t - E_{im}^{-1}(t)\right), \qquad 0 \le t \le 1\,, \tag{1.8}$$

where $M_i = \sum_{j=1}^{m} I(A_{1j}^i \text{ occurs}) = mF_m^i(\infty)$, $X_1^i, X_2^i, \dots, X_{M_i}^i$ are those X observations where A_1^i occurs, $U_j^i = F_i(X_j^i)$, $j = 1, 2, \dots, M_i$,
$$E_{im}(t) = \frac{1}{M_i} \sum_{j=1}^{M_i} I(U_j^i < t)$$

and $E_{im}^{-1}(\cdot)$ is the generalized inverse of $E_{im}(\cdot)$.

2. The main results

Theorem 1.

(a) *On an appropriate probability space there exist l independent sequences $\{B_{im}(\cdot)\}$, $i = 1, 2, \dots, l$ of Brownian bridges such that for $i = 1, 2, \dots, l$ we have*

$$P\left\{ \sup_{0 \le t \le 1} \left| \xi_{im}(t) - B_{im}(t) \right| > C_1 m^{-\frac{1}{2}} \log m \right\} \le D_1 m^{-\eta_1}, \qquad (2.1)$$

where $\eta_1 > 1$ and $C_1 = C_1(\eta_1)$ and D_1 are positive constants, independent of m.

(b) *On an appropriate probability space there exist l independent Kiefer processes $K_i(\cdot, \cdot)$, $i = 1, 2, \dots, l$ such that for $i = 1, 2, \dots, l$ we have*

$$P\left\{ \sup_{0 \le t \le 1} \left| \xi_{im}(t) - m^{-\frac{1}{2}} K_i(t, m) \right| > C_2 m^{-\frac{1}{2}} (\log m)^2 \right\} \le D_2 m^{-\eta_2}, \qquad (2.2)$$

where $\eta_2 > 1$ and $C_2 = C_2(\eta_2)$ and D_2 are positive constants, independent of m.

Theorem 2. *On the same probability space of* (b) *of Theorem 1 we have*

$$\sup_{0 \le t \le 1} \left| u_{im}(t) - m^{-\frac{1}{2}} K_i(t, m) \right| \overset{\text{a.s.}}{=} \mathcal{O}\left(m^{-\frac{1}{4}} (\log \log m)^{\frac{1}{4}} (\log m)^{\frac{1}{2}} \right), \qquad (2.3)$$

for $i = 1, 2, \dots, l$, where $K_i(\cdot, \cdot)$ are as in (2.2).

P r o o f o f T h e o r e m 1. We will prove (2.1) on the probability space of Theorem 3.1 of B u r k e *et al.* (1981). The proof of (2.2) is similar to that of (2.1). Let $\alpha_m^i(\cdot)$ be as in (1.4). Observe that

$$\xi_{im}(t) = \sqrt{\frac{m}{M_i}} \left\{ \alpha_m^i\left(F_i^{-1}(t)\right) - t \alpha_m^i(\infty) \right\}. \qquad (2.4)$$

Let $\{B_m^i(\cdot)\}$ be the Gaussian processes of Theorem 3.1 of B u r k e *et al.* (1981). For $i = 1, 2, \dots, l$, define

$$B_{im}(t) = \tau_i^{-\frac{1}{2}} \left\{ B_m^i\left(F_i^{-1}(t)\right) - t B_m^i(\infty) \right\}. \qquad (2.5)$$

By the first 4 equations of page 92 of B u r k e *et al.* (1981) we have for each m

$$EB_{im}(t) = 0,$$
$$EB_{im}(t) B_{im}(s) = t \wedge s - ts,$$

and

$$EB_{im}(t) B_{jm}(s) = 0.$$

Consequently, $\{B_{im}(t)\}$, $i = 1, 2, \ldots, l$, are independent sequences of Brownian bridges.

By Lemma 1 of Mirzkhmedov and Tursunov (1992) we have

$$\Pr\left\{\left|\frac{M_i}{m} - \tau_i\right| > \varepsilon_m\right\} \le \gamma_m\,, \tag{2.6}$$

where $\gamma_m = 2m^{-6}$ and

$$\varepsilon_m = 4\left\{\tau_i(1-\tau_i)m^{-1}\log m\right\}^{\frac{1}{2}}.$$

By (3.4) of Burke *et al.* (1981) we have

$$\Pr\left\{\max_{1\le i\le l}\sup_{x\in\mathbb{R}}\left|\alpha_m^i(x) - B_m^i(x)\right| > Cm^{-\frac{1}{2}}\log m\right\} \le Dm^{-\eta}, \tag{2.7}$$

where $\eta > 2$ and $C = C(\eta)$ and D are absolute positive constants.

By (2.4) and (2.5) we have

$$\begin{aligned} P_m(\lambda) = {} & \Pr\left\{\sup_{0\le t\le 1}\left|\xi_{im}(t) - B_{im}(t)\right| > 4\lambda\right\} \\ & \le 2\Pr\left\{\sqrt{\frac{m}{M_i}}\sup_{0\le t\le 1}\left|\alpha_m^i\big(F_i^{-1}(t)\big) - B_m^i\big(F_i^{-1}(t)\big)\right| > \lambda\right\} \\ & \quad + 2\Pr\left\{\left|\sqrt{\frac{m}{M_i}} - \frac{1}{\sqrt{\tau_i}}\right|\sup_{x\in\mathbb{R}}\left|B_m^i(x)\right| > \lambda\right\} \\ & = 2P_{m1}(\lambda) + 2P_{m2}(\lambda)\,. \end{aligned} \tag{2.8}$$

As to P_{m1}, we use (2.6) to obtain

$$\begin{aligned} P_{m1}(\lambda) & \le \gamma_m + \Pr\left\{\sqrt{\frac{m}{M_i}}\sup_{x\in\mathbb{R}}\left|\alpha_m^i(x) - B_m^i(x)\right| > \lambda\,,\ \left|\frac{M_i}{m} - \tau_i\right| \le \varepsilon_m\right\} \\ & \le \gamma_m + \Pr\left\{\sup_{x\in\mathbb{R}}\left|\alpha_m^i(x) - B_m^i(x)\right| > \lambda\sqrt{\tau_i - \varepsilon_m}\right\}\,. \end{aligned} \tag{2.9}$$

Let $B(\cdot)$ be a Brownian bridge. It is well known that

$$\Pr\left\{\sup_{0\le t\le 1}\left|B(t)\right| > r\right\} \le C^*\exp\{-2r^2\}\,, \tag{2.10}$$

where C^* is a constant. By (2.6) and (2.10) we have

$$
\begin{aligned}
P_{m2}(\lambda) &\le \Pr\left\{\left|\sqrt{\frac{m}{M_i}}-\frac{1}{\sqrt{\tau_i}}\right|\sup_{0\le t\le 1}|B(t)|>\lambda\right\} \\
&\le \Pr\left\{\left|\sqrt{\frac{m}{M_i}}-\frac{1}{\sqrt{\tau_i}}\right|\sup_{0\le t\le 1}|B(t)|>\lambda\,,\left|\frac{M_i}{m}-\tau_i\right|\le\varepsilon_m\right\}+\gamma_m \\
&\le \Pr\left\{\frac{\left|\frac{M_i}{m}-\tau_i\right|}{\sqrt{\frac{\tau_i M_i}{m}}+\frac{M_i}{m}}\sup_{0\le t\le 1}|B(t)|>\lambda\sqrt{\tau_i}\,,\left|\frac{M_i}{m}-\tau_i\right|\le\varepsilon_m\right\}+\gamma_m \\
&\le \Pr\left\{\sup_{0\le t\le 1}|B(t)|>\frac{\lambda\sqrt{\tau_i}\left(\sqrt{\tau_i\left(\tau_i-\varepsilon_m\right)}+\left(\tau_i-\varepsilon_m\right)\right)}{\varepsilon_m}\right\}+\gamma_m \qquad (2.11)\\
&\le C^*\exp\left\{-\frac{2\lambda^2\tau_i\left(\sqrt{\tau_i\left(\tau_i-\varepsilon_m\right)}+\left(\tau_i-\varepsilon_m\right)\right)^2}{\varepsilon_m^2}\right\}+\gamma_m\,.
\end{aligned}
$$

Let C and D be as in (2.7). By taking in (2.9) and (2.11) $\lambda=\left(Cm^{-\frac{1}{2}}\log m\right)/\sqrt{\tau_i-\varepsilon_m}$ and using (2.7), we get

$$P_{m1}\le Dm^{-\eta}+\gamma_m\,, \tag{2.12}$$

and

$$P_{m2}\le C^*\exp\left\{-\frac{C^2\tau_i\log m}{8}\right\}\,. \tag{2.13}$$

By (2.8), (2.12) and (2.13) we get (2.1). □

Proof of Theorem 2.

First, we show that

$$\sup_{0<t\le 1}\left|t-E_{im}^{-1}(t)\right|\overset{\text{a.s.}}{=}\mathcal{O}\left(m^{-\frac{1}{2}}(\log\log m)^{\frac{1}{2}}\right)\,. \tag{2.14}$$

By a Lemma of Horváth (1984)

$$\sup_{0<t\le 1}\left|t-E_{im}^{-1}(t)\right|\le\sup_{0<t\le 1}\left|E_{im}(t)-t\right|.$$

Hence, it suffices to show that

$$\sup_{0<t\le 1}\left|E_{im}(t)-t\right|\overset{\text{a.s.}}{=}\mathcal{O}\left(m^{-\frac{1}{2}}(\log\log m)^{\frac{1}{2}}\right)\,. \tag{2.15}$$

Let $Q^i(\cdot)$ be the generalized inverse of $F^i(\cdot)$. It is easy to show that

$$E_{im}(t)-t=\frac{m}{M_i}\left\{F_m^i\left(Q^i(\tau_i t)\right)-F^i\left(Q^i(\tau_i t)\right)\right\}+\frac{mt}{M_i}\left\{\tau_i-\frac{M_i}{m}\right\}\,,$$

from which we obtain

$$\begin{aligned}\sup_{0<t\le 1}\left|E_{im}(t)-t\right| &\le \frac{m}{M_i}\sup_{x\in\mathbb{R}}\left|F^i_m(x)-F^i(x)\right| + \frac{m}{M_i}\left|\frac{M_i}{m}-\tau_i\right| \\ &\le \left\{\frac{1}{\tau_i}+\left|\frac{m}{M_i}-\frac{1}{\tau_i}\right|\right\}\cdot a_{im}\,,\end{aligned}\tag{2.16}$$

where

$$a_{im}=\sup_{x\in\mathbb{R}}\left|F^i_m(x)-F^i(x)\right| + \left|\frac{M_i}{m}-\tau_i\right|\,.\tag{2.17}$$

By Theorem 3.1 of Burke *et al.* (1981) and the discussion following their Corollary 9.1 we have

$$\sup_{x\in\mathbb{R}}\left|F^i_m(x)-F^i(x)\right| \overset{\text{a.s.}}{=} \mathcal{O}\Big(m^{-\frac12}(\log\log m)^{\frac12}\Big)\,.\tag{2.18}$$

It is well known that

$$\left|\frac{M_i}{m}-\tau_i\right| \overset{\text{a.s.}}{=} \mathcal{O}\Big(m^{-\frac12}(\log\log m)^{\frac12}\Big)\,.\tag{2.19}$$

By (2.18), (2.19) and (2.17) we get

$$a_{im} \overset{\text{a.s.}}{=} \mathcal{O}\Big(m^{-\frac12}(\log\log m)^{\frac12}\Big)\,.\tag{2.20}$$

By (2.16), (2.20) and the SLLN we obtain (2.15).

By (2.14) and the proof of (3.1.8) of Csörgő (1983) (see also, Csörgő and Révész (1975, 1978 and 1981 (Theorem 4.5.3))) we obtain (2.3). □

Remark 1. From (2.1), (2.2) and (2.3) we see that each of the uniform (conditional) empirical (resp. quantile) processes $\xi_{im}(\cdot)$ (resp. $u_{im}(\cdot)$), $i=1,2,\dots,l$, enjoys the "same" asymptotic theory as that of the ordinary uniform empirical (resp. quantile) processes. In addition, the processes $\xi_{im}(\cdot)$ (resp. $u_{im}(\cdot)$), $i=1,2,\dots,l$, are asymptotically independent.

In the next section we will discuss some of the immediate applications of Theorems 1 and 2. The asymptotic theory of some of these applications requires some intermediate results which are well known in the case of ordinary empirical and quantile processes. We will not attempt to prove here versions of these results for the processes $\xi_{im}(\cdot)$ and $u_{im}(\cdot)$, $i=1,2,\dots,l$. Instead, we will assume that all the required intermediate results are available at our disposal. This way we will be able to focus our attention on the possible applications and avoid getting distracted by technical proofs which will be presented elsewhere.

3. Combined two-sample tests

Let $\{Y_j, A_{2j}^1, A_{2j}^2, \ldots, A_{2j}^l;\ j = 1, 2, \ldots, n\}$ be independent replicas of $\{Y, A_2^1, A_2^2, \ldots, A_2^l\}$ which are independent of the sample $\{X_j, A_{1j}^1, A_{1j}^2, \ldots, A_{1j}^l;\ j = 1, 2, \ldots, m\}$ of Section 1. Assume that $A_2^1, A_2^2, \ldots, A_2^l$ form a partition of Ω, $G(t) = P\{Y < t\}$, $t \in \mathbb{R}$ and $P\{A_2^i\} = P\{A_1^i\} = \tau_i$, $i = 1, 2, \ldots, l$.

For $i = 1, 2, \ldots, l$, define

$$G_i(t) = P\left\{Y < t \,|\, A_2^i\right\}, \tag{3.1}$$

$$G_{in}(t) = \frac{1}{N_i} \sum_{j=1}^{N_i} I(Y_j^i < t), \tag{3.2}$$

and

$$\beta_{in}(t) = N_i^{\frac{1}{2}} \left(G_{in}(t) - G_i(t)\right), \qquad t \in \mathbb{R}, \tag{3.3}$$

where $N_i = \sum_{j=1}^{n} I(A_{2j}^i$ occurs) and Y_j^i, $j = 1, 2, \ldots, N_i$, are those Y observations for which A_2^i occurs.

In this section we consider the problem of testing

$$H_\circ : F(t) = G(t) \qquad \text{for all} \quad t \in \mathbb{R}. \tag{3.4}$$

We follow Albers and Akritas (1987) and group the data into the l separate groups $\{X_1^i,\ X_2^i, \ldots, X_{M_i}^i,\ Y_1^i, Y_2^i, \ldots, Y_{N_i}^i\}$, $i = 1, 2, \ldots, l$. For each of these groups we select an appropriate two-sample test statistic. The resulting l statistics are then combined into an overall test statistic for the whole data. In the rest of this section we will present some specific tests. We will also assume that $B_1(\cdot), \ldots, B_l(\cdot)$ are independent Brownian bridges.

3.1 Kolmogorov-Smirnov tests.

Define

$$D_{n,m}^i = \sqrt{\frac{N_i M_i}{N_i + M_i}} \sup_{-\infty < x < \infty} \left|F_{im}(x) - G_{in}(x)\right|,$$

and

$$D_{n,m} = \max_{1 \le i \le l} D_{n,m}^i .$$

Under $H_\circ$ of (3.4) we have

$$D_{n,m} \xrightarrow{\mathcal{D}} \max_{1 \le i \le l} \sup_{0 \le t \le 1} |B_i(t)| := D_l .$$

Critical values of D_l for $l \ge 1$ can be obtained from the result

$$P\{D_l > x\} = 1 - \left\{P\left\{\sup_{0 \le t \le 1} |B(t)| \le x\right\}\right\}^l$$

and Table 1 of Shorack and Wellner (1986 page 143).

3.2 Cramér-von-Mises tests.

Define

$$W^2_{n,m,i} = \frac{N_i M_i}{N_i + M_i} \int \{F_{im}(x) - G_{in}(x)\}^2 dH_{n,m,i}(x),$$

and

$$W^2_{n,m} = \sum_{i=1}^{l} W^2_{n,m,i},$$

where

$$H_{n,m,i}(x) = \frac{N_i G_{i,n}(x) + M_i F_{im}(x)}{N_i + M_i}.$$

Under $H_\circ$ of (3.4) we have

$$W^2_{n,m} \xrightarrow{\mathcal{D}} \sum_{i=1}^{l} \int B_i^2(t)\, dt := W_l^2 .$$

Tables of the distribution function of W_l^2 for $l \geq 1$ are given in Kiefer (1959).

3.3 Anderson-Darling tests.

Define

$$A^2_{n,m,i} = \frac{N_i M_i}{N_i + M_i} \int \frac{\{F_{im}(x) - G_{in}(x)\}^2}{H_{n,m,i}(x)\big(1 - H_{n,m,i}(x)\big)}\, dH_{n,m,i}(x),$$

and

$$A^2_{n,m} = \sum_{i=1}^{l} A^2_{n,m,i}.$$

Under $H_\circ$ of (3.4) we have

$$A^2_{n,m} \xrightarrow{\mathcal{D}} \sum_{i=1}^{l} \int \frac{B_i^2(t)}{t(1-t)}\, dt := A_l^2 .$$

Tables for A_l^2 are given in Scholz and Stephens (1987).

3.4 Rank tests.

Let r_j^i be the rank of X_j^i among the combined sample $\{X_1^i, X_2^i, \dots, X_{M_i}^i, Y_1^i, Y_2^i, \dots, Y_{N_i}^i\}$ of the ith group. Let $J_i(\cdot)$ be an appropriate score function satisfying some regularity conditions (like, for example, the conditions in

A l y *et al.* (1987)). Without loss of generality, we will assume that $\int J_i(t)\,dt = 0$ for $i = 1, 2, \dots, l$. Define

$$T_{n,m,i} = \frac{1}{M_i} \sum_{j=1}^{M_i} J_i\left(\frac{r_j^i}{N_i + M_i + 1}\right),$$

and

$$T_{n,m} = \sum_{i=1}^{l} \sqrt{\frac{(N_i + M_i)M_i}{lN_i\sigma_i^2}}\; T_{n,m,i}\,,$$

where

$$\sigma_i^2 = 2 \int_0^1 \int_0^y x(1-y)\, dJ_i(x)\, dJ_i(y)\,.$$

Under $H_\circ$ of (3.4) we have

$$T_{n,m} \xrightarrow{\mathcal{D}} N(0,1)\,.$$

Note that the case $l = 2$ follows from Theorem 1 of A l b e r s and A k r i t a s (1987).

Remark 2. We discuss here the case of randomly censored data from the right. Let $\{(X_j^\circ, W_{1j}),\ j = 1, 2, \dots, m\}$ and $\{(Y_j^\circ, W_{2j}),\ j = 1, 2, \dots, n\}$ be two independent bivariate random samples such that the W_{1j}'s are independent of the X_j°'s, the W_{2j}'s are independent of the Y_j°'s, $\Pr\{W_{ij} < x\} = \Psi(x)$, $\Pr\{X_j^\circ < x\} = F^\circ(x)$ and $\Pr\{Y_j^\circ < x\} = G^\circ(x)$. In the random censorship model on the right we only observe $\{(X_j, \delta_{1j}),\ j = 1, 2, \dots, m\}$ and $\{(Y_j, \delta_{2j}),\ j = 1, 2, \dots, n\}$, where $X_j = X_j^\circ \wedge W_{1j}$, $Y_j = Y_j^\circ \wedge W_{2j}$, $\delta_{1j} = I(X_j = X_j^\circ)$, $\delta_{2j} = I(Y_j = Y_j^\circ)$ and $a \wedge b = \min(a, b)$. Note that this setup corresponds to the case when $l = 2$,

$$\begin{aligned} A_j^1 &= \{\omega\colon X_j(\omega) = X_j^\circ(\omega)\}\,,\\ A_j^2 &= \{\omega\colon Y_j(\omega) = Y_j^\circ(\omega)\}\,,\\ 1 - F(x) &= \bigl(1 - F^\circ(x)\bigr)\bigl(1 - \Psi(x)\bigr)\,, \end{aligned}$$

and

$$1 - G(x) = \bigl(1 - G^\circ(x)\bigr)\bigl(1 - \Psi(x)\bigr)\,.$$

The results of this section specialized to the case $l = 2$ will continue to hold in the present setup.

REFERENCES

[1] ALBERS, W.—AKRITAS, M.: *Combined rank tests for the two-sample problem with randomly censored data*, J. Amer. Statist. Assoc. **82** (1987), 648–655.

[2] ALY, E.-E.—CSÖRGŐ, M.—HORVÁTH, L.: *P-P plots, rank processes and Chernoff-Savage Theorems*, in: New Perspectives in Theoretical and Applied Statistics (Puri, M. L. *et al.*, eds.), John Wiley and Sons, New York, 1987.

[3] BURKE, M. D.—CSÖRGŐ, S.—HORVÁTH, L.: *Strong approximations of some biometric estimates under random censorship*, Z. Wahrsch. verw. Gebiete **56** (1981), 87–112.

[4] CSÖRGŐ, M.: *Quantile Processes with Statistical Applications*, Regional Conference Series in Applied Mathematics, Vol. 42, SIAM, Philadelphia, 1983.

[5] CSÖRGŐ, M.—RÉVÉSZ, P.: *Some notes on the empirical distribution function and the quantile process*, in: Colloq. Math. Soc. J. Bolyai, Limit Theorems of Probability Theory (P. Révész, ed.), North-Holland, Amsterdam, Vol. 11, 1975, pp. 59–71.

[6] CSÖRGŐ, M.—RÉVÉSZ, P.: *Strong approximations of the quantile process*, Ann. Statist. **6** (1978), 882–894.

[7] CSÖRGŐ, M.—RÉVÉSZ, P.: *Strong Approximations in Probability and Statistics*, Academic Press, New York, 1981.

[8] GILL, R.: *Censoring and stochastic integrals*, Mathematical Centre, Tracts, Vol. 124, Mathematical Centrum, Amsterdam, 1980.

[9] HORVÁTH, L.: *Strong approximations of extended renewal processes*, Ann. Probab. **12** (1984), 1149–1166.

[10] KIEFER, J.: *K-sample analogues of the Kolmogorov-Smirnov and Cramer-von Mises tests*, Ann. Math. Statist. **30** (1959), 420–447.

[11] MIRZKHMEDOV, H. A.—TURSUNOV, T. G.: *On the rate of convergence of an empirical process constructed from a sample of random size*, Theory Probab. Math. Statist. **44** (1992), 89–93.

[12] SCHOLZ, F. M.—STEPHENS, M. A.: *K-sample Anderson-Darling tests*, J. Amer. Statist. Assoc. **82** (1987), 918–924.

[13] SHORACK, G. R.—WELLNER, J. A.: *Empirical Processes with Applications to Statistics*, J. Wiley, New York, 1986.

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