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STABILITY CLASSES OF SECOND-ORDER LINEAR RECURRENCES MODULO 2^k

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ABSTRACT. We classify the 2^k-blocks of second-order recurrence sequences with parameter $b \equiv 3 \pmod 4$ and identify stability classes modulo 2.

1. Introduction

Let a and b be two fixed integers. Each pair of integers w_0 and w_1 determines a second-order recurrence $w(a,b)$ such that, for $i \geq 2$,

$$w_i = aw_{i-1} + bw_{i-2}\,. \tag{1.1}$$

It is well known that if m is an integer and $(b,m) = 1$, then the corresponding sequence $(\overline{w}_i)$ of residues modulo m is purely periodic. There remain many interesting open questions concerning the reduced sequences $(\overline{w}_i)$, among them the computation of the period and the distribution of frequencies of residues.

In recent years significant progress has been made in understanding the frequency distributions of residues of second-order recurrences when the modulus m is a power of a prime. Considerable work has concerned the key concept of sequence *stability*, first introduced by Eliot Jacobson in [8]. Suppose that $m = p^k$ is a power of a prime p. Define $\lambda = \lambda_w(p^k)$ to be the (least) period of $(\overline{w}_i)$ and, for each residue d modulo p^k, $\nu_w(d,p^k)$ to be the number of times that the residue d appears in a single cycle of the recurrence $(\overline{w}_i)$. Let $\Omega_w(p^k)$ be the image of the frequency distribution function $\nu_w(d,p^k)$, i.e.,

$$\Omega_w(p^k) = \{\nu_w(d,p^k) \colon d \in \mathbb{Z}\}\,.$$

In [8], Jacobson observed that when $w(a,b)$ is the Fibonacci sequence, the sets $\Omega_w(2^k)$ are eventually constant as a function of k. He codified this property with the following definition.

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Definition 1.1. A sequence (w) is *stable modulo* p, or simply p*-stable*, if there is a positive integer N such that $\Omega_w(p^k) = \Omega_w(p^N)$ for all $k \ge N$.

Once a sequence is known to be p-stable, computation of its residue distribution for all powers of p requires only a finite computational procedure.

Much of the work on sequence stability has concentrated on two classes of second-order recurrences, the generalized Fibonacci and Lucas sequences $u(a,b)$ and $v(a,b)$, which obey the recursion (1.1) and have initial terms $u_0 = 0$, $u_1 = 1$ and $v_0 = 2$, $v_1 = a$, respectively. These sequences are sometimes referred to in the literature as the Lucas Sequences of the First Kind (LSFK) and the Lucas Sequences of the Second Kind (LSSK). In a series of papers [2], [3], [1], and [4], Walter Carlip and Eliot Jacobson examined the 2-stability of the generalized Fibonacci sequences $u(a,b)$, and, in [9], Mark Morgan examined the 2-stability of the Lucas sequence $v(1,1)$. In several articles Carroll, Jacobson, and Somer [7], Carlip, Jacobson, and Somer [5], and Carlip and Somer [6, 10] have studied p-stability for odd primes p. In particular, the treatise [6] was not restricted to the generalized Fibonacci and Lucas sequences, and provided a broad generalization to *blocks* of second-order recursive sequences.

In this paper, the first in a series of papers in which we intend to apply the methods of [6] and [1] to sequences modulo powers of 2, we examine the 2-stability of sequences satisfying (1.1) for which a is odd and $b \equiv 3 \pmod 4$. In particular, in § 5 and § 6, we describe the block structure of the family of regular second-order recurrences, $\mathscr{F}(a,b)$ when a is odd and $b \equiv 3 \pmod 4$. In § 8, we offer our main results, a characterization of the 2-stability of blocks in terms of parameters r, s, and t introduced in § 4. Finally, in § 9, we describe the 2-stability of blocks in terms of the defining parameters a and b for the recursion (1.1). The situation when $b \equiv 3 \pmod 8$ turns out to be particularly simple: there are exactly two blocks, one that contains the generalized Fibonacci sequence $u(a,b)$ and another that contains the generalized Lucas sequence $v(a,b)$. In this case, the 2-stability of any sequence is determined by the 2-stability of $u(a,b)$ and $v(a,b)$. When $b \equiv 7 \pmod 8$, there are always at least four blocks, and the classification is considerably richer.

It should be noted that, as is often the case, the prime 2 behaves somewhat differently than the better behaving primes $p > 2$ (hence the common rejoinder that 2 is the only "odd" prime). Consequently, the methods of [6] require some modification to apply here. Thus, for example, the parameters e and f that play a crucial role in the classification of p-blocks for odd p in [6] are not well suited to the job of classifying the 2-blocks and are replaced by the parameters r, s, and t.

2. Regular sequences

Throughout this paper we are interested in 2-regular recurrence sequences.

Definition 2.1. A second-order recurrence $w(a,b)$ is said to be 2-*regular* if

$$\begin{vmatrix} w_0 & w_1 \\ w_1 & w_2 \end{vmatrix} = w_0 w_2 - w_1^2 \not\equiv 0 \pmod{2}. \tag{2.1}$$

Equivalently, $w(a,b)$ is regular if the initial state vectors (w_0, w_1) and (w_1, w_2) are linearly independent modulo 2.

Lemma 2.2. *If a and b are odd, then $w(a,b)$ fails to be 2-regular if and only if $2 \mid (w_0, w_1)$.*

Proof. This follows immediately from (1.1) and (2.1). □

Note that this characterization is true only for $p = 2$. See [6] for a discussion of the more complicated situation when p is odd.

Definition 2.3. Let $\mathscr{F}(a,b)$ denote the set of all 2-regular recurrences $w(a,b)$.

Definition 2.4. As mentioned in the introduction, we single out two classes of second-order recurrences. Define $u = u(a,b)$ to be the second-order recurrence that obeys (1.1) and has initial terms $u_0 = 0$ and $u_1 = 1$. Similarly, define $v = v(a,b)$ to be the second-order recurrence that obeys (1.1) and has initial terms $v_0 = 2$ and $v_1 = a$.

Note that $u(a,b)$ is always 2-regular and, under our standard hypothesis that a is odd, $v(a,b)$ is also 2-regular.

3. Periods and restricted periods

Definition 3.1. Define $\lambda_w(2^k)$ to be the *period* of $w(a,b)$ modulo 2^k, that is, the least positive integer λ such that for all n

$$w_{n+\lambda} \equiv w_n \pmod{2^k}.$$

Similarly, define $h_w(2^k)$ to be the *restricted period* of $w(a,b)$ modulo 2^k, that is, the least positive integer h such that for some integer M and for all n

$$w_{n+h} \equiv M w_n \pmod{2^k}.$$

The integer $M = M_w(2^k)$, defined up to congruence modulo 2^k, is called the *multiplier* of $w(a,b)$ modulo 2^k.

It is well known that $h_w(2^k) \mid \lambda_w(2^k)$ and that $E_w(2^k) = \lambda_w(2^k)/h_w(2^k)$ is the multiplicative order in $(\mathbb{Z}/2^k\mathbb{Z})^*$ of the multiplier $M_w(2^k)$. Moreover, if $h = h_w(2^k)$ and $M = M_w(2^k)$, then for all n

$$w_{n+ih} \equiv M^i w_n \pmod{2^k}. \tag{3.1}$$

Lemma 3.2. *For each $k \geq 1$, all recurrences $w(a,b) \in \mathscr{F}(a,b)$ have the same period and restricted period modulo 2^k.*

Proof. This follows from the regularity by a simple argument using Cramer's rule. See, e.g., [6], p. 695. □

In Theorem 4.5 and Theorem 4.8 below, we provide the periods and restricted periods modulo 2^k, for $k \geq 1$, of the regular second-order recurrences satisfying our main hypothesis that a is odd and $b \equiv 3 \pmod 4$. These depend upon the relationships among the three parameters r, s, and t introduced in the next section.

4. The parameters r, s, and t

In [1], Carlip and Jacobson introduced three parameters, r, s, and t, that are useful for classifying second-order recurrence sequences with respect to their distributions modulo powers of two. Since these parameters play an important role in our analysis here, we reproduce their definitions.

Definition 4.1.

$$2^{r+1} \parallel (bu_5 - 1), \quad 2^{t+1} \parallel u_6, \quad \text{and} \quad 2^{s+1} \parallel (u_7 - 1). \tag{4.1}$$

In rare instances, one or more of the parameters r, s, and t may not exist. Such sequences are classified in [1] and we refer the reader to the Addendum of that paper for a discussion of the distribution of $u(a,b)$ in this case. In general, the distribution of $w(a,b) \in \mathscr{F}(a,b)$ when any of r, s, or t is not defined may be handled in a similar manner.

Before proving several basic properties of these parameters we require an elementary lemma, which we often refer to as the *intertwining lemma.*

Lemma 4.2 (Intertwining Lemma). *For any nonnegative integers i and j,*

$$w_{i+j} = bw_{i-1}u_j + w_i u_{j+1}.$$

Proof. The lemma follows easily by induction on j. □

An immediate consequence of the intertwining lemma is the useful relation

$$u_{2i+1} = b(u_i)^2 + (u_{i+1})^2. \tag{4.2}$$

For much of the remainder of the paper, we fix the following main hypothesis.

Hypothesis 4.3 (Main Hypothesis). *Assume that* $w(a,b) \in \mathscr{F}(a,b)$, *and that* a *is odd and* $b \equiv 3 \pmod 4$. *Define* r, s, *and* t *as in* (4.1).

In the following lemma we collect some facts about the parameters r, s, and t, some of which were proven in [1].

Lemma 4.4. *Assume Hypothesis* (4.3). *Then:*

(a) $2^t \parallel u_3$ *and* $t \geq 2$;

(b) *if* $b \equiv 3 \pmod 8$, *then* $t = 2$;

(c) *if* $b \equiv 7 \pmod 8$, *then* $t \geq 3$;

(d) $r \geq 2$ *and* $s \geq 2$;

(e) $s < t$ *if and only if* $r = s$;

(f) $s > t$ *if and only if* $r = t$; *and*

(g) $s = t$ *if and only if* $r > s$ *and* $r > t$.

Proof. (a) This is Lemma 2.1(a) of [1].

(b) Since $u_3 = a^2 + b \equiv 1 + 3 \equiv 4 \pmod 8$, the result follows from (a).

(c) Since $u_3 = a^2 + b \equiv 1 + 7 \equiv 0 \pmod 8$, the result again follows from (a).

(d) This is a slight strengthening of Lemma 2.1(b) of [1]. By (b), (c), and (4.2),

$$u_5 = b(u_2)^2 + (u_3)^2 \equiv b + 0 \pmod 8.$$

Consequently $bu_5 - 1 \equiv b^2 - 1 \equiv 0 \pmod 8$, and it follows that $r \geq 2$.

Similarly, by (b), (c), and (4.2),

$$u_7 = b(u_3)^2 + (u_4)^2 \equiv (u_4)^2 \pmod 8.$$

Since u_4 is odd, $(u_4)^2 \equiv 1 \pmod 8$, and $u_7 - 1 \equiv 0 \pmod 8$. It follows that $s \geq 2$.

(e),(f),(g) All three statements follow from Lemma 2.1(c), (d) and (e) of [1]. □

With these facts in hand, we turn to the question of computing the period and restricted period of sequences satisfying Hypothesis 4.3.

Theorem 4.5. *Suppose that* $w(a,b)$ *satisfies Hypothesis 4.3.*

(a) *If* $s < t$, *then* $\lambda_w(2^k) = 3 \cdot 2^{k-s}$ *for all* $k > s$.

(b) *If* $s \geq t$, *then* $\lambda_w(2^k) = 3 \cdot 2^{k-t}$ *for all* $k > t$.

Proof. By Lemma 3.2, it suffices to show that (a) and (b) are true for the sequence $u(a,b)$. This is done in Proposition 3.2 of [1]. □

LEMMA 4.6. *Assume Hypothesis 4.3. If m is a nonnegative integer, then* $2^{t+m} \parallel u_{3\cdot 2^m}$.

Proof. This is Lemma 2.2 of [1]. $\square$

LEMMA 4.7. *Assume Hypothesis 4.3. For all nonnegative integers m, $u_\ell \equiv 0 \pmod 2$ if and only if $3 \mid \ell$. Moreover, if $3 \mid \ell$ and $2^m \parallel \ell$, then $2^{t+m} \parallel u_\ell$.*

Proof. The first assertion follows from the observation that a complete cycle of $u(a,b)$ modulo 2 is given by $0, 1, 1$.

To prove the second assertion, write $\ell = 3 \cdot 2^m \cdot d$, with d odd, and assume that d has base two expansion

$$d = d_0 + d_1 2 + d_2 2^2 + \cdots + d_n 2^n,$$

with $d_0 \neq 0$ and $d_n \neq 0$. Proceed by induction on n. If $n = 0$, then the result is simply Lemma 4.6. Now suppose that $n \geq 1$. Then the intertwining lemma (Lemma 4.2) yields

$$\begin{aligned} u_\ell &= u_{3\cdot 2^m \cdot d} \\ &= u_{3\cdot 2^m \cdot (d_0 + d_1 2 + d_2 2^2 + \cdots + d_n 2^n)} \\ &= u_{3\cdot 2^m + 3\cdot 2^{m+1} \cdot (d_1 + d_2 2 + \cdots + d_n 2^{n-1})} \\ &= u_{3\cdot 2^m + 1} u_{3\cdot 2^{m+1} \cdot (d_1 + d_2 2 + \cdots + d_n 2^{n-1})} + b u_{3\cdot 2^m} u_{3\cdot 2^{m+1} \cdot (d_1 + d_2 2 + \cdots + d_n 2^{n-1}) - 1}. \end{aligned}$$

By the induction hypothesis and the first assertion of the theorem, $2^{t+m+1} \parallel u_{3\cdot 2^m+1} u_{3\cdot 2^{m+1}\cdot(d_1+\cdots+d_n 2^{n-1})}$, while $2^{t+m} \parallel b u_{3\cdot 2^m} u_{3\cdot 2^{m+1}\cdot(d_1+\cdots+d_n 2^{n-1})-1}$. It follows that $2^{t+m} \parallel u_\ell$, as desired. $\square$

THEOREM 4.8. *Assume Hypothesis 4.3. Then $h_w(2^k) = 3 \cdot 2^{k-t}$ for all $k > t$, and $h_w(2^k) = 3$ when $k \leq t$.*

Proof. By Lemma 3.2, it suffices to prove the theorem for $u(a,b)$. Moreover, since $u_0 = 0$, the restricted period of $u(a,b)$ modulo 2^k is simply the rank of appearance of 2^k, i.e., the least index $m > 0$ such that $2^k \mid u_m$.

Since, by Lemma 4.4(a), $2^t \mid u_3$, we see that $h_w(2^k) = 3$ when $k \leq t$. On the other hand, if $k > t$, Lemma 4.7 implies that the first term of $u(a,b)$ that is divisible by 2^k is $u_{3\cdot 2^{k-t}}$, as desired. $\square$

Finally, we offer the following lemmas from [1], along with two new lemmas of the same flavor.

LEMMA 4.9. *Assume Hypothesis 4.3. If $s < 2t$ and $k > s$, then $u_{3\cdot 2^{k-s}+1} \equiv 1 + 2^k \pmod{2^{k+1}}$.*

Proof. This is Lemma 2.3 of [1]. $\square$

Lemma 4.10. *Assume Hypothesis 4.3. If* $r < 2t$ *and* $k > r$*, then* $bu_{3\cdot 2^{k-r}-1} \equiv 1 + 2^k \pmod{2^{k+1}}$.

Proof. This is Lemma 2.5 of [1]. □

Lemma 4.11. *Assume Hypothesis 4.3. If* $n \geq 1$ *and* $s \geq t$*, then* $u_{3\cdot 2^n+1} \equiv 1 \pmod{2^{t+n}}$.

Proof. This is Lemma 2.7 of [1]. □

Lemma 4.12. *Assume Hypothesis 4.3. If* $s \geq 2t$ *and* $k \geq 1$*, then* $u_{3\cdot 2^k+1} \equiv 1 \pmod{2^{2t+k}}$.

Proof. Proceed by induction on k. By definition of s, the result is true when $k = 1$. Suppose that $k > 1$ and $u_{3\cdot 2^{k-1}+1} \equiv 1 \pmod{2^{2t+k-1}}$. Then $(u_{3\cdot 2^{k-1}+1})^2 \equiv 1 \pmod{2^{2t+k}}$. Also, by Lemma 4.6, $2^{t+k-1} \parallel u_{3\cdot 2^{k-1}}$, so $(u_{3\cdot 2^{k-1}})^2 \equiv 0 \pmod{2^{2t+k}}$. Now (4.2) implies that

$$\begin{aligned} u_{3\cdot 2^k+1} &= b(u_{3\cdot 2^{k-1}})^2 + (u_{3\cdot 2^{k-1}+1})^2 \\ &\equiv 1 \pmod{2^{2t+k}}. \end{aligned}$$

□

Lemma 4.13. *Assume Hypothesis 4.3. If* $r \geq 2t$ *and* $k \geq 1$*, then* $bu_{3\cdot 2^k-1} \equiv 1 \pmod{2^{2t+k}}$.

Proof. Proceed by induction on k. By definition of r, the result is true when $k = 1$. Suppose that $k > 1$ and $bu_{3\cdot 2^{k-1}-1} \equiv 1 \pmod{2^{2t+k-1}}$. Then $bu_{3\cdot 2^{k-1}-1} + 1 \equiv 0 \pmod 2$. Moreover, by Lemma 4.6, $2^{t+k-1} \parallel u_{3\cdot 2^{k-1}}$, and hence $(u_{3\cdot 2^{k-1}})^2 \equiv 0 \pmod{2^{2t+k}}$. Now (4.2) implies that

$$\begin{aligned} bu_{3\cdot 2^k-1} - 1 = bu_{2(3\cdot 2^{k-1}-1)+1} - 1 &= b^2(u_{3\cdot 2^{k-1}-1})^2 + b(u_{3\cdot 2^{k-1}})^2 - 1 \\ &= (bu_{3\cdot 2^{k-1}-1} - 1)(bu_{3\cdot 2^{k-1}-1} + 1) + b(u_{3\cdot 2^{k-1}})^2 \\ &\equiv 0 \pmod{2^{2t+k}}. \end{aligned}$$

The lemma follows immediately. □

5. Blocks

One of the primary tools used in [6] to study second-order recurrence sequences is an equivalence relation used to partition the set of recurrence sequences into equivalence classes known as *blocks*. In this section we examine the ramifications of this theory for sequences modulo powers of 2.

Definition 5.1. For each $k \geq 1$, we define a relation on the set of second-order recurrences with fixed parameters a and b by $w(a,b) \sim_k w'(a,b)$ if $w'(a,b)$ is an odd (hence invertible modulo 2) multiple of a translation of $w(a,b)$ modulo 2^k, that is, $w(a,b) \sim_k w'(a,b)$ if and only if there exist integers m and c such that $2 \nmid c$ and, for all nonnegative integers n,

$$w'_n \equiv c w_{n+m} \pmod{2^k}. \tag{5.1}$$

Since c is odd, it is easy to see that $\sim_k$ defines an equivalence relation on the set of two-term recurrence relations with fixed parameters a and b. The equivalence classes of $\sim_k$ are called the 2^k-*blocks*, or simply the *blocks* when k is evident.

It is also clear that regularity is preserved by $\sim_k$, i.e., if $w(a,b) \sim_k w'(a,b)$, then $w(a,b)$ is 2-regular if and only if $w'(a,b)$ is 2-regular. Thus, the equivalence relation $\sim_k$ restricts to $\mathscr{F}(a,b)$ and, without risk of confusion, we can refer to *regular blocks*. Furthermore, if $w(a,b)$ and $w'(a,b)$ satisfy (5.1), and d is any integer, then $\nu_w(d,2^k) = \nu_{w'}(cd,2^k)$, and therefore, for every n,

$$\Omega_w(2^k) = \Omega_{w'}(2^k). \tag{5.2}$$

In the following theorem we determine the number of regular 2^k-blocks in $\mathscr{F}(a,b)$.

Theorem 5.2. *Assume Hypothesis 4.3. Then for all $k \geq t$, there are exactly 2^{t-1} regular 2^k-blocks, and each 2^k-block contains $3 \cdot 2^{2k-t-1}$ distinct sequences modulo 2^k.*

Proof. Since a second-order recurrence $w(a,b)$ is determined by its two initial terms, there are $(2^k)^2$ distinct recurrences modulo 2^k. Of these, exactly $(2^{k-1})^2$ satisfy $2 \mid (w_0, w_1)$ and, by Lemma 2.2, are not 2-regular. It follows that $\mathscr{F}(a,b)$ contains $(2^k)^2 - (2^{k-1})^2 = 3 \cdot 2^{2(k-1)}$ distinct sequences modulo 2^k.

If $w(a,b)$ is a 2-regular recurrence, then $w(a,b)$ has 2^{k-1} distinct odd multiples and $h(2^k)$ distinct translations that are not nontrivial multiples of the original sequence. It follows that the 2^k-block containing $w(a,b)$ has exactly $2^{k-1}h(2^k)$ distinct sequences modulo 2^k. Moreover, since, by Theorem 4.8, $h(2^k) = 3 \cdot 2^{k-t}$, a regular 2^k block contains $3 \cdot 2^{2k-t-1}$ recurrences.

We now see that the number of regular 2^k blocks in $\mathscr{F}(a,b)$ is

$$T_k = \frac{3 \cdot 2^{2(k-1)}}{3 \cdot 2^{2k-t-1}} = 2^{t-1}. \qquad \square$$

Corollary 5.3. *For all $k \geq t$, $w(a,b)$ and $w'(a,b)$ lie in the same 2^k-block if and only if they lie in the same 2^t-block.*

Proof. If $w(a,b)$ and $w'(a,b)$ lie in the same 2^k-block, it is obvious that they lie in the same 2^t-block. The converse follows from Theorem 5.2. $\square$

Corollary 5.4. *If $w(a,b)$ and $w'(a,b)$ lie in the same 2^t-block, then $w(a,b)$ is 2-stable if and only if $w'(a,b)$ is 2-stable.*

Proof. This follows from Corollary 5.3 and (5.2). □

In view of Corollary 5.4, we may, with no ambiguity, refer to *stable blocks* of sequences.

6. Principal divisors

One of the consequences of Lemma 4.7 is that we can identify the precise power of two that divides each element of $u(a,b)$. Since every sequence $w(a,b)$ in the same 2^k-block as $u(a,b)$ is a translation of an odd multiple of $u(a,b)$, we can easily determine the exact power of two that divides each element of $w(a,b)$. In particular, for every $k \geq t$, there is an element of $u(a,b)$, and hence of $w(a,b)$, that is exactly divisible by 2^k. Moreover, if a sequence $w(a,b)$ contains a term that is divisible by 2^t, Corollary 5.3 tells us that $w(a,b)$ lies in the same 2^k-block as $u(a,b)$ for all $k \geq t$, and hence $w(a,b)$ contains terms divisible by 2^k for all $k \geq t$.

For sequences $w(a,b)$ that lie outside of the 2^t-block containing $u(a,b)$, the situation is strikingly different. In particular, no element of $w(a,b)$ can be divisible by 2^t, since otherwise $w(a,b)$ would be a multiple of a translation of $u(a,b)$ modulo 2^t. It follows that sequences in $\mathscr{F}(a,b)$ may be characterized by the largest power of two dividing any element in the sequence. This motivates the following definitions.

Definition 6.1. Let m be a nonnegative integer and $w(a,b) \in \mathscr{F}(a,b)$. We say that m is a *divisor* of the sequence $w(a,b)$ if $m \mid w_n$ for some n. We say that 2^k is the *principal 2-divisor*, or simply the *principal divisor* of $w(a,b)$ when the prime 2 is evident, if 2^k is the largest power of 2 dividing any element of $w(a,b)$. In the case of $u(a,b)$, we say that the principal divisor does not exist or, informally, that the principal divisor is infinite.

It is self-evident that any two sequences in the same block have the same principal divisor. It is often convenient to choose a block representative $w(a,b)$ with the property that w_0 is divisible by the principal divisor. Clearly this is always possible, and we refer to such a representative as a *principal representative* of the block.

Armed with these definitions we can now describe precisely the powers of two that divide each term in a sequence $w(a,b)$ lying outside the 2^k-block of $u(a,b)$.

Lemma 6.2. *Assume Hypothesis 4.3, and suppose that $k \geq t$. If $w(a,b)$ is a principal representative of a 2^k-block that does not contain $u(a,b)$, then $w(a,b)$*

has principal divisor 2^m for some m satisfying $1 \le m < t$ and, moreover, $2^m \parallel w_{3n}$ for all nonnegative integers n.

P r o o f . That $m < t$ follows from the fact that $w(a,b)$ does not lie in the same block as $u(a,b)$. Moreover, since a and b are odd, two consecutive odd terms of $w(a,b)$ must be followed by an even term, so $m > 0$.

To show that $2^m \parallel w_{3n}$ for all $n \ge 0$, proceed by induction on n. For $n = 0$ the result follows from the definition of a principal representative of a block. By way of induction, assume that $2^m \parallel w_{3n}$ for some $n \ge 0$. It is easy to see that a complete cycle of $w(a,b)$ modulo 2 is given by $0, 1, 1$. It follows that w_{3n+1} is odd. By the intertwining lemma (Lemma 4.2) we have

$$w_{3n+3} = w_{(3n+1)+2} = b w_{3n} u_2 + w_{3n+1} u_3 \,.$$

Since bu_2 and w_{3n+1} are odd, $2^t \parallel w_{3n+1} u_3$, while $2^m \parallel b w_{3n} u_2$. Since $m < t$, we conclude that $2^m \parallel w_{3n+3}$, as desired. □

While sequences in the same block always have the same principal divisor, the converse is not true, as we see in the next theorem.

THEOREM 6.3. *If $k \ge t$, then there is a unique 2^k-block having principal divisor infinity, namely the block containing $u(a,b)$. If $1 \le m < t$, then there are exactly 2^{t-m-1} distinct blocks with principal divisor 2^m.*

P r o o f . First note that if a sequence $w(a,b)$ has infinite principal divisor, then some term of $w(a,b)$ is congruent to zero modulo 2^k, and it is clear that $w(a,b)$ is a multiple of a translation of $u(a,b)$. The first statement of the theorem follows immediately.

Now suppose that $m < t$. Since, by Theorem 5.2, each 2^k-block contains $3 \cdot 2^{2k-t-1}$ sequences, we need only count the number of distinct sequences modulo 2^k that have principal divisor 2^m. Since any two consecutive terms determine the entire sequence modulo 2^k and, by Lemma 6.2, one of the first three terms must be exactly divisible by 2^m and followed by an odd term, the number of sequences modulo 2^k with principal divisor 2^m is $3 \cdot 2^{k-m-1} \cdot 2^{k-1} = 3 \cdot 2^{2k-m-2}$. It now follows that the number of 2^k-blocks having principal divisor 2^m is

$$\frac{3 \cdot 2^{2k-m-2}}{3 \cdot 2^{2k-t-1}} = 2^{t-m-1} \,,$$

as desired. □

Note that since $1 + 1 + 2 + 4 + \cdots + 2^{t-2} = 2^{t-1}$, Theorem 6.3 yields an alternate proof of Theorem 5.2.

7. Stability classes: Preliminary results

In this section we begin our study of stability by considering sequences for which a is odd and $b \equiv 3 \pmod 4$. In our first result, we observe that the main theorem of [1] applies to each sequence in the 2^t-block of the generalized Fibonacci sequence $u(a,b)$.

Theorem 7.1. *Assume Hypothesis 4.3. Then the 2^t-block of $u(a,b)$ is 2-stable unless $r \geq 2t$ or $s \geq 2t$.*

Proof. By Corollary 5.4, if $w(a,b)$ lies in the same 2^t-block as $u(a,b)$, then $w(a,b)$ is 2-stable if and only if $u(a,b)$ is 2-stable. By Theorem 3.1 of [1], $u(a,b)$ is 2-stable when $r < 2t$ and $s < 2t$. □

We note that the conditions $r < 2t$ and $s < 2t$ are characterized in terms of a and b in [1].

The blocks that do not contain $u(a,b)$ require a little more work. In the next lemmas we repeatedly apply the intertwining lemma to obtain congruences that will be used in this analysis. First we augment our main hypothesis as follows.

Hypothesis 7.2. *Assume Hypothesis 4.3. Let $w(a,b) \in \mathscr{F}(a,b)$ be a principal representative of a block that does not contain the generalized Fibonacci sequence $u(a,b)$ and has principal divisor 2^m with $1 \leq m < t$.*

Lemma 7.3. *Assume Hypothesis 7.2, and suppose that $s < t$.*

(a) *If $1 \leq m < t-s$, $k > s+m$, and $n \equiv 0 \pmod 3$, then*

$$w_{n+3\cdot 2^{k-s-m}} \equiv w_n + 2^k \pmod{2^{k+1}}. \tag{7.1}$$

(b) *If $t-s < m \leq t-1$, $k > t$, and $n \equiv 0 \pmod 3$, then*

$$w_{n+3\cdot 2^{k-t}} \equiv w_n + 2^k \pmod{2^{k+1}}. \tag{7.2}$$

Proof. (a) Since $w(a,b)$ is a principal representative of its block and $n \equiv 0 \pmod 3$, Lemma 6.2 implies that $2^m \parallel w_n$ and w_{n-1} is odd. By Lemma 4.6, $2^{t+k-s-m} \parallel u_{3\cdot 2^{k-s-m}}$ and $u_{3\cdot 2^{k-s-m}+1}$ is odd. Also, by Lemma 4.9, $u_{3\cdot 2^{k-s-m}+1} \equiv 1 + 2^{k-m} \pmod{2^{k-m+1}}$. Finally, by the intertwining lemma,

$$\begin{aligned} w_{n+3\cdot 2^{k-s-m}} &= bw_{n-1}u_{3\cdot 2^{k-s-m}} + w_n u_{3\cdot 2^{k-s-m}+1} \\ &\equiv w_n u_{3\cdot 2^{k-s-m}+1} \pmod{2^{k+1}} \\ &\equiv w_n + 2^k \pmod{2^{k+1}}. \end{aligned}$$

(b) By Lemma 4.6, $2^k \parallel u_{3\cdot 2^{k-t}}$ and, by Lemma 4.9, $u_{3\cdot 2^{k-t}+1} \equiv 1 \pmod{2^{k-t+s}}$. Thus, in this case, the intertwining lemma yields:

$$\begin{aligned} w_{n+3\cdot 2^{k-t}} &= bw_{n-1}u_{3\cdot 2^{k-t}} + w_n u_{3\cdot 2^{k-t}+1} \\ &\equiv 2^k + w_n \pmod{2^{k+1}}. \end{aligned}$$

□

LEMMA 7.4. *Assume Hypothesis 7.2, and suppose that* $s \geq t$. *If* $k > t$ *and* $n \equiv 0 \pmod 3$, *then*

$$w_{n+3\cdot 2^{k-t}} \equiv w_n + 2^k \pmod{2^{k+1}}. \tag{7.3}$$

Proof. Since $w(a,b)$ is a principal representative of its block and $n \equiv 0 \pmod 3$, Lemma 6.2 implies that $2^m \parallel w_n$ and w_{n-1} is odd. By Lemma 4.11, $u_{3\cdot 2^{k-t}+1} \equiv 1 \pmod{2^k}$ and, by Lemma 4.6, $2^k \parallel u_{3\cdot 2^{k-t}}$. As in the previous lemma, the intertwining lemma now yields:

$$\begin{aligned} w_{n+3\cdot 2^{k-t}} &= bw_{n-1}u_{3\cdot 2^{k-t}} + w_n u_{3\cdot 2^{k-t}+1} \\ &\equiv 2^k + w_n \pmod{2^{k+1}}. \end{aligned}$$

□

LEMMA 7.5. *Assume Hypothesis 7.2, and suppose that* $s < 2t$.

(a) *If* $k > s$, $s \leq t$ *and* $n \equiv 1 \pmod 3$, *then*

$$w_{n+3\cdot 2^{k-s}} \equiv w_n + 2^k \pmod{2^{k+1}}. \tag{7.4}$$

(b) *If* $k > s$, $t < s < 2t$, $s - t < m \leq t - 1$, *and* $n \equiv 1 \pmod 3$, *then*

$$w_{n+3\cdot 2^{k-s}} \equiv w_n + 2^k \pmod{2^{k+1}}. \tag{7.5}$$

(c) *If* $k > t + m$, $t < s < 2t$, $1 \leq m < s - t$ *and* $n \equiv 1 \pmod 3$, *then*

$$w_{n+3\cdot 2^{k-t-m}} \equiv w_n + 2^k \pmod{2^{k+1}}. \tag{7.6}$$

Proof. (a),(b) Since $w(a,b)$ is a principal representative of its block and $n \equiv 1 \pmod 3$, w_n is odd and Lemma 6.2 implies that $2^m \parallel w_{n-1}$. By Lemma 4.6, $2^{k-s+t} \parallel u_{3\cdot 2^{k-s}}$ and, by Lemma 4.9, $u_{3\cdot 2^{k-s}+1} \equiv 1 + 2^k \pmod{2^{k+1}}$. Once again, the intertwining lemma yields:

$$\begin{aligned} w_{n+3\cdot 2^{k-s}} &= bw_{n-1}u_{3\cdot 2^{k-s}} + w_n u_{3\cdot 2^{k-s}+1} \\ &\equiv w_n(1 + 2^k) \pmod{2^{k+1}} \\ &\equiv w_n + 2^k \pmod{2^{k+1}}. \end{aligned}$$

(c) By Lemma 4.6, $2^{k-m} \parallel u_{3\cdot 2^{k-t-m}}$ and, by Lemma 4.9, $u_{3\cdot 2^{k-t-m}+1} \equiv 1 + 2^{k+s-t-m} \pmod{2^{k+s-t-m+1}}$. Since $s - t - m \geq 1$, we see that $u_{3\cdot 2^{k-t-m}+1} \equiv 1 \pmod{2^{k+1}}$, and the intertwining lemma yields:

$$\begin{aligned} w_{n+3\cdot 2^{k-t-m}} &= bw_{n-1}u_{3\cdot 2^{k-t-m}} + w_n u_{3\cdot 2^{k-t-m}+1} \\ &\equiv 2^k + w_n \pmod{2^{k+1}}. \end{aligned}$$

□

Lemma 7.6. *Assume Hypothesis 7.2, and suppose that* $s \geq 2t$. *If* $k > t+m$ *and* $n \equiv 1 \pmod 3$, *then*

$$w_{n+3\cdot 2^{k-t-m}} \equiv w_n + 2^k \pmod{2^{k+1}}. \tag{7.7}$$

Proof. Since $w(a,b)$ is a principal representative of its block and $n \equiv 1 \pmod 3$, w_n is odd and Lemma 6.2 implies that $2^m \parallel w_{n-1}$. By Lemma 4.6, $2^{k-m} \parallel u_{3\cdot 2^{k-m-t}}$ and, by Lemma 4.12, $u_{3\cdot 2^{k-t-m}+1} \equiv 1 \pmod{2^{k+t-m}}$. Since $1 \leq m \leq t-1$, we see that $u_{3\cdot 2^{k-t-m}+1} \equiv 1 \pmod{2^{k+1}}$. As usual, the intertwining lemma yields:

$$\begin{aligned} w_{n+3\cdot 2^{k-t-m}} &= bw_{n-1}u_{3\cdot 2^{k-t-m}} + w_n u_{3\cdot 2^{k-t-m}+1} \\ &\equiv 2^k + w_n \pmod{2^{k+1}}. \end{aligned}$$

□

Lemma 7.7. *Assume Hypothesis 7.2, and suppose that* $r < 2t$.

(a) *If* $k > r$, $r \leq t$ *and* $n \equiv 2 \pmod 3$, *then*

$$w_{n+3\cdot 2^{k-r}} \equiv w_n + 2^k \pmod{2^{k+1}}. \tag{7.8}$$

(b) *If* $k > r$, $t < r < 2t$, $r-t < m \leq t-1$, *and* $n \equiv 2 \pmod 3$, *then*

$$w_{n+3\cdot 2^{k-r}} \equiv w_n + 2^k \pmod{2^{k+1}}. \tag{7.9}$$

(c) *If* $k > t+m$, $t < r < 2t$, $1 \leq m < r-t$ *and* $n \equiv 2 \pmod 3$, *then*

$$w_{n+3\cdot 2^{k-t-m}} \equiv w_n + 2^k \pmod{2^{k+1}}. \tag{7.10}$$

Proof. (a),(b) Since $w(a,b)$ is a principal representative of its block and $n \equiv 2 \pmod 3$, w_n is odd and Lemma 6.2 implies that $2^m \parallel w_{n+1}$. By Lemma 4.6, $2^{k-r+t} \parallel u_{3\cdot 2^{k-r}}$ and, by Lemma 4.10, $bu_{3\cdot 2^{k-r}-1} \equiv 1+2^k \pmod{2^{k+1}}$. Again, the intertwining lemma yields:

$$\begin{aligned} w_{n+3\cdot 2^{k-r}} &= bw_n u_{3\cdot 2^{k-r}-1} + w_{n+1}u_{3\cdot 2^{k-r}} \\ &\equiv w_n(1+2^k) \pmod{2^{k+1}} \\ &\equiv w_n + 2^k \pmod{2^{k+1}}. \end{aligned}$$

(c) By Lemma 4.6, $2^{k-m} \parallel u_{3\cdot 2^{k-m-t}}$ and, by Lemma 4.10, $bu_{3\cdot 2^{k-t-m}-1} \equiv 1+2^{k+r-t-m} \pmod{2^{k+r-t-m+1}}$. Since $r-t-m \geq 1$, we see that $bu_{3\cdot 2^{k-t-m}-1} \equiv 1 \pmod{2^{k+1}}$. Finally, the intertwining lemma yields:

$$\begin{aligned} w_{n+3\cdot 2^{k-t-m}} &= bw_n u_{3\cdot 2^{k-t-m}-1} + w_{n+1}u_{3\cdot 2^{k-t-m}} \\ &\equiv w_n + 2^k \pmod{2^{k+1}}. \end{aligned}$$

□

Lemma 7.8. *Assume Hypothesis 7.2, and suppose that $r \geq 2t$. If $k > t + m$ and $n \equiv 2 \pmod 3$, then*

$$w_{n+3\cdot 2^{k-t-m}} \equiv w_n + 2^k \pmod{2^{k+1}}.$$

Proof. Since $w(a,b)$ is a principal representative of its block and $n \equiv 2 \pmod 3$, w_n is odd and Lemma 6.2 implies that $2^m \parallel w_{n+1}$. By Lemma 4.13, $bu_{3\cdot 2^{k-t-m}-1} \equiv 1 \pmod{2^{k+t-m}}$ and, by Lemma 4.6, $2^{k-m} \parallel u_{3\cdot 2^{k-m-t}}$. Since $1 \leq m \leq t-1$, we see that $u_{3\cdot 2^{k-t-m}-1} \equiv 1 \pmod{2^{k+1}}$. One last time, the intertwining lemma yields:

$$\begin{aligned} w_{n+3\cdot 2^{k-t-m}} &= bw_n u_{3\cdot 2^{k-t-m}-1} + w_{n+1} u_{3\cdot 2^{k-t-m}} \\ &\equiv w_n + 2^k \pmod{2^{k+1}}. \end{aligned}$$ □

8. Stability classes

In this section we provide a complete description in terms of the parameters r, s, and t of the 2-stable classes when $b \equiv 3 \pmod 4$. In our main theorems, Theorem 8.3, Theorem 8.4, Theorem 8.5, Theorem 8.6, and Theorem 8.7, below, we identify 2-stable classes and characterize classes that may fail to be 2-stable.

The situation when $b \equiv 3 \pmod 8$ is less complicated than when $b \equiv 7 \pmod 8$; as we shall see in Theorem 8.1 and Theorem 8.2 there are only two blocks when $b \equiv 3 \pmod 8$, while there are at least four blocks when $b \equiv 7 \pmod 8$. Moreover, when $b \equiv 3 \pmod 8$, the generalized Fibonacci sequence $u(a,b)$ and the generalized Lucas sequence $v(a,b)$ lie in different blocks, and hence their stability determines the stability of all sequences in $\mathscr{F}(a,b)$.

In [1], Carlip and Jacobson identify a large number of cases for which $u(a,b)$ is 2-stable and conjecture that $u(a,b)$ is not stable in the remaining cases. The situation for $v(a,b)$ has received less attention; the only 2-stability result for the generalized Lucas sequence is Mark Morgan's proof [9] that $v(1,1)$ is not 2-stable. We intend to prove in a future paper that there is always at least one class of sequences in $\mathscr{F}(a,b)$ that is not 2-stable. Together with Theorem 8.4, below, which argues that either $u(a,b)$ or $v(a,b)$ must be 2-stable when $b \equiv 3 \pmod 8$, that result will completely determine 2-stability in this case.

Theorem 8.1. *Assume Hypothesis 4.3, and suppose that $b \equiv 3 \pmod 8$. Then $\mathscr{F}(a,b)$ decomposes into exactly two blocks modulo 2^k when $k \geq 2$. Moreover, the sequences $u(a,b)$ and $v(a,b)$ lie in different classes.*

Proof. By Lemma 4.4(b) and Theorem 5.2, there are $2^{t-1} = 2$ distinct 2^k-blocks when $k \geq 2$.

To see that $v(a,b)$ lies in a different 4-block than $u(a,b)$, observe that a single cycle of $v(a,b)$ modulo 4 is

$$\begin{aligned} &2,1,3,2,3,1 && \text{when } a \equiv 1 \pmod 4, \quad \text{and} \\ &2,3,3 && \text{when } a \equiv 3 \pmod 4, \end{aligned}$$

from which it follows that the principal 2-divisor in both cases is 2. Since $u(a,b)$ has infinite 2-divisor, it follows that $u(a,b)$ and $v(a,b)$ lie in different 4-blocks. □

Theorem 8.2. *Assume Hypothesis 4.3 and suppose that* $b \equiv 7 \pmod 8$. *Then* $\mathscr{F}(a,b)$ *decomposes into* 2^{t-1} *distinct* 2^t*-blocks. Moreover,* 2^{t-1} *is always at least* 4, *and may be arbitrarily large.*

Proof. The first assertion is simply Theorem 5.2, and the fact that $2^{t-1} \geq 4$ follows from Lemma 4.4(c). It is easy to construct sequences with arbitrarily large values of t. For example, if $a = 1$ and $b = 2^n - 1$, for $n \geq 3$, then $b \equiv 7 \pmod 8$ and $u_3 = a^2 + b = 2^n$, so $t = n$. □

In Theorem 7.1, we observed that the 2^t-block containing $u(a,b)$ is 2-stable when $s < 2t$ and $r < 2t$. In the next theorems we consider every possible remaining relationship between s, t, and r and, in each case, show that every block, except those with one particular principal divisor, is 2-stable. In each case, we apply a counting argument similar to that used in [1]. We begin by considering the case that is complementary to that in Theorem 7.1.

Theorem 8.3. *Assume Hypothesis 4.3. If* $s \geq 2t$ *or* $r \geq 2t$, *then every* 2^t*-block is 2-stable, except possibly the unique* 2^t*-block containing* $u(a,b)$.

Proof. Let $w(a,b) \in \mathscr{F}(a,b)$ and assume that $w(a,b)$ does not lie in the same 2^t-block as $u(a,b)$. Then, by Lemma 6.2, $w(a,b)$ has principal 2-divisor 2^m, with $1 \leq m \leq t-1$.

First suppose that $s \geq 2t$. Then, by Lemma 4.4(f), $r = t$. For all sufficiently large k, Lemma 7.4, Lemma 7.6, and Lemma7.7.(a) tell us that

$$\begin{aligned} w_{n+3\cdot 2^{k-t}} &\equiv w_n + 2^k \pmod{2^{k+1}} && \text{when } n \equiv 0 \pmod 3, \\ w_{n+3\cdot 2^{k-t-m}} &\equiv w_n + 2^k \pmod{2^{k+1}} && \text{when } n \equiv 1 \pmod 3, \quad \text{and} \\ w_{n+3\cdot 2^{k-r}} &\equiv w_n + 2^k \pmod{2^{k+1}} && \text{when } n \equiv 2 \pmod 3. \end{aligned}$$

Now suppose that $r \geq 2t$. Then, by Lemma 4.4(f) and (e), $s = t$. Now, for all sufficiently large k, Lemma 7.4, Lemma 7.5(a), and Lemma 7.8 tell us that

$$\begin{aligned} w_{n+3\cdot 2^{k-t}} &\equiv w_n + 2^k \pmod{2^{k+1}} && \text{when } n \equiv 0 \pmod 3, \\ w_{n+3\cdot 2^{k-s}} &\equiv w_n + 2^k \pmod{2^{k+1}} && \text{when } n \equiv 1 \pmod 3, \quad \text{and} \\ w_{n+3\cdot 2^{k-t-m}} &\equiv w_n + 2^k \pmod{2^{k+1}} && \text{when } n \equiv 2 \pmod 3. \end{aligned}$$

In both cases, we see that each instance of a residue d modulo 2^{k+1} in one cycle of $w(a,b)$ modulo 2^{k+1} can be paired with a unique instance of the residue $d+2^k$ modulo 2^{k+1} in the same cycle. Therefore $\nu_w(d,2^{k+1}) = \nu_w(d+2^k,2^{k+1})$. On the other hand, by Theorem 4.5, the period of $w(a,b)$ modulo 2^{k+1} is double the period modulo 2^k. It follows that $\nu_w(d,2^k) = \nu_w(d,2^{k+1}) = \nu_w(d+2^k,2^{k+1})$. Finally, we conclude that no new residue frequencies occur modulo 2^{k+1}, that is $\Omega_w(2^k) = \Omega_w(2^{k+1})$. Since this is true for all sufficiently large k, it follows that $w(a,b)$ is 2-stable. □

We are now in a position to describe the 2-stability of blocks when $b \equiv 3 \pmod 8$. Recall from Theorem 8.1 that when $b \equiv 3 \pmod 8$ there are exactly two 2^t-blocks, one containing $u(a,b)$ and the other containing $v(a,b)$. In Theorem 8.4 we will show that at least one of these blocks must always be 2-stable.

THEOREM 8.4. *Assume Hypothesis 4.3, and suppose that $b \equiv 3 \pmod 8$. If $r < 2t$ and $s < 2t$, then the 2^t-block containing $u(a,b)$ is 2-stable. If $r \geq 2t$ or $s \geq 2t$, then the 2^t-block containing $v(a,b)$ is 2-stable.*

P r o o f . The theorem follows from Theorem 8.1, Theorem 7.1, and Theorem 8.3. □

For the remainder of this section we examine the situation when $b \equiv 7 \pmod 8$.

THEOREM 8.5. *Assume Hypothesis 4.3, and suppose that $b \equiv 7 \pmod 8$. If $s < t$, then every 2^t-block is stable, except possibly those blocks that have a principal 2-divisor equal to 2^{t-s}. Moreover, there are exactly 2^{s-1} such blocks.*

P r o o f . Let $w(a,b) \in \mathscr{F}(a,b)$ and assume that $s < t$. Note that if $w(a,b)$ lies in the same 2^t-block as $u(a,b)$, then, by Theorem 7.1 and Lemma 4.4(e), $w(a,b)$ is 2-stable. Assume, then, that $w(a,b)$ does not lie in the same 2^t-block as $u(a,b)$. Then, by Lemma 6.2 and the preceding discussion, $w(a,b)$ has principal 2-divisor 2^m, with $1 \leq m \leq t-1$.

Now suppose $m \neq t-s$. Then, by Lemma 4.4(e), $r = s < t$. Hence, for all sufficiently large k, Lemma 7.3 tells us that either

$$w_{n+3\cdot 2^{k-s-m}} \equiv w_n + 2^k \pmod{2^{k+1}} \qquad \text{when } n \equiv 0 \pmod 3, \quad \text{or}$$
$$w_{n+3\cdot 2^{k-t}} \equiv w_n + 2^k \pmod{2^{k+1}} \qquad \text{when } n \equiv 0 \pmod 3,$$

accordingly, as $m < t-s$ or $m > t-s$. Furthermore, Lemma 7.5(a) and Lemma 7.7(a) tell us that

$$w_{n+3\cdot 2^{k-s}} \equiv w_n + 2^k \pmod{2^{k+1}} \qquad \text{when } n \equiv 1 \pmod 3, \quad \text{and}$$
$$w_{n+3\cdot 2^{k-r}} \equiv w_n + 2^k \pmod{2^{k+1}} \qquad \text{when } n \equiv 2 \pmod 3.$$

As in the proof of Theorem 8.3, residues d and $d+2^k$ in one cycle of $w(a,b)$ modulo 2^{k+1} may be paired, and no new residue frequencies are introduced. It follows that $w(a,b)$ is 2-stable, as desired.

The fact that there are exactly 2^{s-1} blocks with principal divisor 2^{t-s} follows from Theorem 6.3. □

THEOREM 8.6. *Assume Hypothesis 4.3, and suppose that* $b \equiv 7 \pmod 8$. *If* $t < s < 2t$, *then every* 2^t*-block is stable, except possibly those blocks that have a principal* 2*-divisor equal to* 2^{s-t}. *Moreover, there are exactly* 2^{2t-s-1} *such blocks.*

Proof. Let $w(a,b) \in \mathscr{F}(a,b)$ and assume that $t < s < 2t$. Note that if $w(a,b)$ lies in the same 2^t-block as $u(a,b)$, then, by Theorem 7.1 and Lemma 4.4(f), $w(a,b)$ is 2-stable. Assume, then, that $w(a,b)$ does not lie in the same 2^t-block as $u(a,b)$. Then, by Lemma 6.2 and the preceding discussion, $w(a,b)$ has principal 2-divisor 2^m, with $1 \le m \le t-1$.

Now suppose that $m \ne s-t$. Then, by Lemma 4.4(f), $r = t$. Hence, for all sufficiently large k, Lemma 7.4 and Lemma 7.7.(a) tell us that

$$w_{n+3\cdot 2^{k-t}} \equiv w_n + 2^k \pmod{2^{k+1}} \qquad \text{when } n \equiv 0 \pmod 3, \quad \text{and}$$
$$w_{n+3\cdot 2^{k-r}} \equiv w_n + 2^k \pmod{2^{k+1}} \qquad \text{when } n \equiv 2 \pmod 3.$$

Moreover, Lemma 7.5 tells us that either

$$w_{n+3\cdot 2^{k-s}} \equiv w_n + 2^k \pmod{2^{k+1}} \qquad \text{when } n \equiv 1 \pmod 3, \quad \text{or}$$
$$w_{n+3\cdot 2^{k-t-m}} \equiv w_n + 2^k \pmod{2^{k+1}} \qquad \text{when } n \equiv 1 \pmod 3,$$

accordingly, as $m > s-t$ or $m < s-t$.

As in the proof of Theorem 8.3, residues d and $d+2^k$ in one cycle of $w(a,b)$ modulo 2^{k+1} may be paired, and no new residue frequencies are introduced. It follows that $w(a,b)$ is 2-stable, as desired.

The fact that there are exactly 2^{2t-s-1} blocks with principal divisor 2^{s-t} follows from Theorem 6.3. □

THEOREM 8.7. *Assume Hypothesis 4.3, and suppose that* $b \equiv 7 \pmod 8$. *If* $s = t$ *and* $r < 2t$, *then every* 2^t*-block is stable, except possibly those blocks that have a principal* 2*-divisor equal to* 2^{r-t}. *Moreover, there are exactly* 2^{2t-r-1} *such blocks.*

Proof. Let $w(a,b) \in \mathscr{F}(a,b)$ and assume that $s = t$ and $r < 2t$. Note that if $w(a,b)$ lies in the same 2^t-block as $u(a,b)$, then, by Theorem 7.1, $w(a,b)$ is 2-stable. Assume, then, that $w(a,b)$ does not lie in the same 2^t-block as $u(a,b)$. Then, by Lemma 6.2 and the preceding discussion, $w(a,b)$ has principal 2-divisor 2^m, with $1 \le m \le t-1$.

Now suppose that $m \neq r-t$. Then, by Lemma 4.4(g), $t<r$. Hence, for all sufficiently large k, Lemma 7.4 and Lemma 7.5(a) tell us that

$$w_{n+3\cdot 2^{k-t}} \equiv w_n + 2^k \pmod{2^{k+1}} \qquad \text{when } n \equiv 0 \pmod{3}, \qquad \text{and}$$
$$w_{n+3\cdot 2^{k-s}} \equiv w_n + 2^k \pmod{2^{k+1}} \qquad \text{when } n \equiv 1 \pmod{3}.$$

Moreover, Lemma 7.7 tells us that either

$$w_{n+3\cdot 2^{k-r}} \equiv w_n + 2^k \pmod{2^{k+1}} \qquad \text{when } n \equiv 2 \pmod{3}, \qquad \text{or}$$
$$w_{n+3\cdot 2^{k-t-m}} \equiv w_n + 2^k \pmod{2^{k+1}} \qquad \text{when } n \equiv 2 \pmod{3},$$

accordingly, as $m > r-t$ or $m < r-t$.

As in the proof of Theorem 8.3, residues d and $d+2^k$ in one cycle of $w(a,b)$ modulo 2^{k+1} may be paired, and no new residue frequencies are introduced. It follows that $w(a,b)$ is 2-stable, as desired.

The fact that there are exactly 2^{2t-r-1} blocks with principal divisor 2^{r-t} follows from Theorem 6.3. □

We note that in each of the proofs in this section, we required k to be "sufficiently large" to obtain the required congruences. In fact, the precise value of k that is required follows from the lemmas in § 7 and can be used to compute the *index of stability*, i.e., the least value N such that $\Omega_w(p^k) = \Omega_w(p^N)$ for all $k \geq N$, for each 2-stable block of sequences. We observe that it is sufficient to take $k > \max\{s+m,\, t+m,\, r\}$ to obtain stability, and remark that, as in [3], arbitrarily large values of k may be required in some cases.

9. The parameters a and b

In our main theorems of the previous section we characterized the 2-stability of blocks in terms of the parameters r, s, and t. In this section we elaborate on these results by applying techniques from [1] to identify 2-stable blocks in terms of the parameters a and b occurring in the fundamental recurrence relation (1.1). Clearly, for any pair of odd integers a and b, there are corresponding integers r, s, and t, and if $b \equiv 3 \pmod{4}$, then one of our main theorems of § 8 can be applied to describe the 2-stability of the corresponding blocks.

We begin by examining the situation when $b \equiv 3 \pmod{8}$.

Theorem 9.1. *Assume Hypothesis 4.3, and suppose that $b \equiv 3 \pmod{8}$.*

(a) *If either*

$$b \equiv 3 \pmod{16} \quad \textit{and} \quad a \equiv \pm 5 \pmod{16} \ \textit{or} \ a \equiv \pm 7 \pmod{16}, \ \textit{or} \tag{9.1}$$
$$b \equiv 11 \pmod{16} \quad \textit{and} \quad a \equiv \pm 3 \pmod{16} \ \textit{or} \ a \equiv \pm 7 \pmod{16}, \tag{9.2}$$

then the 2^t-block containing $u(a,b)$ is 2-stable.

(b) *If either*

$$b \equiv 3 \pmod{16} \quad \textit{and} \quad a \equiv \pm 1 \pmod{16} \textit{ or } a \equiv \pm 3 \pmod{16}, \textit{ or} \tag{9.3}$$

$$b \equiv 11 \pmod{16} \quad \textit{and} \quad a \equiv \pm 1 \pmod{16} \textit{ or } a \equiv \pm 5 \pmod{16}, \tag{9.4}$$

then the 2^t-block containing $v(a,b)$ is 2-stable.

Proof. (a) By Lemma 4.3 and Lemma 4.4 of [1], (9.1) implies that $s < 2t$ and $r < 2t$. Similarly, by Lemma 4.6 and Lemma 4.7 of [1], (9.2) implies that $s < 2t$ and $r < 2t$. Thus, in both cases, Theorem 8.4 implies that the 2^t-block containing $u(a,b)$ is 2-stable.

(b) By Lemma 4.3 and Lemma 4.4 of [1], (9.3) implies that either $s \geq 2t$ or $r \geq 2t$. Similarly, by Lemma 4.6 and Lemma 4.7 of [1], (9.4) implies that either $s \geq 2t$ or $r \geq 2t$. Thus, in both cases, Theorem 8.4 implies that the 2^t-block containing $v(a,b)$ is 2-stable. □

The analysis of the situation when $b \equiv 7 \pmod{16}$ is divided into several theorems corresponding to different congruence classes of a. In Theorems 9.2, 9.3 and 9.4 we handle the cases in which $a \equiv \pm 3 \pmod 8$, $a \equiv \pm 1 \pmod{16}$, and $a \equiv \pm 7 \pmod{16}$, respectively.

THEOREM 9.2. *Assume Hypothesis 4.3. If*

$$b \equiv 7 \pmod{16} \qquad \textit{and} \qquad a \equiv \pm 3 \pmod 8, \tag{9.5}$$

then $t \geq 4$, $r = s = 2$, and every 2^t-block is stable, except possibly those blocks that have principal divisor equal to 2^{t-2}. Moreover, there are exactly two such blocks.

Proof. We apply the technique described in [1] to compute t and s. By the proof of Lemma 4.9 of [1], (9.5) implies that $t \geq 4$. Furthermore, by the proof of Lemma 4.1 and Lemma 4.9 of [1], (9.5) implies that $s = 2$. Thus, in both cases, $s < t$ and, by Lemma 4.4(a), $r = s = 2$. Finally, by Theorem 8.5, we conclude that every 2^t-block is stable, except possibly those blocks that have principal divisor equal to 2^{t-2}, and that there are exactly two such blocks. □

THEOREM 9.3. *Assume Hypothesis 4.3, and suppose that $a \equiv \pm 1 \pmod{16}$.*

(a) *If*

$$b \equiv 7 \pmod{16} \qquad \textit{and} \qquad a \equiv \pm 1 \pmod{32}, \tag{9.6}$$

then $s = 4$, $t = r = 3$, and every 2^t-block is stable, except possibly those blocks that have principal divisor equal to 2. Moreover, there are exactly two such blocks.

(b) *If either*

$$b \equiv 7 \pmod{32} \quad \textit{and} \quad a \equiv \pm 15 \pmod{64}, \quad \textit{or} \tag{9.7}$$
$$b \equiv 23 \pmod{32} \quad \textit{and} \quad a \equiv \pm 17 \pmod{64}, \tag{9.8}$$

then $s = 5$, $t = r = 3$, *and every* 2^t*-block is stable, except possibly the unique block that has principal divisor equal to* 2^2.

(c) *If either*

$$b \equiv 7 \pmod{32} \quad \textit{and} \quad a \equiv \pm 17 \pmod{64}, \quad \textit{or} \tag{9.9}$$
$$b \equiv 23 \pmod{32} \quad \textit{and} \quad a \equiv \pm 15 \pmod{64}, \tag{9.10}$$

then $s \geq 6$, $t = r = 3$, *and every* 2^t*-block is stable, except possibly the unique block containing* $u(a,b)$.

Proof. Suppose that $b \equiv 7 \pmod{16}$. By the proof of Lemma 4.9 of [1], $t = 3$ when $a \equiv \pm 1 \pmod{16}$. To derive the remaining conclusions of the theorem we apply the technique used in [1] to evaluate s.

(a) By the proof of Lemma 4.1 of [1], $s = 4$ when (9.6) holds. By Lemma 4.4(f), $r = t = 3$. Finally, since $t < s < 2t$, the remaining conclusions follow from Theorem 8.6.

(b) By the proof of Lemma 4.1 of [1], $s = 5$ when either (9.7) or (9.8) holds. By Lemma 4.4(f), $r = t = 3$. Again, $t < s < 2t$, and the remaining conclusions follow from Theorem 8.6.

(c) By the proof of Lemma 4.1 of [1], $s \geq 6$ when either (9.9) or (9.10) holds. By Lemma 4.4(f), $r = t = 3$. Since $2t \leq s$, the remaining conclusions follow from Theorem 8.3. □

THEOREM 9.4. *Assume Hypothesis 4.3, and suppose that* $a \equiv \pm 7 \pmod{16}$.

(a) *If either*

$$b \equiv 7 \pmod{32} \quad \textit{and} \quad a \equiv \pm 7 \pmod{32}, \quad \textit{or} \tag{9.11}$$
$$b \equiv 23 \pmod{32} \quad \textit{and} \quad a \equiv \pm 9 \pmod{32}, \tag{9.12}$$

then $r = 4$, $t = s = 3$, *and every* 2^t*-block is stable, except possibly those blocks that have principal divisor equal to* 2. *Moreover, there are exactly two such blocks.*

(b) *If either*

$$b \equiv 7 \pmod{64} \quad \textit{and} \quad a \equiv \pm 9 \pmod{64}, \quad \textit{or} \tag{9.13}$$
$$b \equiv 23 \pmod{64} \quad \textit{and} \quad a \equiv \pm 25 \pmod{64}, \quad \textit{or} \tag{9.14}$$
$$b \equiv 39 \pmod{64} \quad \textit{and} \quad a \equiv \pm 23 \pmod{64}, \quad \textit{or} \tag{9.15}$$
$$b \equiv 55 \pmod{64} \quad \textit{and} \quad a \equiv \pm 7 \pmod{64}, \tag{9.16}$$

then $r = 5$, $t = s = 3$, *and every* 2^t*-block is stable, except possibly the unique block that has principal divisor equal to* 2^2.

(c) *If either*

$$
\begin{array}{llllr}
b \equiv 7 \pmod{64} & \text{and} & a \equiv \pm 23 \pmod{64}, & \text{or} & (9.17)\\
b \equiv 23 \pmod{64} & \text{and} & a \equiv \pm 7 \pmod{64}, & \text{or} & (9.18)\\
b \equiv 39 \pmod{64} & \text{and} & a \equiv \pm 9 \pmod{64}, & \text{or} & (9.19)\\
b \equiv 55 \pmod{64} & \text{and} & a \equiv \pm 25 \pmod{64}, & & (9.20)
\end{array}
$$

then $r \geq 6$, $t = s = 3$, *and every* 2^t*-block is stable, except possibly the unique block containing* $u(a, b)$.

P r o o f . Suppose that $b \equiv 7 \pmod{16}$. By the proof of Lemma 4.9 of [1], $t = 3$ when $a \equiv \pm 7 \pmod{16}$. In a manner similar to that used in the previous theorem, we apply the technique used in [1] to evaluate r.

(a) By the proof of Lemma 4.2 of [1], $r = 4$ when (9.11) or (9.12) holds. By Lemma 4.4(e), (f), and (g), it follows that $t = s = 3$. Since $r < 2t$, the final conclusion now follows from Theorem 8.7.

(b) By the proof of Lemma 4.2 of [1], $r = 5$ when (9.13), (9.14), (9.15), or (9.16) holds. By Lemma 4.4(e), (f), and (g), it again follows that $t = s = 3$. Again $r < 2t$, and the final conclusion follows from Theorem 8.7.

(c) By the proof of Lemma 4.2 of [1], $r \geq 6$ when (9.17), (9.18), (9.19), or (9.20) holds. By Lemma 4.4(e), (f), and (g), it again follows that $t = s = 3$. This time, $r \geq 2t$ and the final conclusion follows from Theorem 8.3. □

We now turn to the case in which $b \equiv 15 \pmod{16}$. Again the analysis is broken into several theorems corresponding to different congruence classes of a. In Theorems 9.5 and 9.6 we handle the cases in which $a \equiv \pm 3 \pmod{8}$ and $a \equiv \pm 7 \pmod{16}$, respectively.

THEOREM 9.5. *Assume Hypothesis 4.3. If*

$$b \equiv 15 \pmod{16} \qquad \text{and} \qquad a \equiv \pm 3 \pmod{8}, \tag{9.21}$$

then $t = 3$, $s = r = 2$, *and every* 2^t*-block is stable, except possibly those blocks that have a principal divisor equal to* 2. *Moreover, there are exactly two such blocks.*

P r o o f . Suppose that $b \equiv 15 \pmod{16}$. By the proof of Lemma 4.12 of [1], $t = 3$ when $a \equiv \pm 3 \pmod{8}$. By the proof of Lemma 4.2 of [1], $r = 2$ when (9.21) holds, and therefore Lemma 4.4(e), (f), and (g) imply that $r = s = 2$. Since $s < t$, the final conclusion follows from Theorem 8.5. □

THEOREM 9.6. *Assume Hypothesis 4.3. If*

$$b \equiv 15 \pmod{16} \qquad \text{and} \qquad a \equiv \pm 7 \pmod{16}, \tag{9.22}$$

then $t \geq 4$, $s = r = 3$, and every 2^t-block is stable, except possibly those blocks that have a principal divisor equal to 2^{t-3}. Moreover, there are exactly four such blocks.

Proof. Suppose that $b \equiv 15 \pmod{16}$. By the proof of Lemma 4.12 of [1], $t \geq 4$ when $a \equiv \pm 7 \pmod{16}$. By the proof of Lemma 4.2 of [1], $r = 3$ when (9.22) holds, and therefore Lemma 4.4(e), (f), and (g) imply that $r = s = 3$. Since $s < t$, the final conclusion follows from Theorem 8.5. ☐

In the final theorems of this paper, we discuss the case in which $b \equiv 15 \pmod{16}$ and $a \equiv \pm 1 \pmod{16}$. As with the other values for a and b, the main theorems of § 8 can be used to provide a precise description of which blocks corresponding to given a and b may fail to be 2-stable. However, a general characterization of the 2-stability in terms of a and b in this case is quite complicated and depends on delicate computations involving the exact power of 2 dividing $b + 1$ and either $a + 1$ or $a - 1$.

First we consider that case in which $a \equiv 1 \pmod{16}$.

THEOREM 9.7. *Assume Hypothesis 4.3, and suppose that $b \equiv 15 \pmod{16}$ and $a \equiv 1 \pmod{16}$. Choose nonnegative integers α and β and odd integers x and y such that*

$$b = -1 + 2^{\alpha+4}x \qquad \text{and} \qquad a = 1 + 2^{\beta+4}y, \tag{9.23}$$

and positive integers θ, γ, and δ such that

$$2^\theta \parallel \left(x + y\left(1 + (2^{\beta+3}y)\right)\right), \quad 2^\gamma \parallel (x + y), \quad \text{and} \quad 2^\delta \parallel (x - y). \tag{9.24}$$

(a) *If $\alpha < \beta - 1$, then $s = \alpha + 5$, $t = r = \alpha + 4$, and every 2^t-block is stable, except possibly those blocks that have a principal divisor equal to 2. Moreover, there are exactly $2^{\alpha+2}$ such blocks.*

(b) *If $\alpha = \beta - 1$ and $\gamma \neq \alpha + 2$, then $t = r = \alpha + 4$, $t < s < 2t$, and every 2^t-block is stable, except possibly those blocks that have a principal divisor equal to 2^{s-t}. Moreover, there are exactly 2^{2t-s-1} such blocks.*

(c) *If $\alpha = \beta - 1$ and $\gamma = \alpha + 2$, then $t = r = \alpha + 4$, $s \geq 2t$, and every 2^t-block is stable, except possibly the unique block containing $u(a, b)$.*

(d) *If $\alpha = \beta$ and $\delta \neq \alpha + 3$, then $t = s = \alpha + 4$, $r < 2t$, and every 2^t-block is stable, except possibly those blocks that have a principal divisor equal to 2^{r-t}. Moreover, there are exactly 2^{2t-r-1} such blocks.*

(e) *If $\alpha = \beta$ and $\delta = \alpha + 3$, then $t = s = \alpha + 4$, $r \geq 2t$, and every 2^t-block is stable, except possibly the unique block containing $u(a, b)$.*

(f) *If* $\alpha = \beta + 1$, *then* $t = \alpha + 4 + \theta$, $r = s = \alpha + 3$, *and every* 2^t*-block is stable, except possibly those blocks that have a principal divisor equal to* $2^{\theta+1}$. *Moreover, there are exactly* $2^{\alpha+2}$ *such blocks.*

(g) *If* $\alpha > \beta + 1$, *then* $t = \beta + 5$, $r = s = \beta + 4$, *and every* 2^t*-block is stable, except possibly those blocks that have a principal divisor equal to* 2. *Moreover, there are exactly* $2^{\beta+3}$ *such blocks.*

Note. If $b = -1$, then α and x are not well-defined. In this case we write $\alpha = \infty$ and $x = 0$ and, by convention, $\alpha > \beta$ for all integers β. Similarly, if $a = 1$, then β and y are not well-defined. In this case we write $\beta = \infty$ and $y = 0$ and, by convention, $\beta > \alpha$ for all integers α. If both conditions $b = -1$ and $a = 1$ occur, then t is not defined, and we refer the reader to the Addendum of [1] for a discussion of $u(1, -1)$. The general case $w(1, -1) \in \mathscr{F}(1, -1)$ may be handled similarly.

Proof. (a) By the proof of Lemma 4.12 of [1], $t = \alpha + 4$ and $s < 2t$ in this case. Moreover, the method used in Lemma 4.1 of [1] shows that $s = \alpha + 5$. Since $t < s$, Lemma 4.4(f) implies that $r = t = \alpha + 4$. The final conclusion now follows from Theorem 8.6.

(b) The proof of Lemma 4.12 of [1] shows that $t = \alpha+4$ and, since $\gamma \neq \alpha+2$, that $s < 2t$. Moreover, the computation in Lemma 4.12 of [1] also shows that $s \geq \alpha+6$, and hence $t < s$. Since $t < s$, Lemma 4.4(f) implies that $r = t = \alpha+4$. Finally, since $t < s < 2t$, the final conclusion follows from Theorem 8.6.

(c) The proof of Lemma 4.12 of [1] shows that $t = \alpha+4$ and, since $\gamma = \alpha+2$, that $s \geq 2t$. Since $t < s$, Lemma 4.4(f) implies that $r = t = \alpha + 4$. The final conclusion now follows from Theorem 8.3.

(d) By the proof of Lemma 4.12 of [1], $t = \alpha + 4$ and $s < 2t$. Moreover, the method used in Lemma 4.1 of [1] shows that $s = \alpha + 4$. Since $\delta \neq \alpha + 3$, the computation in Theorem 4.13 of [1] shows that $r < 2t$. The final conclusion now follows from Theorem 8.7.

(e) By the proof of Lemma 4.12 of [1], $t = \alpha+4$. Moreover, the method used in Lemma 4.1 of [1] shows that $s = \alpha + 4$. Since $\delta = \alpha + 3$, the computation in Theorem 4.13 of [1] shows that $r \geq 2t$. The final conclusion now follows from Theorem 8.3.

(f) By the proof of Lemma 4.12 of [1], $t = \alpha + 4 + \theta$ and $s < 2t$. Moreover, the method used in Lemma 4.1 of [1] shows that $s = \beta+4 = \alpha+3$. Since $s < t$, Lemma 4.4(e) implies that $r = s = \alpha+3$. The final conclusion now follows from Theorem 8.5.

(g) By the proof of Lemma 4.12 of [1], $t = \beta + 5$ and $s < 2t$. Moreover, the method used in Lemma 4.1 of [1] shows that $s = \beta + 4$. Since $s < t$, Lemma 4.4(e) implies that $r = s = \beta + 4$. The final conclusion now follows from Theorem 8.5. $\square$

Remark. We can actually obtain somewhat more information in cases (b) and (d) of the preceding theorem. In particular, if $\alpha = \beta - 1$, as in (b), then the condition that $\gamma \neq \alpha + 2$ is equivalent to $\alpha + 2 < \zeta$ or $\alpha + 2 > \zeta$, where $2^\zeta \parallel (y + x) + 2^{\alpha+5} y(3y + 2^{\beta+4} y^2 + x)$. If $\alpha + 2 < \zeta$, then the arguments in Lemma 4.12 and Lemma 4.1 of [1] show that $s = 2t - 1 = 2\alpha + 7$. In this case, there is a unique block with principal divisor equal to $2^{\alpha+3}$ that may fail to be 2-stable. If $\alpha + 2 > \zeta$, then $s = \alpha + \zeta + 5$ and there are $2^{\alpha-\zeta+2}$ distinct blocks with principal divisor equal to $2^{\zeta+1}$ that may fail to be 2-stable.

Similarly, if $\alpha = \beta$, as in (d), then the condition that $\delta \neq \alpha + 3$ is equivalent to $\alpha + 3 < \xi$ or $\alpha + 3 > \xi$, where $2^\xi \parallel (x - y) + 2^{\alpha+4} xy$. If $\alpha + 3 < \xi$, then the arguments in Lemma 4.13 and Lemma 4.2 of [1] show that $r = 2t - 1 = 2\alpha + 7$. In this case, there is a unique block with principal divisor equal to $2^{\alpha+3}$ that may fail to be 2-stable. If $\alpha + 3 > \xi$, then $r = \alpha + \xi + 4$ and there are $2^{\alpha-\xi+3}$ distinct blocks with principal divisor equal to 2^ξ that may fail to be 2-stable.

Cases (c) and (e) of the previous theorem, the cases in which the unique block containing the generalized Fibonacci sequence $u(a, b)$ is the only block that may fail to be 2-stable, can be restated in a simpler form as follows.

THEOREM 9.8. *Assume Hypothesis 4.3, and suppose that* $b \equiv 15 \pmod{16}$ *and* $a \equiv 1 \pmod{16}$. *Then every* 2^t*-block is stable, except possibly the unique block containing* $u(a, b)$, *if and only if there exists a nonnegative integer* α *such that one of the following two conditions is satisfied:*

$$\begin{gathered} a \equiv 1 + 2^{\alpha+4} \pmod{2^{\alpha+5}}, \qquad b \equiv -1 + 2^{\alpha+4} \pmod{2^{\alpha+5}}, \\ \textit{and} \quad a - b - 2 \equiv 2^{2\alpha+7} \pmod{2^{2\alpha+8}}, \end{gathered} \tag{9.25}$$

$$\begin{gathered} a \equiv 1 + 2^{\alpha+5} \pmod{2^{\alpha+6}}, \qquad b \equiv -1 + 2^{\alpha+4} \pmod{2^{\alpha+5}}, \\ \textit{and} \quad a + 2b + 1 \equiv 2^{2\alpha+7} \pmod{2^{2\alpha+8}}. \end{gathered} \tag{9.26}$$

P r o o f. The theorem follows immediately from Theorem 9.7 and the reformulation of conditions (c) and (e) proven in Lemma 4.12 and Lemma 4.13 of [1]. $\square$

Finally, we consider the case that $a \equiv -1 \pmod{16}$. Note that the statement of Theorem 9.9 is identical to that of Theorem 9.7, except for the minor changes in the definitions of θ, δ, and γ.

THEOREM 9.9. *Assume Hypothesis 4.3, and suppose that* $b \equiv 15 \pmod{16}$ *and* $a \equiv -1 \pmod{16}$. *Choose nonnegative integers* α *and* β *and odd integers* x *and* y *such that*

$$b = -1 + 2^{\alpha+4} x \qquad \textit{and} \qquad a = -1 + 2^{\beta+4} y, \tag{9.27}$$

and positive integers θ, γ, and δ such that

$$2^{\theta} \parallel \bigl(x - y(1-(2^{\beta+3}y))\bigr), \quad 2^{\gamma} \parallel (y-x), \quad \text{and} \quad 2^{\delta} \parallel (x+y). \tag{9.28}$$

(a) *If $\alpha < \beta - 1$, then $s = \alpha + 5$, $t = r = \alpha + 4$, and every 2^t-block is stable, except possibly those blocks that have a principal divisor equal to 2. Moreover, there are exactly $2^{\alpha+2}$ such blocks.*

(b) *If $\alpha = \beta - 1$ and $\gamma \neq \alpha + 2$, then $t = r = \alpha + 4$, $t < s < 2t$, and every 2^t-block is stable, except possibly those blocks that have a principal divisor equal to 2^{s-t}. Moreover, there are exactly 2^{2t-s-1} such blocks.*

(c) *If $\alpha = \beta - 1$ and $\gamma = \alpha + 2$, then $t = r = \alpha + 4$, $s \geq 2t$, and every 2^t-block is stable, except possibly the unique block containing $u(a,b)$.*

(d) *If $\alpha = \beta$ and $\delta \neq \alpha + 3$, then $t = s = \alpha + 4$, $r < 2t$, and every 2^t-block is stable, except possibly those blocks that have a principal divisor equal to 2^{r-t}. Moreover, there are exactly 2^{2t-r-1} such blocks.*

(e) *If $\alpha = \beta$ and $\delta = \alpha + 3$, then $t = s = \alpha + 4$, $r \geq 2t$, and every 2^t-block is stable, except possibly the unique block containing $u(a,b)$.*

(f) *If $\alpha = \beta + 1$, then $t = \alpha + 4 + \theta$, $r = s = \alpha + 3$, and every 2^t-block is stable, except possibly those blocks that have a principal divisor equal to $2^{\theta+1}$. Moreover, there are exactly $2^{\alpha+2}$ such blocks.*

(g) *If $\alpha > \beta + 1$, then $t = \beta + 5$, $r = s = \beta + 4$, and every 2^t-block is stable, except possibly those blocks that have a principal divisor equal to 2. Moreover, there are exactly $2^{\beta+3}$ such blocks.*

Note. As in Theorem 9.7, if $b = -1$, then α and x are not well-defined, and we write $\alpha = \infty$ and $x = 0$, and, by convention, $\alpha > \beta$ for all integers β. Similarly, if $a = -1$, then β and y are not well-defined, and we write $\beta = \infty$ and $y = 0$. Again, by convention, $\beta > \alpha$ for all integers α. If both conditions $b = -1$ and $a = -1$ occur, then r is not defined, and, again, we refer the reader to the Addendum of [1] for a discussion of $u(-1,-1)$. The general case $w(-1,-1) \in \mathscr{F}(-1,-1)$ may be handled similarly.

P r o o f . The proof is essentially the same as that of Theorem 9.7, except that the second cases of Lemma 4.12 and Lemma 4.13 of [1] are used. We leave the verification of the details to the reader. □

Remark. Once again, we can obtain somewhat more information in cases (b) and (d) of the preceding theorem.

If $\alpha = \beta - 1$, as in (b), then the condition that $\gamma \neq \alpha + 2$ is equivalent to $\alpha + 2 < \zeta$ or $\alpha + 2 > \zeta$, where this time $2^{\zeta} \parallel (y-x) + 2^{\alpha+5}y(-3y + 2^{\beta+4}y^2 + x)$. If $\alpha + 2 < \zeta$, then the arguments in Lemma 4.12 and Lemma 4.1 of [1] show that $s = 2t - 1 = 2\alpha + 7$. In this case, there is a unique block with principal divisor equal to $2^{\alpha+3}$ that may fail to be 2-stable. If $\alpha+2>\zeta$, then $s=\alpha+\zeta+5$ and there are $2^{\alpha-\zeta+2}$ distinct blocks with principal divisor equal to $2^{\zeta+1}$ that may fail to be 2-stable.

Similarly, if $\alpha=\beta$, as in (d), then the condition that $\delta\neq\alpha+3$ is equivalent to $\alpha+3<\xi$ or $\alpha+3>\xi$, where this time $2^{\xi}\parallel(-x-y)+2^{\alpha+4}xy$. If $\alpha+3<\xi$, then the arguments in Lemma 4.13 and Lemma 4.2 of [1] show that $r=2t-1=2\alpha+7$. In this case, there is a unique block with principal divisor equal to $2^{\alpha+3}$ that may fail to be 2-stable. If $\alpha+3>\xi$, then $r=\alpha+\xi+4$ and there are $2^{\alpha-\xi+3}$ distinct blocks with principal divisor equal to 2^{ξ} that may fail to be 2-stable.

Once again, cases (c) and (e), in which the unique block containing the generalized Fibonacci sequence $u(a,b)$ is the only block that may fail to be 2-stable, can be restated in a simpler form.

THEOREM 9.10. *Assume Hypothesis 4.3, and suppose that $b\equiv 15 \pmod{16}$ and $a\equiv -1 \pmod{16}$. Then every 2^t-block is stable, except possibly the unique block containing $u(a,b)$, if and only if there exists a nonnegative integer α such that one of the following two conditions is satisfied:*

$$\begin{gathered} a\equiv -1+2^{\alpha+4} \pmod{2^{\alpha+5}}, \qquad b\equiv -1+2^{\alpha+4} \pmod{2^{\alpha+5}}, \\ \textit{and} \quad a+b+2\equiv 2^{2\alpha+7} \pmod{2^{2\alpha+8}}, \end{gathered} \tag{9.29}$$

$$\begin{gathered} a\equiv -1+2^{\alpha+5} \pmod{2^{\alpha+6}}, \qquad b\equiv -1+2^{\alpha+4} \pmod{2^{\alpha+5}}, \\ \textit{and} \quad a-2b-1\equiv 2^{2\alpha+7} \pmod{2^{2\alpha+8}}. \end{gathered} \tag{9.30}$$

Proof. The theorem follows immediately from Theorem 9.9 and the reformulation of conditions (c) and (e) proven in Lemma 4.12 and Lemma 4.13 of [1]. □

REFERENCES

[1] CARLIP, W.—JACOBSON, E.: *A criterion for stability of two-term recurrence sequences modulo 2^k*, Finite Fields Appl. **2** (1996), 369–406.

[2] CARLIP, W.—JACOBSON, E.: *On the stability of certain Lucas sequences modulo 2^k*, Fibonacci Quart. **34** (1996), 298–305.

[3] CARLIP, W.—JACOBSON, E.: *Unbounded stability of two-term recurrence sequences modulo 2^k*, Acta Arith. **74** (1996), 329–346.

[4] CARLIP, W.—JACOBSON, E.: *Stability of two-term recurrence sequences with even parameter*, Finite Fields Appl. **3** (1997), 70–83.

[5] CARLIP, W.—JACOBSON, E.—SOMER, L.: *A criterion for stability of two-term recurrence sequences modulo odd primes*, in: Applications of Fibonacci numbers, Vol. 7, (Graz, 1996) (Dordrecht), Kluwer Acad. Publ., Dordrecht, 1998, pp. 49–60.

[6] CARLIP, W.—SOMER, L.: *Bounds for frequencies of residues of regular second-order recurrences modulo p^r*, in: Number theory in progress, Vol. 2, (Zakopane-Kościelisko, 1997) (Berlin), de Gruyter, Berlin, 1999, pp. 691–719.

[7] CARROLL, D.—JACOBSON, E.—SOMER, L.: *Distribution of two-term recurrence sequences mod p^e*, Fibonacci Quart. **32** (1994), 260–265.

[8] JACOBSON, E. T.: *Distribution of the Fibonacci numbers mod 2^k*, Fibonacci Quart. **30** (1992), 211–215.

[9] MORGAN, M. D.: *The distribution of second order linear recurrence sequences mod 2^m*, Acta Arith. **83** (1998), 181–195.

[10] SOMER, L.—CARLIP, W.: *Stability of second-order recurrences modulo p^r*, Internat. J. Math. Math. Sci. **23** (2000), 225–241.

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