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FORMULAE FOR SOME ARTIN ROOT NUMBERS

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ABSTRACT. Let N be a normal number field with Galois group any dicyclic group of order $4p$, $p \geq 3$ an odd prime. Let L denote the real quadratic subfield of N. We determine the root numbers which appear in the functional equations of the Artin L-functions $L(s, \psi, N/\mathbb{Q})$ associated with the irreducible characters of degree two of the Galois group $\mathrm{Gal}(N/\mathbb{Q})$. We deduce that if p is totally ramified in N and $L = \mathbb{Q}(\sqrt{p})$, then the Dedekind zeta function of N has a zero of order $\geq (p-1)/2$ at $s = 1/2$.

1. Introduction

Let $p \geq 3$ denote any odd prime. There are two non-abelian groups of order $4p$: the *dicyclic group* Q_{4p} defined by the presentation $Q_{4p} = \langle a, b\colon a^{2p} = 1,\ a^p = b^2,\ b^{-1}ab = a^{-1}\rangle$, and the *dihedral group* D_{4p} defined by the presentation $D_{4p} = \langle a, b\colon a^{2p} = 1,\ a^{2p} = b^2 = 1,\ b^{-1}ab = a^{-1}\rangle$. Let H denote the cyclic subgroup of order $2p$ of $G = Q_{4p}$ or $G = D_{4p}$ generated by a. Both these groups G have four irreducible characters of degree one and $p-1$ irreducible characters ψ of degree two and these $p-1$ irreducible characters ψ of degree two of G are induced by the $2(p-1)$ non-quadratic characters χ of degree one of H (and for any non-quadratic character χ of H, the characters $\bar{\chi}$ and χ induce the same irreducible character of degree two of G). Since for any such χ we have the following relation between Artin L-functions: $L(s, \chi, N/L) = L(s, \chi^*, N/\mathbb{Q}) = L(s, \psi, N/\mathbb{Q})$ (where χ^* denotes the character of Q_{4p} induced by χ of H) and since $\chi^* = \psi$ is real valued, we deduce that the Artin root number $W(\chi)$ which appears in the functional equation of the L-function $L(s, \chi, N/L)$ satisfies $W(\chi) = \pm 1$. We also note that since all the irreducible characters of degree two of D_{4p} are characters of irreducible representations over the field of real numbers, then their root numbers are equal to $+1$ (see [FQ, Th. 1]). In contrast, only $(p-1)/2$ out of the $p-1$ irreducible characters of degree 2 of Q_{4p} are characters of irreducible representations over the field of real numbers: those

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induced by the characters χ of H of order dividing p (see [JL, Exercise 23.5]). For these $(p-1)/2$ characters we have $W(\psi)=+1$. The aim of this paper is to compute $W(\psi)$ for the remaining $(p-1)/2$ irreducible characters of degree two of Q_{4p}. More precisely, let N be a *dicyclic number field* of degree $4p$, i.e., N is a number field (considered as a subfield of the field of complex numbers) such that the extension $N/\mathbb{Q}$ is a normal extension of degree $4p$ with Galois group Q_{4p}. Let K denote the subfield of N fixed by the cyclic subgroup generated by a^p and M denote the subfield of N fixed by the cyclic subgroup generated by a^2. We have the following lattice of subfields:

$$\begin{tikzcd}
 & & N \arrow[dd, dash, "p"] \\
 & K \arrow[ur, dash, "2"] \arrow[dd, dash, "p"'] & \\
 & & M \\
 & L \arrow[ur, dash, "2"] & \\
\mathbb{Q} \arrow[ur, dash, "2"] & &
\end{tikzcd}$$

$N/\mathbb{Q}$ dicyclic

N/L cyclic

$M/\mathbb{Q}$ cyclic quartic

$K/\mathbb{Q}$ dihedral

This paper is devoted to proving the following theorem:

Theorem 1. *Let N be dicyclic field of degree $4p$, let χ be any one of the $p-1$ characters on $\mathrm{Gal}(N/L)$ of order $2p$ and let $\left(\frac{t}{p}\right)$ denote Legendre's symbol. Notice that p is the only prime which can be totally ramified in $N/\mathbb{Q}$ and if p is totally ramified in $N/\mathbb{Q}$ then $p\equiv 1 \pmod 4$.*

1. *If p is not totally ramified in $N/\mathbb{Q}$ then $W(\chi)$ is equal to -1 or $+1$ but $W(\chi)=W_N$ does not depend on χ (and there are infinitely many examples of N's for which $W_N=+1$ and infinitely many examples of N's for which $W_N=-1$).*
2. *If $p\equiv 1 \pmod 4$ is totally ramified in $N/\mathbb{Q}$ and if $L=\mathbb{Q}(\sqrt{p})$, then for any t coprime with $2p$ we have $W(\chi^t)=\left(\frac{t}{p}\right)W(\chi)$. Hence, $W(\chi)=-1$ for $(p-1)/2$ of these $p-1$ characters χ and $W(\chi)=+1$ for $(p-1)/2$ of these $p-1$ characters.*

In fact, we will give formulae for these root numbers (see Theorem 7). We refer the reader to [LouPark] for examples of the practical use of such formulae for computing numerically relative class numbers of dicyclic CM-fields of degree $4p$.

2. Gaussian sums and the first formulae for W_χ

2.1 Gaussian sums and root numbers.

For any integral ideal $\mathcal{I}$ of L, for any $\gamma\in L$ such that $\gamma\in\mathcal{I}^{-1}\mathcal{D}^{-1}$ and for any character ϕ on $(A_L/\mathcal{I})^*$ we set

$$W_{\mathcal{I}}(\gamma,\phi) = \Bigg(\sum_{\lambda \pmod{\mathcal{I}}} \phi(\lambda) \exp\Big(2\pi i \operatorname{Tr}_{L/\mathbb{Q}}(\gamma\lambda)\Big) \Bigg) \Big/ \sqrt{N_{L/\mathbb{Q}}(\mathcal{I})}\,. \tag{1}$$

It is well known that if ϕ is primitive then $\big|W_{\mathcal{I}}(\gamma,\phi)\big|^2 = 1$; if $\alpha \in A_L$ and $\mathcal{I}$ are relatively prime then

$$W_{\mathcal{I}}(\alpha\gamma,\phi) = \overline{\phi(\alpha)}\, W_{\mathcal{I}}(\gamma,\phi)\,; \tag{2}$$

and if $\mathcal{I} = \mathcal{I}_1 \cdots \mathcal{I}_r$ is a factorization of $\mathcal{I}$ into a product of ideals $\mathcal{I}_i$ relatively prime in pairs and if $\phi = \phi_1 \cdots \phi_r$ is the corresponding factorization of ϕ then for any $\gamma_i \in \mathcal{I}_i^{-1}\mathcal{D}^{-1}$ satisfying $\gamma - \gamma_i \in \mathcal{I}_i\mathcal{I}^{-1}\mathcal{D}^{-1}$ we have

$$W_{\mathcal{I}}(\gamma,\phi) = \prod_{i=1}^{r} W_{\mathcal{I}_i}(\gamma_i,\phi_i)\,. \tag{3}$$

Let $L(s,\chi)$ be a Hecke L-series associated with a primitive Hecke character χ on a ray class group of a real quadratic number field L. Assume that if χ is ramified at some infinite place of L then χ is ramified at both the infinite places ∞_1 and ∞_2 of L. Let $\mathcal{F}_\chi$ be the finite part of the conductor of χ. Set $a_\chi = 0$ if χ is unramified at both the infinite places and $a_\chi = 1$ if χ is ramified at both the infinite places. Set $A_\chi = d_L N_{L/\mathbb{Q}}(\mathcal{F}_\chi)/\pi^2$ and $F(s,\chi) = A_\chi^{s/2}\Gamma^2\big((s+a_\chi)/2\big)L(s,\chi)$. Let $\tilde{\chi}$ be the primitive character on $\big(A_L/\mathcal{F}_\chi\big)^*$ such that

$$\chi\big((\alpha)\big) = \nu(\alpha)\tilde{\chi}(\alpha)\,,$$

where ν is trivial if $a_\chi = 0$, but $\nu(\alpha)$ is equal to the sign of $N_{L/\mathbb{Q}}(\alpha)$ if $a_\chi = 1$. Choose $\gamma \in \mathcal{F}_\chi^{-1}\mathcal{D}^{-1}$ such that the integral ideal $\mathcal{I} = \big(\gamma\sqrt{d_L}\,\big)\mathcal{F}_\chi$ is relatively prime with $\mathcal{F}_\chi$. Then (see [Sie, Pages 158 and 169])

$$W(\chi) = \chi(\mathcal{I})\nu\big(\gamma\sqrt{d_L}\,\big)W_{\mathcal{F}_\chi}(\gamma,\tilde{\chi}) \tag{4}$$

has absolute value one and does not depend on the choice of γ.

2.2 First formulae for the root numbers.

Let us fix some of the notation we will be using throughout this paper. We let A_L, d_L, $\mathcal{D} = \big(\sqrt{d_L}\,\big)$ and χ_L denote the ring of algebraic integers, the discriminant, the different of the extension $L/\mathbb{Q}$ and the primitive Dirichlet character of conductor d_L associated with a real quadratic number field L, respectively. We let N be a dicyclic field of degree $4p$. Since $K/\mathbb{Q}$ is dihedral of degree $2p$, the conductor of the cyclic extension K/L is given by $\mathcal{F}_{K/L} = (f_{K/L})$ for some rational integer $f_{K/L} \geq 1$ of the form

$$f_{K/L} = p^a \prod_{i=1}^{r} q_i, \quad \text{where } a = 0 \text{ or } a = \begin{cases} 2 & \text{if } p \geq 3 \text{ does not divide } d_L, \\ 1 & \text{if } p \geq 5 \text{ divides } d_L, \\ 1 \text{ or } 2 & \text{if } p = 3 \text{ and } p \text{ divides } d_L. \end{cases}$$

and where the q_i's are distinct primes not equal to p and satisfying $q_i \equiv \chi_L(q_i) \pmod{p}$ (see [Mar] and [LPL]). In our situation L is a subfield of a cyclic quartic field M, hence any rational prime which is ramified in $L/\mathbb{Q}$ is not equal to 3 modulo 4 and the third case above cannot crop up. Let $\mathcal{F}_{M/L}$ be the finite part of the conductor of the extension M/L and let f_M and f_L denote the conductors of the abelian fields M and L, respectively. Since $\mathcal{F}_{M/L}$ is invariant under the action of the Galois group $\mathrm{Gal}(L/\mathbb{Q})$ and since two integral ideals of L which are both invariant under the action of $\mathrm{Gal}(L/\mathbb{Q})$ are equal if and only if they have the same norm, we easily get

$$\mathcal{F}_{M/L} = (f_{M/L})\mathcal{D}, \qquad \text{where} \quad f_{M/L} = f_M/f_L.$$

Moreover we set

$$(\mathcal{D}', d'_L, f'_{M/L}) = \begin{cases} (\mathcal{D}, d_L, f_{M/L}) & \text{if } d_L \equiv 1 \pmod 4, \\ ((2)\mathcal{D}, 4d_L, f_{M/L}/2) & \text{if } d_L \equiv 0 \pmod 4. \end{cases}$$

Hence, $\mathcal{F}_{M/L} = (f'_{M/L})\mathcal{D}'$ and $(f'_{M/L})$ and $\mathcal{D}'$ are relatively prime.

Proposition 2. *Let N be dicyclic field of degree $4p \geq 12$, let $\phi = \phi_{K/L}$ be any one of the $p-1$ characters of order p associated with the cyclic extension K/L and let $\psi = \psi_{M/L}$ be the quadratic character associated with the extension M/L. Hence, $\chi_{N/L} = \chi = \phi\psi$ is one of the $p-1$ characters of order $2p$ associated with the cyclic extension N/L. Set $\varepsilon_N = \varepsilon_M = -1$ if N and M are imaginary and $\varepsilon_N = \varepsilon_M = +1$ if N and M are real.*

1. *Assume that p is not totally ramified in $N/\mathbb{Q}$. Then, $\mathcal{F}_{K/L} = (f_{K/L})$, $\mathcal{F}_{M/L} = \mathcal{D}'(f'_{M/L})$ and $\mathcal{F}_{N/L} = \mathcal{D}'(f)$ where $f = \mathrm{lcm}(f_{K/L}, f'_{M/L})$. Moreover, $(f_{K/L})$ and $\mathcal{D}'$ are relatively prime, $(f'_{M/L})$ and $\mathcal{D}'$ are relatively prime and (f) and $\mathcal{D}'$ are relatively prime. Let $\tilde{\chi} = \chi_{\mathcal{D}'}\chi_{(f)}$ be the associated factorization of $\tilde{\chi}$ as a product of a character $\chi_{\mathcal{D}'}$ on $(A_L/\mathcal{D}')^*$ and a character $\chi_{(f)}$ on $(A_L/(f))^*$. Let $\tilde{\psi} = \psi_{\mathcal{D}'}\psi_{(f'_{M/L})}$ be the associated factorization of $\tilde{\psi}$ as a product of a character $\psi_{\mathcal{D}'}$ on $(A_L/\mathcal{D}')^*$ and a character*

$\psi_{(f'_{M/L})}$ *on* $(A_L/(f_{M/L'}))^*$. *It holds*

$$\begin{aligned}
\chi_{\mathcal{D}'} &= \psi_{\mathcal{D}'} \qquad \text{and} \qquad \chi_{(f)} = \tilde{\phi}\psi_{(f'_{M/L})}, \\
W(\chi) &= \varepsilon_N W_{\mathcal{D}'(f)}\Big(1/f\sqrt{d'_L d_L}, \tilde{\chi}\Big) \\
&= \varepsilon_N \varepsilon_\chi W_{\mathcal{D}'}\Big(1/\sqrt{d'_L d_L}, \chi_{\mathcal{D}'}\Big) W_{(f)}\Big(1/f\sqrt{d_L}, \chi_{(f)}\Big), \\
\text{where } \varepsilon_\chi &= \chi_{\mathcal{D}'}(f)\chi_{(f)}\Big(\sqrt{d'_L}\Big) = \psi_{\mathcal{D}'}(f)\tilde{\phi}\Big(\sqrt{d'_L}\Big)\psi_{(f'_{M/L})}\Big(\sqrt{d'_L}\Big), \\
W(\psi) &= \varepsilon_M W_{(f'_{M/L})\mathcal{D}'}\Big(1/f'_{M/L}\sqrt{Ld'_L d_L}, \tilde{\psi}_{M/L}\Big) \\
&= \varepsilon_M \varepsilon_\psi W_{\mathcal{D}'}\Big(1/\sqrt{d'_L d_L}, \psi_{\mathcal{D}'}\Big) W_{(f'_{M/L})}\Big(1/f'_{M/L}\sqrt{d_L}, \psi_{(f'_{M/L})}\Big), \\
\text{where } \varepsilon_\psi &= \psi_{\mathcal{D}'}(f'_{M/L})\psi_{(f'_{M/L})}\Big(\sqrt{d'_L}\Big), \\
\frac{W(\chi)}{W(\psi)} &= \varepsilon_\chi W_{(f)}\Big(1/f\sqrt{d_L}, \chi_{(f)}\Big)\Big/\varepsilon_\psi W_{(f'_{M/L})}\Big(1/f'_{M/L}\sqrt{d_L}, \psi_{(f'_{M/L})}\Big), \\
\varepsilon_\chi/\varepsilon_\psi &= \chi_{\mathcal{D}'}(f/f'_{M/L})\tilde{\phi}\Big(\sqrt{d'_L}\Big).
\end{aligned}$$

Notice that $f/f'_{M/L} = f_{K/L}/\gcd(f_{K/L}, f'_{M/L}) = f_{K/L}/\gcd(f_{K/L}, f_M)$.

2. *Assume that* $p \equiv 1 \pmod 4$ *is totally ramified in* $N/\mathbb{Q}$ *and that* $L = \mathbb{Q}(\sqrt{p})$. *Set* $f = \operatorname{lcm}(f_{K/L}/p, f_{M/L}) = \operatorname{lcm}(f_{K/L}/p, f_M/p) = (f_{K/L}/p)$ $(f_M/\gcd(f_{K/L}, f_M))$. *Then* $\mathcal{F}_{N/L} = (p)(f)$, *and* (f) *and* $(\sqrt{p}) = \mathcal{D}$ *are relatively prime. Let* $\tilde{\chi} = \chi_{(p)}\chi_{(f)}$ *be the associated factorization of* $\tilde{\chi}$ *as a product of a character* $\chi_{(p)}$ *on* $(A_L/(p))^*$ *and a character* $\chi_{(f)}$ *on* $(A_L/(f))^*$. *It holds*

$$\begin{aligned}
W(\chi) &= W_{(pf)}\Big(1/pf\sqrt{d_L}, \tilde{\chi}\Big) \\
&= \varepsilon_\chi W_{(p)}\Big(1/p\sqrt{d_L}, \chi_{(p)}\Big) W_{(f)}\Big(1/f\sqrt{d_L}, \chi_{(f)}\Big),
\end{aligned}$$

where $\varepsilon_\chi = \chi_{(p)}(f)\chi_{(f)}(p)$.

P r o o f.

1. In (4), we choose $\gamma = 1/f\sqrt{d'_L d_L} \in \mathcal{F}_{N/L}^{-1}\mathcal{D}^{-1}$ and get $\mathcal{I} = (1)$ and the first formula for $W(\chi)$. Take $\gamma_f = \Gamma_f/f\sqrt{d_L} \in (f)^{-1}\mathcal{D}^{-1}$ such that $\gamma - \gamma_f \in (\mathcal{D}')^{-1}\mathcal{D}^{-1}$, take $\gamma_{\mathcal{D}'} = \Gamma_{\mathcal{D}'}/\sqrt{d_L d'_L} \in (\mathcal{D}')^{-1}\mathcal{D}^{-1}$ such that $\gamma - \gamma_{\mathcal{D}'} \in (f)^{-1}\mathcal{D}^{-1}$. Hence, $1 - \sqrt{d'_L}\Gamma_f \in (f)$, $1 - f\,\Gamma_{\mathcal{D}'} \in \mathcal{D}'$. Therefore, $\overline{\chi_{(f)}(\Gamma_f)} = \chi_{(f)}(\sqrt{d'_L})$ and $\overline{\chi_{\mathcal{D}'}(\Gamma_{\mathcal{D}'})} = \chi_{\mathcal{D}'}(f)$. According to (2)–(3) we get the second formula for $W(\chi)$. The proofs for $W(\psi)$ are similar.

2. In (4), we choose $\gamma = 1/pf\sqrt{d_L} \in \mathcal{F}_{N/L}^{-1}\mathcal{D}^{-1}$ and get $\mathcal{I} = (1)$ and the first formula. Take $\gamma_p = \Gamma_p/p\sqrt{d_L} \in (p)^{-1}\mathcal{D}^{-1}$ such that $\gamma - \gamma_p \in (f)^{-1}\mathcal{D}^{-1}$ and take $\gamma_f = \Gamma_f/f\sqrt{d_L} \in (f)^{-1}\mathcal{D}^{-1}$ such that $\gamma - \gamma_f \in (p)^{-1}\mathcal{D}^{-1}$. Hence, $1 - f\,\Gamma_p \in (p)$ and $1 - p\,\Gamma_f \in (f)$. Therefore, $\overline{\chi_{(p)}(\Gamma_p)} = \chi_{(p)}(f)$ and $\overline{\chi_{(f)}(\Gamma_f)} = \chi_{(f)}(p)$. According to (2)–(3) we get the second formula. □

Proposition 3.

1. *$\tilde{\chi}_{K/L}$ is trivial on the image of $\mathbb{Z}$ (see [LPL]), and $\tilde{\chi}_{K/L}\big(\sqrt{d'_L}\,\big) = +1$.*
2. *Assume that $p \equiv 1 \pmod 4$ is totally ramified in $M/\mathbb{Q}$ and $L/\mathbb{Q}(\sqrt{p})$. If a prime $q \neq p$ divides $f_{K/L}$ then $\chi_L(q) = +1$.*
3. *For any $n \in \mathbb{Z}$ we have $\psi_{\mathcal{D}'}(n) = \chi_L(n)$ and $\psi_{(f'_{M/L})}(n) = +1$. Therefore, the characters $\chi_{(f)}$ and $\psi_{(f'_{M/L})}$ in Proposition 2 are trivial on the image of $\mathbb{Z}$, $\varepsilon_\chi/\varepsilon_\psi = \chi_L\big(f/f'_{M/L}\big) = \chi_L\big(f_{K/L}/\gcd(f_{K/L}, f_M)\big)$ in case 1 of Proposition 2. and $\varepsilon_\chi = \chi_{(p)}(f) = \big(\frac{f}{p}\big) = \chi_L(f) = \chi_L\big(f_{K/L}/p\big)\chi_L\big(f_M/\gcd(f_{K/L}, f_M)\big) = \chi_L\big(f_M/\gcd(f_{K/L}, f_M)\big)$ in case 2 of Proposition 2.*
4. *Let $\psi_{M/L}$ be the non-trivial Hecke character associated with a quadratic extension M/L. Then $W\big(\psi_{M/L}\big) = +1$.*
5. *Let $f = \prod_{i=1}^{r} f_i$ be a factorization of $f \geq 1$ into a product of rational integers $f_i \geq 1$ relatively prime in pairs. Let ϕ be a character on $\big(A_L/(f)\big)^*$ and let $\phi = \prod_{i=1}^{r} \phi_i$ be its associated factorization. Set $\varepsilon = \prod_{i=1}^{r} \phi_i(f/f_i)$. Then $W_{(f)}\big(1/f\sqrt{d_L},\, \phi\big) = \varepsilon \prod_{i=1}^{r} W_{(f_i)}\big(1/f_i\sqrt{d_L},\, \phi_i\big)$. Notice that $\varepsilon = +1$ and that all the ϕ_i are trivial on the image of $\mathbb{Z}$ whenever ϕ is trivial on the image of $\mathbb{Z}$.*

Proof.

1. The order of $\tilde{\chi}_{K/L}$ divides p and $\tilde{\chi}_{K/L}$ is trivial on the image of $\mathbb{Z}$. Hence, $\tilde{\chi}_{K/L}\big(\sqrt{d'_L}\,\big)$ is a root of unity of order dividing p and 2, hence is equal to $+1$.
2. Notice that if a prime $q \neq p$ divides $f_{K/L}$ then $q \equiv \chi_L(q) \pmod p$ and $p = d_L \equiv 1 \pmod 4$ imply $\chi_L(q) = +1$.
3. Let $q \geq 2$ be a prime such that $\gcd\big((q), \mathcal{F}_{M/L}\big) = 1$. If q is inert in $L/\mathbb{Q}$ then q is inert in the cyclic quartic extension $M/\mathbb{Q}$, hence $\chi_L(q) = -1 = \psi_{M/L}\big((q)\big) = \tilde{\psi}_{M/L}(q)$. If $(q) = \mathcal{Q}\mathcal{Q}'$ splits in $L/\mathbb{Q}$ then $\psi_{M/L}(\mathcal{Q}) = \psi_{M/L}(\mathcal{Q}') = \pm 1$, hence $\chi_L(q) = +1 = \psi_{M/L}(\mathcal{Q})\psi_{M/L}(\mathcal{Q}') = \psi_{M/L}\big((q)\big) =$

$\tilde{\psi}_{M/L}(q)$. Therefore, the restriction of $\tilde{\psi}_{M/L}$ to $\mathbb{Z}$ is induced by the restriction of χ_L to $\mathbb{Z}$, and we get the desired results.

4. $W(\psi_{M/L})$ is the constant which appears in the functional equation of the ratio ζ_M/ζ_L of Dedekind zeta functions.

5. Set $\gamma = 1/f\sqrt{d_L}$, let $\gamma_i = \Gamma_i/f_i\sqrt{d_L} \in (f_i)^{-1}\mathcal{D}^{-1}$ satisfy $\gamma - \gamma_i \in (f/f_i)^{-1}\mathcal{D}^{-1}$. We have $1 - \Gamma_i(f/f_i) \in (f_i)$. Hence, $\overline{\phi_i(\Gamma_i)} = \phi_i(f/f_i)$. According to (2)–(3) we get the desired result. □

According to Propositions 2 and 3, the computation of $W(\chi_{N/L})$ is reduced to the computation (1) of $W_{(q^n)}(1/q\sqrt{d_L}, \phi)$ for unramified in L primes q and primitive characters ϕ on $(A_L/(q^n))^*$ which are trivial on the image of $\mathbb{Z}$ in this group, and (2) of $W_{(p)}(1/p\sqrt{d_L}, \phi)$ where ϕ is a primitive character of order $2p$ on $(A_L/(p))^*$ and where p is ramified in L.

3. Computation of some Gaussian sums

Lemma 4. *Let $q \geq 2$ be a prime which is inert in L. Let ϕ be a primitive character on $(A_L/(q^n))^*$ which is trivial on the image of $\mathbb{Z}$. Then $W_{(q^n)}(1/q^n\sqrt{d_L}, \phi) = 1$.*

Proof. Set $\omega_L = (d_L + \sqrt{d_L})/2$ and $e(z) = e^{2\pi i z}$. Since q is inert, then

$$\begin{aligned}&\{\lambda = y(x+\omega_L)\colon 1 \leq y < q^n,\ y \not\equiv 0 \pmod q,\ 0 \leq x < q^n\}\\ &\quad \cup \{\lambda = x(1+qy\omega_L)\colon 1 \leq x < q^n,\ x \not\equiv 0 \pmod q,\ 0 \leq y < q^{n-1}\}\end{aligned}$$

is a set of representatives of $(A_L/(q^n))^*$. Since ϕ is trivial on the image of $\mathbb{Z}$, we get

$$\begin{aligned}\tau_{(q^n)}(1/q^n\sqrt{d_L}, \phi) &= \sum_{\lambda \pmod{(q)}} \phi(\lambda)\, e\Big(\mathrm{Tr}_{L/\mathbb{Q}}(\lambda/q^n\sqrt{d_L})\Big)\\ &= \sum_{\substack{1\leq y\leq q^n-1\\ y\not\equiv 0 \pmod q}} \left(\sum_{x=0}^{q^n-1} \phi(x+\omega_L)\right) e(y/q^n)\\ &\quad + \sum_{y=0}^{q^{n-1}-1} \left(\sum_{\substack{1\leq x\leq q^n-1\\ x\not\equiv 0 \pmod q}} e(xy/q^{n-1})\right) \phi(1+qy\omega_L)\,.\end{aligned}$$

Since

$$\sum_{\substack{1\leq x\leq q^n-1\\ x\not\equiv 0 \pmod q}} e\left(xy/q^{n-1}\right) = \begin{cases} 0 & \text{if} \quad y\not\equiv 0 \pmod{q^{n-2}}, \\ -q^{n-1} & \text{if} \quad y\equiv 0 \pmod{q^{n-2}}, \\ & \text{but } y\not\equiv 0 \pmod{q^{n-1}}, \\ q^n-q^{n-1} & \text{if} \quad y\equiv 0 \pmod{q^{n-1}} \end{cases}$$

and since $0=\sum_{y=0}^{q-1}\phi\left(1+q^{n-1}y\omega_L\right)$ (for ϕ is primitive), we obtain

$$\begin{aligned}\tau_{(q^n)}\left(1/q^n\sqrt{d_L},\,\phi\right) &= \sum_{\substack{1\leq y\leq q^n-1\\ y\not\equiv 0 \pmod q}}\left(\sum_{x=0}^{q^n-1}\phi\left(x+\omega_L\right)\right)e\left(y/q^n\right)\\ &\quad+\left(q^n-q^{n-1}\right)-q^{n-1}\sum_{y=1}^{q-1}\phi\left(1+q^{n-1}y\omega_L\right)\\ &= \sum_{\substack{1\leq y\leq q^n-1\\ y\not\equiv 0 \pmod q}}\left(\sum_{x=0}^{q^n-1}\phi\left(x+\omega_L\right)\right)e\left(y/q^n\right)\ +q^n\,.\end{aligned}$$

On the other hand, and by the same token, we obtain

$$\begin{aligned}0=\sum_{\lambda \pmod{(q)}}\phi(\lambda) &= \sum_{\substack{1\leq y\leq q^n-1\\ y\not\equiv 0 \pmod q}}\left(\sum_{x=0}^{q^n-1}\phi\left(x+\omega_L\right)\right)\\ &\quad+\sum_{y=0}^{q^{n-1}-1}\left(\sum_{\substack{1\leq x\leq q^n-1\\ x\not\equiv 0 \pmod q}}1\right)\phi\left(1+qy\omega_L\right)\\ &= \left(q^n-q^{n-1}\right)\sum_{x=0}^{q^n-1}\phi\left(x+\omega_L\right)\end{aligned}$$

(for ϕ primitive implies $0=\sum_{y=0}^{q^{n-1}-1}\phi(1+qy\omega_L)$). Therefore, we obtain $0=\sum_{x=0}^{q^n-1}\phi(x+\omega_L)$ and $\tau_{(q^n)}\left(1/q^n\sqrt{d_L},\,\phi\right)=q^n$. ☐

Lemma 5. *Let $q\geq 2$ be a prime which splits in L. Let ϕ be a primitive character on $\left(A_L/(q^n)\right)^*$ which is trivial on the image of $\mathbb{Z}$. Then $W_{(q^n)}\left(1/q^n\sqrt{d_L},\,\phi\right)=1$.*

P r o o f. Let α' denote the conjugate of α in L. Write $(q)=\mathcal{Q}\mathcal{Q}'$ and let $\phi=\psi\psi'$ be the associated factorization of ϕ. Let $\alpha_q\in A_L$ satisfy $\alpha_q\equiv 1$

(mod $\mathcal{Q}^n$) and $\alpha_q \equiv 0$ (mod ${\mathcal{Q}'}^n$). Then $\alpha'_q \equiv 0$ (mod $\mathcal{Q}^n$) and $\alpha'_q \equiv 1$ (mod ${\mathcal{Q}'}^n$) and for any $\lambda \in A_L$ we have $\psi(\lambda) = \phi(\alpha_q\lambda+\alpha'_q)$ and $\psi'(\lambda) = \phi(\alpha_q + \alpha'_q\lambda) = \phi((\alpha_q\lambda' + \alpha'_q)')$. Since ϕ is trivial on the image of $\mathbb{Z}$ we get $\psi'(\lambda) = \overline{\phi(\alpha_q\lambda' + \alpha'_q)} = \overline{\psi(\lambda')}$. Therefore, for λ_1 and λ_2 in A_L we have $\phi(\alpha_q\lambda_1 + \alpha'_q\lambda_2) = \psi(\lambda_1)\psi'(\lambda_2) = \psi(\lambda_1)\overline{\psi(\lambda'_2)}$ and $\mathrm{Tr}_{L/\mathbb{Q}}((\alpha_q\lambda_1 + \alpha'_q\lambda_2)/q\sqrt{d_L}) = \mathrm{Tr}_{L/\mathbb{Q}}(\alpha_q\lambda_1/q\sqrt{d_L}) + \mathrm{Tr}_{L/\mathbb{Q}}(-\alpha_q\lambda'_2/q\sqrt{d_L})$, and we do get

$$\begin{aligned}\tau_{(q^n)}(1/q^n\sqrt{d_L}, \phi) &= \tau_{\mathcal{Q}^n}(\alpha_q/q^n\sqrt{d_L}, \psi)\tau_{\mathcal{Q}^n}(-\alpha_q/q^n\sqrt{d_L}, \bar{\psi}) \\ &= \left|\tau_{\mathcal{Q}^n}(\alpha_q/q^n\sqrt{d_L}, \psi)\right|^2 = q^n .\end{aligned}$$

□

Lemma 6. *Assume that $p \geq 3$ is ramified in L. Let ϕ be a primitive character on $(A_L/(p))^*$. Let n_ϕ be any rational integer satisfying $\phi(1 + \sqrt{d_L}) = \exp(-4n_\phi\pi i/p)$. Then $W_{(p)}(1/p\sqrt{d_L}, \phi) = \phi(n_\phi)$. In particular, for any t coprime with p we get $W_{(p)}(1/p\sqrt{d_L}, \phi^t) = \phi(t)W_{(p)}(1/p\sqrt{d_L}, \phi)$. Finally, if the order of ϕ is equal to $2p$, then $\phi(n_\phi) = \left(\frac{n_\phi}{p}\right)$ and $\phi(t) = \left(\frac{t}{p}\right)$.*

Proof. Since $\{\lambda = x(1 + y\sqrt{d_L}) : 1 \leq x < p,\ 0 \leq y < p\}$ is a set of representatives of $(A_L/(p))^*$, we have

$$\tau_{(p)}(1/p\sqrt{d_L}, \phi) = \sum_{x=1}^{p-1}\phi(x)\left(\sum_{y=0}^{p-1}\phi(1 + y\sqrt{d_L})e^{4xy\pi i/p}\right).$$

Since $y \mapsto \lambda_x(y) = \phi(1+y\sqrt{d_L})e^{4xy\pi i/p}$ is clearly a morphism from the additive group $\mathbb{Z}/p\mathbb{Z}$ into $\mathbb{C}^*$ which is trivial if and only if $x \equiv n_\phi$ (mod p), we have

$$\sum_{y=0}^{p-1}\lambda_x(y) = \begin{cases} p & \text{if } x = n_\phi, \\ 0 & \text{otherwise}, \end{cases}$$

and we do get $\tau_{(p)}(1/p\sqrt{d_L}, \phi) = p\phi(n_\phi)$.

As for the last assertion, it follows from the fact that since $(A_L/(p))^*$ is cyclic of order $p(p-1)$ then the restriction of ϕ on the image of $\mathbb{Z}$ is a quadratic character modulo p, hence $\phi(n) = \left(\frac{n}{p}\right)$ for any $n \in \mathbb{Z}$. □

4. Formulae for the Artin root number $W(\chi)$

Putting everything together we end up with:

THEOREM 7. *Let N be dicyclic field of degree $4p \geq 12$ and let χ denote any of the $p-1$ characters on* $\mathrm{Gal}(N/L)$ *of order* $2p$.

(1) *If p is not totally ramified in $N/\mathbb{Q}$ then*

$$W(\chi) = \chi_L\big(f_{K/L}/\gcd(f_{K/L}, f_M)\big) \stackrel{\text{def}}{=} W_N$$

does not depend on χ.

(2) *If $p \equiv 1 \pmod 4$ is totally ramified in $N/\mathbb{Q}$ and $L = \mathbb{Q}(\sqrt{p})$, then*

$$W(\chi) = \left(\frac{n_{2p}}{p}\right)\chi_L\big(f_M/\gcd(f_{K/L}, f_M)\big).$$

Here, n_{2p} is defined by means of $\phi_{2p}(1+\sqrt{p}) = \exp(-4n_{2p}\pi i/p)$ where ϕ_{2p} denotes the component of $\tilde{\chi}$ on $\big(A_L/(p)\big)^*$. Hence, $\gcd(t, 2p) = 1$ implies

$$W(\chi^t) = \left(\frac{t}{p}\right)W(\chi),$$

and $W(\chi)$ depends on χ.

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