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TATRA MOUNTAINS
Mathematical Publications

PROBLEM SESSION

1. Let $n = 2m$ be an even positive integer, $\mathbb{Z}_n$ the cyclic group of order n and A any subset of $\mathbb{Z}_n$ of cardinality $|A| = m + 1$. Let

$$2 * A = \{a + b;\, a \in A,\, b \in A,\, a \neq b\},$$
$$3 * A = \{a + b + c;\, a \in A,\, b \in A,\, c \in A,\, a \neq b \neq c \neq a\}.$$

Do the following assertions hold for each subset A of $\mathbb{Z}_n$ of cardinality $m + 1$ and for all sufficiently large n or for infinitely many values of n?

(1) $3 * A = \mathbb{Z}_n,\ (n \geq 12)$;

(2) $\left|\mathbb{Z}_n \setminus (2 * A)\right| \leq 2,\ (n \geq 2)$;

(3) $(2 * A) \cup (3 * A) = \mathbb{Z}_n,\ (n \geq 6)$.

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2. Let K be an algebraic number field, and $f(X)$ a monic polynomial of degree $m \geq 3$ with integral coefficients in K which is irreducible over K. Denote by N the number of $x \in \mathbb{Z}$ for which $f(x)$ is a unit in K. It follows from a result of J. H. Evertse (1997) that

$$N \leq (5 \cdot 10^6 \cdot m)^{r+2}$$

where r denotes the unit rank of K.

Conjecture: If m is sufficiently large, then

$$N < 2m\,. \qquad (*)$$

Remark. $(*)$ would yield an important result concerning irreducible polynomials.

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3. We say that the algebraic integers α, α' are $\mathbb{Z}$-equivalent if $\alpha' - \alpha \in \mathbb{Z}$. In this case $\mathbb{Z}[\alpha'] = \mathbb{Z}[\alpha]$. Denote by O_K the ring of integers of a number field K. Further, for given $n \geq 3$, let $\mathcal{P}(n)$ denote the maximal number of pairwise $\mathbb{Z}$-equivalent $\alpha \in O_K$ with $O_K = \mathbb{Z}[x]$, when K runs through the number fields K of degree n. For $\mathcal{P}(n)$ the best known upper bound, due to G y ö r y (2000), is

$$\mathcal{P}(n) \leq 2^{4n(n-1)+9}.$$

Problem: Does there exist an upper bound for $\mathcal{P}(n)$ which is polynomial in terms of n?

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4. For any real number x let $\|x\| = \min_{n \in \mathbb{Z}} |x - n|$.

Problem: Find the maximum value of the constant C such that the following holds: If $b > 1$ is such that there exists $n_0 \in \mathbb{N}$ with

$$\|b^n\| < \frac{C}{b^{\frac{n+3}{2}}} \qquad \text{for all } n > n_0,$$

then b is a quadratic unit.
Together with Andreas Dress we showed that

$$C \geq \frac{1}{4\sqrt{2}} \qquad \text{but} \quad C < 10.$$

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5. For $n \geq 3$ let $L(n)$ be the smallest integer l such that there exists an algebraic number field K of degree l, and a monic polynomial $f \in \mathbb{Z}[x]$ having a cycle of length n in the field K.

Prove or disprove $L(n) \leq n$.

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6. Let $d > 0$ be a square free integer, $K = \mathbb{Q}(\sqrt{d}), h(d) =$ class number of K, $\frac{T+U\sqrt{d}}{2} =$ fundamental unit of K.

I. Are there infinitely many d for which d divides U? Examples $d = 46, 430, 1817$ and 4 more, less than 10^7.

II. Is there an $\alpha > 0$ such that $\big(d, h(d)\big) < d^{\frac{1}{2}-\alpha}$ for all d?

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