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PARTIAL SUMS OF THE HARMONIC SERIES, p-ADIC L-FUNCTIONS AND BERNOULLI NUMBERS

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ABSTRACT. The divisibility properties of partial sums of the harmonic series (and allied sums) are studied with the help of Bernoulli numbers and the p-adic L-functions. The paper contains some results generalizing the well-known Wolstenholme–Leudesdorf's Theorem and recent investigations of D. Boyd, L. C. Washington and the author.

0. Let p be an odd prime and $l, m, n, s \in \mathbb{N}$. We will study some classical objects — the sums

$$H(n,s) = \sum_{i=1}^{n} i^{-s}, \quad W(m,s) = \sum_{i=1,(i,m)=1}^{m-1} i^{-s} \quad \text{and} \quad P(n,p,s) = \sum_{i=1,(i,p)=1}^{np} i^{-s}.$$

There are many obvious connections between these sums. For example,

$$H(np,s) - p^{-s}H(n,s) = P(n,p,s), \tag{1}$$

the relation useful (in the case $s = 1$) for the studying the set

$$\mathbf{J}_p = \{n \colon H(n,1) \equiv 0 \pmod p\}$$

with a fixed prime p. It is known that the integers $p-1$, p^2-p, p^2-1 belong to the set $\mathbf{J}_p$. This fact is a simple consequence of our results [4], but for the first time it was proved in [2]. Moreover, A. Eswarathasan and E. Levine showed that for certain primes such as $p = 5, 13, 17, 23$ and 67 there are no other elements in $\mathbf{J}_p$. Further, D. W. Boyd [1] conjectured that these numbers p (so-called "harmonic primes") have a density $1/e$ (in the set of all primes).

An interesting article of L. C. Washington [6] is extending these investigations to the case $s \geq 1$. The author gives explicit p-adic expansions of the

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sums $P(n,p,s)$ as power series in n. The coefficients of the expansions are values of p-adic L-functions (for more on the history of the question see [1] or [6]).

The present paper contains some results concerning both the classical Wolstenholme–Leudesdorf's Theorem and the mentioned recent investigations of D. Boyd, L. C. Washington and the author [4].

1. Almost all known generalizations of the Wolstenholme's Theorem are contained in

Theorem 1. ([4]) *If* $u=(\varphi(p^l)-1)s$ *and* $(m,6)=1$ *then*

$$W(m,s)\equiv\begin{cases} m\prod\limits_{p|m}(1-p^{u-1})B_u \pmod{m^2} & \text{for } 2|s\,,\\ (u/2)m\prod\limits_{p|m}(1-p^{u-2})B_{u-1} \pmod{m^2} & \textit{otherwise}, \end{cases}$$

with the product taken over all prime divisors p *of* m. *Here* B_m *are Bernoulli numbers (B.n.) in the "even suffix" notation, i.e.,* $B_0=1$, $B_1=-1/2$, $B_2=1/6$, $B_{2n+1}=0$ *for* $n\ge 1$.

Further, the result is added by

Theorem 2. *Let* $p\ge 5$ *be an odd prime number and* $\varphi(x)$ *be the Euler totient-function. For the fixed natural* s *we denote* $t=\varphi(p^l)-s$ *with* $l\le\varphi(p^l)-s$. *Then*

$$S(m,p,s):=\sum_{i=1,\,(i,p)=1}^{m} i^{-s}\equiv\{B_{t+1}(m+1)-B_{t+1}\}/(t+1)\pmod{p^l}$$

and

$$P(n,p,s)\equiv\{B_{t+1}(np)-B_{t+1}\}/(t+1)\pmod{p^l}. \tag{2}$$

Proof. Fermat–Euler's Theorem gives $i^t\equiv i^{-s}\pmod{p^l}$, $(i,p)=1$. Noting that $i^t\equiv 0\pmod{p^l}$ for $p|i$, by Bernoulli's formula

$$\sum_{i=1}^{x-1} i^m=\{B_{m+1}(x)-B_{m+1}\}/(m+1)\,,\qquad m,x\in\mathbb{N}\,,$$

we obtain

$$S(m,p,s)\equiv\sum_{i=1}^{m} i^t\equiv\{B_{t+1}(m+1)-B_{t+1}\}/(t+1)\pmod{p^l}$$

with $l\le t$, i.e., $s\le\varphi(p^l)-l$.

To obtain the congruence (2) we note that

$$P(n,p,s) = S(np,p,s) = S(np-1,p,s)$$

and

$$S(np-1,p,s) \equiv \{B_{t+1}(np) - B_{t+1}\}/(t+1) \pmod{p^l}.$$

Hence, the congruence (2) was proved. □

Remarks. 1). Note that the last congruence obtained for sums of negative powers of naturals may be treated as analog of the mentioned classical formula of Jacob Bernoulli. Further, the left part of the congruence (2) is independent of l, so that we obtain the p-adic successive approximation of $P(n,p,s)$.

2). The congruence (2) yields

$$P(n,p,s) \equiv \sum_{i=1}^{t} \binom{t}{i-1} (np)^i B_{t+1-i}/i \pmod{p^l},$$

where $\binom{n}{k}$ is a binomial coefficient. Hence, in the case $s=1$, i.e., $t = \varphi(p^l) - l$, it follows that $B_t = 0$, $\mathrm{ord}_p B_{t-1} \geq 0$ (Staudt–Clausen's Theorem! See, e.g., [5]) and

$$P(n,p,1) \equiv (t/2)(np)^2 B_{t-1} \pmod{p^3}.$$

In particular, we conclude that

$$P(n,p,1) \equiv 0 \pmod{p^2}. \tag{3}$$

3). Further, $P(1,p,s) = W(p,s)$, so that it follows

$$W(p,s) \equiv \sum_{i=1}^{t} \binom{t}{i-1} B_{t+1-i}\, p^i/i \pmod{p^l}. \tag{4}$$

The congruence is an extension of Wolstenholme's one

$$W(p,1) \equiv 0 \pmod{p^2},$$

because in the case $s=1$ the congruence (4) yields

$$W(p,1) \equiv tp^2 B_{t-1}/2 \pmod{p^2}$$

and by Staudt–Clausen Theorem $\mathrm{ord}_p B_{t-1} \geq 0$ for $t-1 = \varphi(p^l) - 2$.

Now we can propose a simple proof of Eswarathasan–Levine's Theorem.

THEOREM 3. ([2]). *For any prime $p>3$ the numbers $p-1$, p^2-p, p^2-1 belong to the set* $\mathbf{J}_p$.

P r o o f. (i) $p-1\in\mathbf{J}_p$ because by the congruence (3) we have

$$H(p-1,1)=P(1,p,1)\equiv 0 \pmod{p^2} \tag{5}$$

(ii) $p^2-p\in\mathbf{J}_p$. Indeed, by the identity (1) we find that

$$H(p^2-p,1)=p^{-1}H(p-1,1)+P(p-1,p,1)\,.$$

Then with the help of the congruences (3) and (5) we obtain

$$H(p^2-p,1)=\bigcirc(p)+\bigcirc(p^2)=\bigcirc(p)\,.$$

Here we use the standard notation $\bigcirc(p^k)$ for any integer x for which $\mathrm{ord}_p\, x\geq k$.

(iii) At last, $p^2-1\in\mathbf{J}_p$, because $H(p^2-1,1)=H(p^2,1)-p^{-2}=p^{-1}H(p,1)-p^{-2}+P(p,p,1)=p^{-1}H(p-1,1)+P(p,p,1)=\bigcirc(p)+\bigcirc(p^2)=\bigcirc(p)$.

2. As it is known [3], for the first time the p-adic L-functions were constructed by means of Kummer's congruences for B.n. But there are many papers which study relations between B.n. with the help of the p-adic L-functions. Nevertheless, it is natural to bring direct proofs of these results. In particular, the note is inspired by the mentioned Washington's paper [6] studying connections between p-adic L-functions and partial sums of harmonic series.This paper also contained some useful corollaries for B.n. which we will below consider by elementary methods.

To obtain these corollaries we use the congruence (4) with $l=3$ or 4 and consider the following cases:

I. First, let s be an odd number, so that t is an odd number too, and, in particular, $B_t=B_{t-2}=0$.

$\mathbf{I}_1$. Setting $l=3$ by the congruence (4) we have

$$W(p,s)\equiv tp^2B_{t-1}/2 \pmod{p^3}\,. \tag{6}$$

The congruence (6) is a simple consequence of (4). Indeed, since the von Staudt–Clausen Theorem implies that at most p^1 divides the denominators of the Bernoulli numbers, we can conclude that $\mathrm{ord}_p(B_{t+1-i}\,p^{i-3}/i)\geq 0$ for $i\geq 4$ and $p\geq 3$.

(a) If $1\leq s\leq p-4$, then the congruence (6) yields

$$W(p,s)\equiv -sB_{p^2(p-1)-s-1}p^2/2 \pmod{p^3}\,.$$

By the binary Kummer's congruence

$$B_{p^2(p-1)-s-1} \equiv \big((p^2-1)(p-1)+p-s-2\big)B_{p-s-2}/(p-s-2) \pmod{p}$$

we have

$$W(p,s) \equiv -B_{p-s-2}\,p^2\,s(s+1)/\big(2(s+2)\big) \pmod{p^3}, \quad p \ge s+4,$$

and Corollary 1 ([6]) is obtained.

(b) For an odd prime number $p \ge 3$ and $s = p-2$ we have $t-1 = (p-1)(p^2-1)$. Then from the congruence (6) it follows

$$W(p,p-2) \equiv p^2\big(p^2(p-1)-p+2\big)B_{t-1}/2 \pmod{p^3}$$

or

$$W(p,p-2) \equiv p^2/2 + p^2\,B_{t-1} \pmod{p^3}.$$

In particular, by the von Staudt–Clausen Theorem we obtain Corollary 3 ([6])

$$W(p,p-2) \equiv -p \pmod{p^2}, \qquad p \ge 3.$$

$\mathbf{I}_2$. In the case $l > 3$ the congruence (4) yields

$$W(p,s) \equiv tp^2\,B_{t-1}/2 + p^4\,t(t-1)(t-2)\,B_{t-3}/24 \pmod{p^4},$$

because $\mathrm{ord}_p\,(B_{t+1-i}\,p^{i-4}/i) \ge 0$ for $i \ge 6$. Noting that $t-3 = \varphi(p^4)-s-3 \equiv -(s+3) \pmod{(p-1)}$, i.e., $p-1$ does not divide $t-3$ for $1 \le s \le p-6$, we have

$$W(p,s) \equiv tp^2\,B_{t-1}/2 \pmod{p^4}, \qquad p \ge s+6.$$

Now, noting that $t-1 = \varphi(p^4)-p(p-1)+p(p-1)-s-1$ and using Kummer's congruence

$$B_{t-1} \equiv -(s+1)\,B_{p(p-1)-s-1}/(p^2-p-s-1) \pmod{p^2},$$

we obtain

$$W(p,s) \equiv p^2\,B_{p(p-1)-s-1}s(s+1)/\big(2(p^2-p-s-1)\big) \pmod{p^4}$$

or

$$W(p,s) \equiv -p^2\,B_{p(p-1)-s-1}s(s+1)/\big(2(p+s+1)\big) \pmod{p^4}.$$

Therefore,

$$W(p,s) \equiv 0 \pmod{p^4} \iff B_{p(p-1)-s-1} \equiv 0 \pmod{p^2} \tag{7}$$

for $1 \le s \le p-6$.

Corollary 4 ([6]) is formulated in terms of zeros of the p-adic L-functions and the Teichmüller character ω (see, e.g., [5]). For an odd integer s and $p > s+5$ the first part of the Corollary 4 can be written in the form:

A p-adic L-function belonging to the character ω^{-1-s} has a zero β with

$$\beta \equiv s+2 \pmod p \implies W(p,s) \equiv 0 \pmod{p^4}.$$

Hence, returning to the equivalence (7) we obtain

A p-adic L-function belonging to the character ω^{-1-s} has a zero β with

$$\beta \equiv s+2 \pmod p \implies B_{p(p-1)-s-1} \equiv 0 \pmod{p^2}.$$

II. Let s be an even integer and $2 \le s \le p-5$. Taking $l=4$ and turning to the congruence (4), we conclude that

$$W(p,s) \equiv p\,B_t + t(t-1)p^3\,B_{t-2}/6 \pmod{p^4},\ p>5.$$

Indeed, $B_{t-1} = B_{t-3} = 0$ and $\mathrm{ord}_p(B_{t+1-i}\,p^{i-4}/i) \ge 0$ for $i \ge 5$. So by the Kummer's congruence

$$B_{t-2} \equiv (s+2)\,B_{p-s-3}/(s+3) \pmod p,$$

where $t-2 = \varphi(p^4) - s - 2 \equiv p-s-3 \pmod{(p-1)}$, i.e., $p-1$ is not a divisor of $t-2$, it follows that

$$W(p,s) \equiv p\,B_t + s(s+1)(s+2)p^3\,B_{p-s-2}/\big(6(s+3)\big) \pmod{p^4}$$

or

$$W(p,s) \equiv p\,B_t \pmod{p^3}.$$

Now, since

$$B_t \equiv tB_{p(p-1)-s}/(p+s) \pmod{p^2}$$

with $t = \varphi(p^4) - p(p-1) + p(p-1) - s$, we have

$$W(p,s) \equiv 0 \pmod{p^3} \iff B_{p(p-1)-s} \equiv 0 \pmod{p^2}. \tag{8}$$

For an even integer s with $s+4<p$, the second part of the Corollary 4 ([6]) can be given in the form:

A p-adic L-function belonging to the character ω^{-s} has a zero β

$$\text{with } \beta \equiv s+1 \pmod p \implies W(p,s) \equiv 0 \pmod{p^3}.$$

Therefore, with the help of the equivalence (8) we conclude that

A p-adic L-function belonging to the character ω^{-s} has a zero β

$$\text{with } \beta \equiv s+1 \pmod p \implies B_{p(p-1)-s} \equiv 0 \pmod{p^2}.$$

III. At last, let $l=2$ and $2 \le s \le p-3$, $2|s$. As above, turning again to the congruence (4), we have $\mathrm{ord}_p(B_{t+1-i}\,p^{i-2}/i) \ge 0$ for $i \ge 3$, so that

$W(p,s) \equiv p\,B_t \pmod{p^2}$ with $t = \varphi(p^2) - s$. Finally, by a suitable Kummer congruence we obtain

$$W(p,s) \equiv ps\,B_{p-s-1}/(s+1) \pmod{p^2}, \qquad p > s+2,$$

the Corollary 2 of the paper [6]. □

Remark. In a similar way it is also possible to consider sums of the form

$$\sum_{i=1,(i,p)=1}^{nF} \chi(i)\, i^{\pm s}$$

where χ is a Dirichlet character and F is any multiple of p and the conductor of χ.

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