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SEQUENTIAL SELECTION OF OBSERVATIONS IN RANDOMLY GENERATED EXPERIMENTS

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ABSTRACT. We consider a situation where experimental conditions are observed sequentially but are uncontrolled. The length of the observed sequence is N, and only $n < N$ experiments can be performed. The problem is to select on-line the most informative ones. More precisely, we assume that parameters $\boldsymbol{\theta} \in \mathbb{R}^p$ have to be estimated in a regression model with observations $y_k = \eta(\boldsymbol{\theta}, \xi_k) + \varepsilon_k$, with $\{\varepsilon_k\}$ a sequence of i.i.d. errors. The experimental conditions ξ_k are i.i.d. random variables, independent of $\{\varepsilon_k\}$ and are observed sequentially. The decision of observing y_k or not must be made on-line. The criterion to be maximized is $\det[\boldsymbol{\Omega}^{-1} + \mathbf{M}]$, with $\mathbf{M}$ the Fisher information matrix, calculated at a prior nominal value $\hat{\boldsymbol{\theta}}^0$ of the model parameters, and $\boldsymbol{\Omega}$ a given positive-definite matrix. When θ is a scalar parameter, the problem corresponds to the maximization of the sum of n i.i.d. random variables $x_k = \left[z(\hat{\theta}^0, \xi_k)\right]^2$, with $z(\theta, \xi) = \partial\eta(\theta, \xi)/\partial\theta$, which are observed sequentially in a sequence of length N. The optimal decision rule, which is solution of a stochastic dynamic-programming problem, is compared to the simpler (but suboptimal) open-loop feedback-optimal rule. The situation is far more complex when $\boldsymbol{\theta}$ is p-dimensional, because the information is not additive. However, the computation of the expected value of the determinant of the informatio matrix allows one to construct the open-loop feedback-optimal rule. Forced certainty-equivalence is proposed for the case where the distribution of the ξ_k's is unknown and estimated on-line. Certainty equivalence is also forced in the case where θ can be estimated on-line: at the step k, the prior nominal value $\hat{\theta}^0$ is replaced by an estimate $\hat{\boldsymbol{\theta}}^{k-1}$. Illustrative examples are presented.

1. Introduction and problem statement

We consider a regression model with observations

$$y_k = \eta(\boldsymbol{\theta}, \xi_k) + \varepsilon_k ,$$

where $\{\varepsilon_k\}$ is a sequence of i.i.d. random variables, assumed for simplicity to be distributed $\mathcal{N}(0, \sigma^2)$, and $\boldsymbol{\theta} \in \Theta$ is the vector of parameters to be estimated,

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with Θ an open subset of $\mathbb{R}^p$. The experimental conditions ξ_k are uncontrolled but observed. We assume that $\{\xi_k\}$ is a sequence of i.i.d. random variables, independent of $\{\varepsilon_k\}$. The function $\eta(\cdot,\xi)$ is assumed continuously differentiable in Θ.

Only n observations can be taken, but N experimental conditions are proposed sequentially. For each of them, one has to decide on-line whether the observation should be taken or not. Let ν denote the probability measure of the experimental conditions ξ_k, $\hat{\boldsymbol{\theta}}^0 \in \Theta$ denote a prior nominal value for $\boldsymbol{\theta}$, and $\{u_k\}$ denote the decision sequence, with $u_k = 1$ if y_k is observed, $u_k = 0$ otherwise. One wishes to maximize

$$EJ(\hat{\boldsymbol{\theta}}^0, \boldsymbol{\xi}_1^N, \boldsymbol{u}_1^N) = E\Big\{\det\Big[\boldsymbol{\Omega}^{-1} + \mathbf{M}(\hat{\boldsymbol{\theta}}^0, \boldsymbol{\xi}_1^N, \boldsymbol{u}_1^N)\Big]\Big\} \tag{1}$$

with the constraint

$$\sum_{k=1}^{N} u_k = n\,, \tag{2}$$

where

$$\mathbf{M}(\hat{\boldsymbol{\theta}}^0, \boldsymbol{\xi}_1^N, \boldsymbol{u}_1^N) = \frac{1}{\sigma^2}\sum_{k=1}^{N} u_k \boldsymbol{z}(\hat{\boldsymbol{\theta}}^0, \xi_k)\, \boldsymbol{z}^T(\hat{\boldsymbol{\theta}}^0, \xi_k)$$

is the Fisher information matrix, with $\boldsymbol{z}(\boldsymbol{\theta},\xi) = \partial\eta(\boldsymbol{\theta},\xi)/\partial\boldsymbol{\theta}$. In (1) the expectation is w.r.t. the measure ν for ξ. Here and in what follows we use the notation $\boldsymbol{x}_1^N$ for the sequence $\{x_1,\ldots,x_N\}$. To simplify the notations, we shall omit the dependence in $\hat{\boldsymbol{\theta}}^0$, and denote $\boldsymbol{z}_k = \boldsymbol{z}(\hat{\boldsymbol{\theta}}^0,\xi_k)$, $\mathbf{M}(\boldsymbol{z}_1^N, \boldsymbol{u}_1^N) = \mathbf{M}(\hat{\boldsymbol{\theta}}^0, \boldsymbol{\xi}_1^N, \boldsymbol{u}_1^N)$. The matrix $\boldsymbol{\Omega}^{-1}$ may correspond to some prior information on $\boldsymbol{\theta}$, as used in a Bayesian framework. The measure ν for ξ_k induces a measure for $\boldsymbol{z}_k$, denoted μ_z, and is assumed to be such that the expectation $E\{\mathbf{M}(\boldsymbol{z}_1^N, \boldsymbol{u}_1^N)\}$ exists for any sequence $\boldsymbol{u}_1^N$ satisfying (2).

For any sequence $\boldsymbol{u}_1^N$ and any step j, $1 \le j \le N$, let a_j be the number of observations already taken, that is $a_j = \sum_{k=1}^{j-1} u_k$ with $a_1 = 0$, and define $\mathcal{I}_j$ as $\mathcal{I}_j = \Big(j, a_j, \mathbf{M}(\boldsymbol{z}_1^{j-1}, \boldsymbol{u}_1^{j-1})\Big)$. The decision u_j at step j may be *adaptive*: it may depend on $\mathcal{I}_j$ and $\boldsymbol{z}_j$ (which is known before the observation is taken). The choice of a decision rule is thus a *sequential design* problem. The construction of optimal and suboptimal solutions of such problems is usual in stochastic control (see, e.g., [7] for a presentation of optimality concepts).

The optimal solution at step j is obtained by solving the following stochastic dynamic–programming problem (see, e.g., [1]):

$$\max_{u_j\in\mathcal{U}_j}\Bigg[E_{\mathbf{z}_{j+1}}\Bigg\{\max_{u_{j+1}\in\mathcal{U}_{j+1}}\Bigg[E_{\mathbf{z}_{j+2}}\Bigg\{\max_{u_{j+2}\in\mathcal{U}_{j+1}}\Bigg[\ldots$$
$$\ldots E_{\mathbf{z}_{N-1}}\Bigg\{\max_{u_{N-1}\in\mathcal{U}_{N-1}}\Bigg[E_{\mathbf{z}_N}\Bigg\{\max_{u_N\in\mathcal{U}_N}\Bigg[\det\Big[\mathbf{\Omega}^{-1}+\mathbf{M}\big(\mathbf{z}_1^N,\mathbf{u}_1^N\big)\Big]\Bigg]\Bigg\}\Bigg]\Bigg\}\ldots$$
$$\Bigg]\Bigg\}\Bigg]\Bigg\}\Bigg], \qquad (3)$$

where $E_{\mathbf{z}_k}\{\cdot\}$ denotes the expectation w.r.t. $\mathbf{z}_k$, distributed with the measure μ_z, and

$$\mathcal{U}_k=\begin{cases}\{0\} & \text{if } a_k\ge n\,,\\ \{1\} & \text{if } a_k+N-k+1\le n\,,\\ \{0,1\} & \text{otherwise}\,.\end{cases}$$

A simpler approach, called Open–Loop Feedback–Optimal (OLFO) in control theory, assumes that at each step j, all the decisions $\mathbf{u}_j^N$ may only depend on $\mathcal{I}_j$ and $\mathbf{z}_j$. The (suboptimal) OLFO solution is then obtained by solving the following optimization problem:

$$\max_{\mathbf{u}_j^N}\Bigg[E_{\mathbf{z}_{j+1}^N}\Big\{\det\big[\mathbf{\Omega}^{-1}+\mathbf{M}\big(\mathbf{z}_1^N,\mathbf{u}_1^N\big)\big]\Big\}\Bigg], \qquad (4)$$

with the constraint (2).

The case where θ is scalar is considered in Section 2. The optimal solution can then be constructed by backward recurrence in j. An illustrative example is presented. The multidimensional case is considered in Section 3. The OLFO solution is constructed and illustrated by an example.

2. Estimation of a scalar parameter

When θ is a scalar, the criterion is $J\big(\mathbf{z}_1^N,\mathbf{u}_1^N\big)=\mathbf{\Omega}^{-1}+\frac{1}{\sigma^2}\sum_{k=1}^{N}u_kx_k$, where $x_k=z_k^2$ and $\mathbf{\Omega}^{-1}$ is a scalar. The information $\mathcal{I}_j$ can be restricted to $\mathcal{I}_j=(j,a_j)$, and one simply has to maximize $E\Big\{\sum_{k=1}^{N}u_kx_k\Big\}$.

We show in [5] that, when ν is known, the OLFO solution is obtained by comparing x_k to a fixed threshold equal to $E\{x\}$, whereas the threshold for the optimal solution satisfies a nonlinear recurrence equation (backward with respect to k).

When ν is unknown, it can be estimated on-line by some measure $\hat{\nu}_j$. An important feature of the problem is that decisions are *neutral* with respect to estimation (see [2, 4] for a definition of neutrality in control problems): the sequence $\{\xi_k\}$ is observed whatever the decision sequence $\{u_k\}$ is, and decisions have no effect on the accuracy of the estimation of the distribution. However, the construction of the optimal decision rule seems particularly difficult in this case, and we shall restrict our attention to suboptimal policies based on *Forced Certainty Equivalence* (FCE): at step j, the decision is determined for $\hat{\nu}_j$, the empirical distribution of the ξ_k's. In practise, we shall initialize the construction of $\hat{\nu}$ by M random samples of ξ. Each calculation using $\hat{\nu}_j$ is thus based on $j+M$ samples. We show in [5] that, in spite of the neutrality property, this FCE policy is suboptimal.

FCE can also be used when the model response $\eta(\cdot,\xi)$ is nonlinear in θ, and θ can be estimated on-line. One can then replace at step j the prior estimate $\hat{\theta}^0$ by the current estimated value $\hat{\theta}^{j-1}$, based on the observations already taken, if any. It may correspond to LS or Bayesian estimation, e.g., with the normal prior $\mathcal{N}(\hat{\theta}^0,\mathbf{\Omega})$. We assume that $\hat{\theta}^j \in \Theta$ for all j (a projection on Θ may be used if necessary), and that the measure ν is such that $E\{z^2(\theta,\xi)\}$ exists for all $\theta \in \Theta$. Note that this last assumption is not required in practise when the empirical distribution of $x = z^2(\theta^{j-1},\xi)$ is used at step j for FCE decisions.

EXAMPLE 1. We consider a simple illustrative model, where one has to estimate the maximum of a quadratic model:

$$\eta(\theta,\xi_k) = -(\xi_k - \theta)^2 \, .$$

The true maximum is at $\theta = 2$, the ξ_k's are i.i.d. $\mathcal{N}(1,1)$. The observation errors ε_k are i.i.d. $\mathcal{N}(0,\sigma^2)$, with $\sigma = 0.1$. $N = 100$ values of ξ_k are proposed, and $n = 5$ observations can be taken. Five random samples $\xi_{-4},\ldots,\xi_0$ are used to initialize the construction of the empirical distribution of the ξ_k's in the FCE decision rules. LS estimation is used for $\hat{\theta}^j$ (which is solution of a third order polynomial equation). The initial value $\hat{\theta}^0$ (used for the FCE rule before any observation is taken) is 3.

We performed $m = 100$ independent repetitions of the experiment to compare the performances of the policies when FCE is used. The two criteria used for the comparison are the information per observation, given by

$$J(\boldsymbol{u}_1^N) = \frac{1}{n\sigma^2}\sum_{k=1}^{N} u_k z^2(\theta,\xi) = \frac{4}{n\sigma^2}\sum_{k=1}^{N} u_k(\xi_k - \theta)^2 \, ,$$

and the squared error for the estimation of θ, given by

$$S(\boldsymbol{u}_1^N) = \left(\theta - \hat{\theta}^N\right)^2 .$$

TABLE 1. Average performances of the optimal and OLFO decision rules, with FCE, for the maximization of J (Example 1, 100 repetitions).

	J	S
optimal	$3.04\ 10^3$	$7.14\ 10^{-5}$
OLFO	$1.55\ 10^3$	$1.54\ 10^{-4}$

The result are given in Table 1.

The exact optimal expected value of J, obtained for the optimal rule with the true distribution of the ξ_k's, is $3.48\ 10^3$. The exact expected value of J obtained with the OLFO rule and the true distribution of the ξ_k's is $1.83\ 10^3$. Using FCE thus does not seem to much deteriorate the performances of the two policies in terms of J. These results can be compared to the expected value of J obtained when n random experiments are performed: one gets $E\{J\} = 8.0\ 10^2$, hence the interest of using the optimal or OLFO strategy.

Let $J_i(R)$ and $S_i(R)$ denote the performance of rule R ($R = R_1$ for optimal, R_2 for OLFO) in the ith repetition. We can test if the performance of rule 1 is significantly better than that of rule 2 by computing the difference of performances in each experiment, that is $\Delta_J^{1,2}(i) = J_i(R_1) - J_i(R_2)$ (which gives m independent realizations of $\Delta_J^{1,2}$), and then the empirical mean $E_J^{1,2}$ and empirical standard deviation $\Sigma_J^{1,2}$ of $\Delta_J^{1,2}$. The ratio

$$\rho_J^{1,2} = \sqrt{m}\,\frac{E_J^{1,2}}{\Sigma_J^{1,2}}, \tag{5}$$

is approximately distributed $\mathcal{N}(0,1)$ if the two decision rules have similar performance. We get $\rho_J^{1,2} = 20.2$, and conclude that performance of rule 1 is significantly better than that of rule 2 in terms of J. The same calculations for $\Delta_S^{1,2}(i) = S_i(R_1) - S_i(R_2)$ give a ratio $\rho_S^{1,2} = -10.6$, so that the performance of the optimal rule is also significantly better than that of the OLFO rule in terms of S.

3. Estimation of a parameter vector

When $p = \dim(\boldsymbol{\theta}) > 1$, the problem (3) cannot be solved analytically, and we restrict our attention to suboptimal solutions. We shall denote

$$\mathbf{Q}_j = \mathbf{\Omega}^{-1} + \mathbf{M}(\boldsymbol{z}_1^j, \boldsymbol{u}_1^j), \qquad \overline{\mathbf{M}} = E\{\boldsymbol{z}\boldsymbol{z}^T/\sigma^2\}.$$

Consider the OLFO rule, solution of (4). Since $\boldsymbol{u}_j^N$ only depends on $\mathcal{I}_j$ and $\boldsymbol{z}_j$,

$\mathbf{u}_{j+1}^{N}$ may be considered as fixed, with $\sum_{l=j}^{N} u_l = n - a_j$. The decision rule is thus:

$$\widehat{u}_j = \begin{cases} 0 & \text{if } E_{\mathbf{z}_{j+1}^{n+j-a_j}}\left\{\det\left[\mathbf{Q}_{j-1} + \mathbf{z}_j\mathbf{z}_j^T/\sigma^2 + \sum_{l=j+1}^{n+j-a_j} \mathbf{z}_l\mathbf{z}_l^T/\sigma^2\right]\right\} \leq \\ & E_{\mathbf{z}_j^{n+j-a_j}}\left\{\det\left[\mathbf{Q}_{j-1} + \sum_{l=j}^{n+j-a_j} \mathbf{z}_l\mathbf{z}_l^T/\sigma^2\right]\right\} \text{ or } a_j \geq n\,, \\ 1 & \text{otherwise}\,. \end{cases} \tag{6}$$

Note that $\widehat{u}_j$ is independent of N. Its computation requires the evaluation of expectations of determinants. Their analytical expression is given by the following theorem.

THEOREM 1. *Assume that the vectors $\mathbf{z}_i \in \mathbb{R}^p$ are i.i.d., with a probability measure μ_z such that $\overline{\mathbf{M}}$ exists. Then for any regular matrix $\mathbf{Q}$:*

$$E_{\mathbf{z}_1^k}\left\{\det\left[\mathbf{Q} + \sum_{l=1}^{k} \mathbf{z}_l\mathbf{z}_l^T/\sigma^2\right]\right\} = \det[\mathbf{Q}] \times \left[1 + \sum_{l=1}^{k} \binom{k}{l} P_l(\mathbf{t}_1^l)\right], \tag{7}$$

with $t_l = \mathrm{tr}\left[(\mathbf{Q}^{-1}\overline{\mathbf{M}})^l\right]$ and

$$P_l(\mathbf{t}_1^l) = \begin{cases} l! \displaystyle\sum_{1^{n_1}\ldots l^{n_l}} \frac{(-1)^{l+\sum_{i=1}^{l} n_i}}{(1^{n_1}\cdots l^{n_l})(n_1!\cdots n_l!)}\, t_1^{n_1}\cdots t_l^{n_l} & \text{if } l \leq p\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where the summation $\sum_{1^{n_1}\ldots l^{n_l}}$ is over all integers $n_1, \ldots, n_l$ such that $\sum_{i=1}^{l} i n_i = l$.

The proof is by induction on k. One has for the first values of l (for $p \geq 4$):

$$\begin{aligned} P_1(t_1) &= t_1\,, \\ P_2(\mathbf{t}_1^2) &= t_1^2 - t_2\,, \\ P_3(\mathbf{t}_1^3) &= t_1^3 - 3t_1t_2 + 2t_3\,, \\ P_4(\mathbf{t}_1^4) &= t_1^4 - 6t_1^2t_2 + 8t_1t_3 + 3t_2^2 - 6t_4\,. \end{aligned}$$

Note that, up to the sign factor $(-1)^{l+\sum_{i=1}^{l} n_i}$, the polynomials $P_l(\mathbf{t}_l)/l!$ coincide with the cycle index of symmetric groups (see, e.g., [8], p. 465) and with the coefficients of the characteristic polynomial of the matrix $\mathbf{Q}^{-1}\overline{\mathbf{M}}$, as calculated, e.g., by Faddeev's method (see [3], Vol. 1, p. 88). The value of $\widehat{u}_j$, given by (6), is easily obtained by applying Theorem 1 with $\mathbf{Q} = \mathbf{Q}_{j-1}$ and $\mathbf{Q} = \mathbf{Q}_{j-1} + \mathbf{z}_j\mathbf{z}_j^T/\sigma^2$. Again, FCE can be used when the distribution of the ξ_k's (and thus of the $\mathbf{z}_k$'s) is unknown.

EXAMPLE 2. Consider the linear regression model, quadratic in the design variable ξ_k:

$$\eta(\boldsymbol{\theta}, \xi_k) = \theta_0 + \theta_1 \xi_k + \theta_2 \xi_k^2 .$$

The true value of $\boldsymbol{\theta} = (\theta_0, \theta_1, \theta_2)$ is $(-4, 4, -1)$, so that the response is the same as in Example 1, and has its maximum at 2. The observation errors ε_k are i.i.d. $\mathcal{N}(0, \sigma^2)$, with $\sigma = 0.1$, and the ξ_k's are i.i.d. $\mathcal{N}(1,1)$. $N = 100$ values of ξ_k are proposed, and $n = 10$ observations can be taken. Five random samples $\xi_{-4}, \ldots, \xi_0$ are used to initialize the construction of the empirical distribution of the ξ_k's in the FCE decision rules. Bayesian estimation is used for $\hat{\boldsymbol{\theta}}^j$, with a normal prior $\mathcal{N}(\hat{\boldsymbol{\theta}}^0, \boldsymbol{\Omega})$ ($\hat{\boldsymbol{\theta}}^0 = (0, 3, -0.5)$ and $\boldsymbol{\Omega} = 10^6 \mathbf{I}$).

We performed $m = 1000$ independent repetitions of the experiment to compare the performances of different rules. Three criteria are used for the comparison: the same as in Example 1,

$$J(\boldsymbol{u}_1^N) = \frac{4}{n\sigma^2} \sum_{k=1}^{N} u_k \left(\xi_k - \frac{-\theta_1}{2\theta_2} \right)^2 ,$$

where $-\theta_1/(2\theta_2) = 2$ is the point where $\eta(\boldsymbol{\theta}, \xi)$ is maximum, the squared error for the estimation of this point, given by

$$S(\boldsymbol{u}_1^N) = \left[\theta_1/(2\theta_2) - \hat{\theta}_1^N/(2\hat{\theta}_2^N) \right]^2 ,$$

and the determinant of the information matrix per sample:

$$D(\boldsymbol{u}_1^N) = \det \left[\mathbf{M}(\boldsymbol{z}_1^N, \boldsymbol{u}_1^N)/n \right] .$$

Three rules are compared, using FCE for each of them: 1) the optimal rule for the maximization of J, 2) the OLFO rule for the maximization of J, 3) the OLFO rule for the maximization of $\det\left[\boldsymbol{\Omega}^{-1} + \mathbf{M}(\boldsymbol{z}_1^N, \boldsymbol{u}_1^N)\right]$. The results are given in Table 2.

TABLE 2. Average performances of rules 1), 2) and 3) (Example 2, 1000 repetitions).

	J	S	D
1)	2.69 10^3	2.09 10^{-2}	3.58 10^6
2)	1.65 10^3	4.75 10^{-3}	1.71 10^6
3)	1.22 10^3	5.81 10^{-5}	7.36 10^6

The exact optimal expected value of J, obtained for the optimal rule 1) with the true distribution of the ξ_k's, is $2.92 \cdot 10^3$. The exact expected value of J obtained with the OLFO rule 2) and true distribution of the ξ_k's is $1.83 \cdot 10^3$. As in Example 1, using FCE does not seem to much deteriorate the performances of the two policies in terms of J. One can compare these results to the expected values of J and D obtained when n random experiments are performed. Direct calculation gives $E\{J\} = 8.0\ 10^2$. From a result in [6], one has

$$E\Big\{ \det\big[\mathbf{M}(\boldsymbol{z}_1^N, \boldsymbol{u}_1^N)/n\big]\Big\} = \frac{n!}{n^p(n-p)!} \det[\overline{\mathbf{M}}]\,,$$

which gives $E\{D\} = 1.44\ 10^6$.

Similarly to Example 1, we denote the performance of decision rule R in the ith repetition by $J_i(R)$, $S_i(R)$ and $D_i(R)$. We then test if the performance of rule R_j is significantly better than that of rule R_k by computing the difference of performances in each experiment, that is $\Delta_J^{j,k}(i) = J_i(R_j) - J_i(R_k)$, $\Delta_S^{j,k}(i) = S_i(R_j) - S_i(R_k)$, and $\Delta_D^{j,k}(i) = D_i(R_j) - D_i(R_k)$, and their means and standard deviations. The values of the ratios $\rho_J^{j,k}$, $\rho_S^{j,k}$ and $\rho_D^{j,k}$, computed as in (5), are given in Table 3.

Table 3. Comparison of the performances of rules 1), 2) and 3) (Example 2, 1000 repetitions: if two rules have similar average performance, then ρ is approximately $\mathcal{N}(0,1)$).

j,k	1,2	1,3	2,3
$\rho_J^{j,k}$	67.4	112.7	33.6
$\rho_S^{j,k}$	1.9	2.5	7.6
$\rho_D^{j,k}$	7.8	-13.7	-29.6

We conclude from Table 3 that rule 1) has much better performance than rules 2) and 3) in terms of J. However, since there are three unknown parameters in the model, the experiment obtained with rules 1) or 2) may contain little information, and rule 3) has much better performance than rules 1) and 2) in terms of S or D.

Further work will concern the improvement of the OLFO rule, e.g., by an on–line adjustment of the threshold in (6).

REFERENCES

[1] BERTSEKAS, D.: *Dynamic Programming. Deterministic and Stochastic Models*, Prentice-Hall, Englewood Cliffs, 1987.

[2] FEL'DBAUM, A.: *Optimal Control Systems*, Academic Press, New York, 1965.

[3] GANTMACHER, F.: *Théorie des Matrices*, Dunod, Paris, 1966.

[4] PATCHELL, J.—JACOBS, O.: *Separability, neutrality and certainty equivalence*, Internat. J. Control **13** (1971), 337–342.

[5] PRONZATO, L.: *Optimal selection of information with restricted storage capacity*, Tech. Rep. 97–01, Laboratoire I3S, CNRS-URA 1376, Sophia Antipolis, 06560 Valbonne, France, January 1997.

[6] PRONZATO, L.: *On a property of the expected value of a determinant*, Statist. Probab. Lett. **39** (1998), 161–165.

[7] RUNGGALDIER, W.: *Concepts of optimality in stochastic control*, in: Reliability and Decision Making (R. Barlow et al., eds.), Elsevier, Amsterdam, 1993, pp. 101–114.

[8] VAN LINT, J.—WILSON, R.: *A Course in Combinatorics*, Cambridge University Press, Cambridge, 1992.

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