Tatra Mt. Math. Publ. **20** (2000), 111–117

TATRA MOUNTAINS
Mathematical Publications

BERNOULLI NUMBERS AND BINARY TREES

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ABSTRACT. We generalize the idea of S. C. Woon (1997) of a binary tree for generating Bernoulli numbers. We introduce a construction of a binary tree for generating any complex sequence and give a few examples concerning the generalized Bernoulli numbers.

S. C. Woon [2] introduced a binary tree whose nodes are rational numbers and the sum of the nodes at the nth level of the tree is equal to $B_n/n!$, where B_n is the nth Bernoulli number. The tree is generated by two simple arithmetic operators for computing the two children of each node. Woon's proof is analytical, using a certain continuation of the Riemann zeta function. We show, with quite different and elementary methods, that Woon's tree is in fact a particular case of a more general construction which may be applied to any complex sequence.

Let Ω be the set of all m-tuples $(k_1, \dots, k_m)$ with $m, k_1, \dots, k_m$ positive integers. We define operators $P, I : \Omega \to \Omega$ as follows:

$$P(k_1, k_2, \dots, k_m) = (1, k_1, k_2, \dots, k_m),$$
$$I(k_1, k_2, \dots, k_m) = (k_1 + 1, k_2, \dots, k_m).$$

Operator P changes an m-tuple into an $(m+1)$-tuple by putting 1 at the beginning and operator I increases the first number in the given m-tuple by 1.

DEFINITION 1. The *general PI-tree* is a binary tree whose nodes are m-tuples $(k_1, \dots, k_m) \in \Omega$, the initial node being (1), and the two children

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2000 Mathematics Subject Classification: 11B68.
Keywords: Bernoulli numbers, binary trees.

Research supported by the Grant Agency of the Czech Republic: Algebraic and Analytic Methods in Number Theory, No. 201/97/0433.

of a given node $(k_1, \dots, k_m)$ are $P(k_1, \dots, k_m)$ and $I(k_1, \dots, k_m)$.

$$\begin{tikzcd}[column sep=small]
 & & & (1) \arrow[dll, "P"'] \arrow[drr, "I"] & & & \\
 & (1,1) \arrow[dl, "P"'] \arrow[dr, "I"] & & & & (2) \arrow[dl, "P"'] \arrow[dr, "I"] & \\
(1,1,1) & & (2,1) & & (1,2) & & (3) \\
 & & & \cdots & & &
\end{tikzcd}$$

Definition 2. Let $(c_n)_{n=1}^{\infty}$ be a sequence of complex numbers. The *PI-tree of the sequence* $(c_n)_{n=1}^{\infty}$ is the binary tree obtained from the general *PI*-tree by replacing the nodes $(k_1, \dots, k_m)$ with products $c_{k_1} \dots c_{k_m}$.

$$\begin{tikzcd}[column sep=small]
 & & & c_1 \arrow[dll, "P"'] \arrow[drr, "I"] & & & \\
 & c_1 c_1 \arrow[dl, "P"'] \arrow[dr, "I"] & & & & c_2 \arrow[dl, "P"'] \arrow[dr, "I"] & \\
c_1 c_1 c_1 & & c_2 c_1 & & c_1 c_2 & & c_3 \\
 & & & \cdots & & &
\end{tikzcd}$$

Theorem. *Let $(c_n)_{n=1}^{\infty}$ be a sequence of complex numbers. Let $(x_n)_{n=1}^{\infty}$ be a sequence defined inductively by the formula*

$$x_n = \sum_{j=1}^{n-1} c_j x_{n-j} + c_n \,. \tag{1}$$

Then x_n is equal to the sum of the nodes at the nth level of the PI-tree of the sequence $(c_n)_{n=1}^{\infty}$, counting the initial node as the first level.

Proof. First we observe that, in the general *PI*-tree, the nth level is formed by all the m-tuples $(k_1, \dots, k_m) \in \Omega$ with $m \ge 1$ and $k_1 + \dots + k_m = n$, and each such m-tuple appears exactly once. Therefore, in the *PI*-tree of the sequence $(c_n)_{n=1}^{\infty}$, the sum of the nodes at the nth level is

$$\sum_{\substack{k_1 + \dots + k_m = n \\ m \ge 1}} c_{k_1} \dots c_{k_m} \,,$$

and we have to establish that this sum is equal to x_n. But this equality holds for $x_1 = c_1$, and supposing it holds for all $\nu < n$, we get

$$x_n = \sum_{j=1}^{n-1} c_j x_{n-j} + c_n = \sum_{j=1}^{n-1} c_j \left(\sum_{\substack{k_1 + \dots + k_m = n-j \\ m \ge 1}} c_{k_1} \dots c_{k_m} \right) + c_n$$

$$= \sum_{\substack{k_1+\cdots+k_m=n \\ m\geq 2}} c_{k_1} \ldots c_{k_m} + c_n = \sum_{\substack{k_1+\cdots+k_m=n \\ m\geq 1}} c_{k_1} \ldots c_{k_m} .$$

The theorem is thus proved. □

In view of the above theorem, the *PI*-tree of the sequence $(c_n)_{n=1}^{\infty}$ will also be called the *PI*-tree for generating x_n. The *PI*-tree for generating x_n exists and is unique for any complex sequence $(x_n)_{n=1}^{\infty}$, since formula (1) uniquely determines $(c_n)_{n=1}^{\infty}$ for any given $(x_n)_{n=1}^{\infty}$.

Our aim now is to construct the *PI*-trees for generating certain sequences. Description of these trees will be easier with the following extension of the definition of operators P and I. Operators P and I were defined to act only on the m-tuples of positive integers. However, if we treat the nodes of the *PI*-tree of a sequence $(c_n)_{n=1}^{\infty}$, i.e., the products $c_{k_1} \ldots c_{k_m}$, as finite sequences of symbols, and all the c_n are distinct, we may define

$$P(c_{k_1} c_{k_2} \ldots c_{k_m}) = c_1 c_{k_1} c_{k_2} \ldots c_{k_m} ,$$
$$I(c_{k_1} c_{k_2} \ldots c_{k_m}) = c_{k_1+1} c_{k_2} \ldots c_{k_m} .$$

Operator P puts c_1 at the beginning of an m-tuple and operator I increases the first index by 1. The *PI*-tree of the sequence $(c_n)_{n=1}^{\infty}$ of distinct complex numbers can thus be defined as a binary tree whose nodes are $c_{k_1} \ldots c_{k_m}$ with $m, k_1, \ldots, k_m$ positive integers, each node has two children, $P(c_{k_1} \ldots c_{k_m})$ and $I(c_{k_1} \ldots c_{k_m})$, and the initial node is c_1.

COROLLARY 1. *The PI-tree for generating $B_n/n!$.*

The Bernoulli numbers satisfy, for $n \geq 1$, the following formula,

$$(n+1)B_n = -\sum_{j=0}^{n-1} \binom{n+1}{j} B_j ,$$

from which we easily obtain

$$\frac{B_n}{n!} = \sum_{j=1}^{n-1} \frac{-1}{(j+1)!} \cdot \frac{B_{n-j}}{(n-j)!} + \frac{-1}{(n+1)!} .$$

This is, however, formula (1) for $B_n/n!$ with $c_n = \frac{-1}{(n+1)!}$. Therefore, the *PI*-tree for generating $B_n/n!$ has the nodes

$$c_{k_1} \ldots c_{k_m} = \frac{(-1)^m}{(k_1+1)! \ldots (k_m+1)!} ,$$

and each node has two children,

$$P\left(\frac{(-1)^m}{(k_1+1)!\,(k_2+1)! \ldots (k_m+1)!} \right) = \frac{(-1)^{m+1}}{2!\,(k_1+1)!\,(k_2+1)! \ldots (k_m+1)!} ,$$

$$I\left(\frac{(-1)^m}{(k_1+1)!\,(k_2+1)!\ldots(k_m+1)!}\right)=\frac{(-1)^m}{(k_1+2)!\,(k_2+1)!\ldots(k_m+1)!}\,,$$

and the initial node is $c_1=\frac{-1}{2!}$. The sum of the nodes at the nth level is $B_n/n!$. It is the tree introduced, as mentioned at the beginning, with different methods by S. C. W o o n [2].

$$\begin{tikzcd}[column sep=small]
 & & & \frac{-1}{2!} \arrow[dll, "P"'] \arrow[drr, "I"] & & & \\
 & \frac{1}{2!2!} \arrow[dl, "P"'] \arrow[dr, "I"] & & & & \frac{-1}{3!} \arrow[dl, "P"'] \arrow[dr, "I"] & \\
\frac{-1}{2!2!2!} \arrow[d] \arrow[d] & & \frac{1}{3!2!} \arrow[d] & & \frac{1}{2!3!} \arrow[d] & & \frac{-1}{4!} \arrow[d] \\
 & & & \cdots & & &
\end{tikzcd}$$

If the sequence $(x_n)_{n=1}^{\infty}$ is defined as coefficients in a power expansion of a certain function, then $(c_n)_{n=1}^{\infty}$ for formula (1) can also be found as coefficients of a power expansion. Let

$$f(t)=1+\sum_{n=1}^{\infty}x_nt^{\alpha n},$$

where α is some positive integer, and put

$$g(t)=\sum_{n=1}^{\infty}c_nt^{\alpha n}.$$

Formula (1) now implies $f(t)=f(t)g(t)+1$, and therefore

$$g(t)=1-\frac{1}{f(t)}\,. \tag{2}$$

Thus the previous result for the Bernoulli numbers could also be obtained from

$$f(t)=\frac{t}{e^t-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}\,t^n=1+\sum_{n=1}^{\infty}\frac{B_n}{n!}\,t^n,$$

$$g(t)=\sum_{n=1}^{\infty}c_nt^n=1-\frac{1}{f(t)}=1-\frac{e^t-1}{t}=\sum_{n=1}^{\infty}\frac{-1}{(n+1)!}\,t^n,$$

and $c_n=\frac{-1}{(n+1)!}$.

Corollary 2. *The PI-tree for generating $B_{n,\chi_0}/n!$.*

The generalized Bernoulli numbers B_{n,χ_0}, where χ_0 is the trivial Dirichlet character, are defined by

$$f(t)=\frac{te^t}{e^t-1}=\sum_{n=0}^{\infty}\frac{B_{n,\chi_0}}{n!}\,t^n=1+\sum_{n=1}^{\infty}\frac{B_{n,\chi_0}}{n!}\,t^n.$$

Using formula (2), we obtain

$$g(t) = \sum_{n=1}^{\infty} c_n t^n = 1 - \frac{1}{f(t)} = 1 - \frac{e^t - 1}{te^t} = 1 - \frac{e^{-t} - 1}{t} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} t^n,$$

and $c_n = \frac{(-1)^{n+1}}{(n+1)!}$. The *PI*-tree for generating $B_{n,\chi_0}/n!$ has the nodes

$$c_{k_1} \dots c_{k_m} = \frac{(-1)^{(k_1+1)+\dots+(k_m+1)}}{(k_1+1)! \dots (k_m+1)!},$$

and each node has two children,

$$P\left(\frac{(-1)^{(k_1+1)+(k_2+1)+\dots+(k_m+1)}}{(k_1+1)!\,(k_2+1)! \dots (k_m+1)!}\right) = \frac{(-1)^{2+(k_1+1)+(k_2+1)+\dots+(k_m+1)}}{2!\,(k_1+1)!\,(k_2+1)! \dots (k_m+1)!},$$
$$I\left(\frac{(-1)^{(k_1+1)+(k_2+1)+\dots+(k_m+1)}}{(k_1+1)!\,(k_2+1)! \dots (k_m+1)!}\right) = \frac{(-1)^{(k_1+2)+(k_2+1)+\dots+(k_m+1)}}{(k_1+2)!\,(k_2+1)! \dots (k_m+1)!},$$

and the initial node is $c_1 = \frac{1}{2!}$. The sum of the nodes at the nth level is $B_{n,\chi_0}/n!$.

$$\begin{matrix}
 & & & \frac{1}{2!} & & & \\
 & {\scriptstyle P}\swarrow & & & & \searrow{\scriptstyle I} & \\
 & \frac{1}{2!2!} & & & & \frac{-1}{3!} & \\
{\scriptstyle P}\swarrow & & \searrow{\scriptstyle I} & & {\scriptstyle P}\swarrow & & \searrow{\scriptstyle I} \\
\frac{1}{2!2!2!} & & \frac{-1}{3!2!} & & \frac{-1}{2!3!} & & \frac{1}{4!} \\
\swarrow\searrow & & \swarrow\searrow & & \swarrow\searrow & & \swarrow\searrow \\
 & & & \cdots & & &
\end{matrix}$$

The *generalized Bernoulli numbers* $B_{n,\chi}$, where χ is a nontrivial Dirichlet character with a conductor f, are defined by

$$\sum_{a=1}^{f} \chi(a) \frac{te^{at}}{e^{ft} - 1} = \sum_{n=0}^{\infty} \frac{B_{n,\chi}}{n!} t^n,$$

from which we get, since $B_{0,\chi} = 0$,

$$\sum_{a=1}^{f} \frac{\chi(a)e^{at}}{B_{1,\chi}(e^{ft} - 1)} = \sum_{n=0}^{\infty} \frac{B_{n+1,\chi}}{B_{1,\chi}(n+1)!} t^n. \tag{3}$$

Numbers $B_{n+1,\chi}/\left[B_{1,\chi}(n+1)\right]$ are for $f = 3$ denoted D_n and called the *D-numbers*, and for $f = 4$ denoted E_n and called the *Euler numbers* [1].

Corollary 3. *The PI-tree for generating $D_{2n}/(2n)!$.*

By (3),

$$\sum_{n=0}^{\infty} \frac{D_n}{n!} t^n = \frac{e^t - e^{2t}}{\frac{-1}{3}\left(e^{3t} - 1\right)} = \frac{3}{e^t + e^{-t} + 1} = f(t).$$

Since $f(t)$ is even and $f(0)=1$, we have the expansion

$$f(t)=1+\sum_{n=1}^{\infty}\frac{D_{2n}}{(2n)!}\,t^{2n}\,.$$

Therefore, using formula (2),

$$g(t)=\sum_{n=1}^{\infty}c_n t^{2n}=1-\frac{1}{f(t)}=1-\frac{e^t+e^{-t}+1}{3}=\sum_{n=1}^{\infty}\frac{-2}{3}\frac{1}{(2n)!}\,t^{2n},$$

and $c_n=\frac{-2}{3}\frac{1}{(2n)!}$. The *PI*-tree for generating $D_{2n}/(2n)!$ has the nodes

$$c_{k_1}\ldots c_{k_m}=\left(\frac{-2}{3}\right)^m\frac{1}{(2k_1)!\ldots(2k_m)!}\,,$$

each node has two children,

$$P\left(\left(\frac{-2}{3}\right)^m\frac{1}{(2k_1)!\,(2k_2)!\ldots(2k_m)!}\right)=\left(\frac{-2}{3}\right)^{m+1}\frac{1}{2!\,(2k_1)!\,(2k_2)!\ldots(2k_m)!}\,,$$
$$I\left(\left(\frac{-2}{3}\right)^m\frac{1}{(2k_1)!\,(2k_2)!\ldots(2k_m)!}\right)=\left(\frac{-2}{3}\right)^{m}\frac{1}{(2k_1+2)!\,(2k_2)!\ldots(2k_m)!}\,,$$

and the initial node is $c_1=\frac{-2}{3}\frac{1}{2!}$. The sum of the nodes at the nth level is $D_{2n}/(2n)!$.

$$\begin{tikzcd}[column sep=tiny]
 & & & \frac{-2}{3}\frac{1}{2!} \arrow[dl, "P"'] \arrow[dr, "I"] & & & \\
 & & \left(\frac{-2}{3}\right)^2\frac{1}{2!2!} \arrow[dl, "P"'] \arrow[d, "I"] & & \frac{-2}{3}\frac{1}{4!} \arrow[d, "P"'] \arrow[dr, "I"] & & \\
 & \left(\frac{-2}{3}\right)^3\frac{1}{2!2!2!} & \left(\frac{-2}{3}\right)^2\frac{1}{4!2!} & & \left(\frac{-2}{3}\right)^2\frac{1}{2!4!} & \frac{-2}{3}\frac{1}{6!} &
\end{tikzcd}$$

$$\cdots$$

Corollary 4. *The PI-tree for generating $E_{2n}/(2n)!$.*

Again by (3),

$$\sum_{n=0}^{\infty}\frac{E_n}{n!}\,t^n=\frac{e^t-e^{3t}}{\frac{-1}{2}\left(e^{4t}-1\right)}=\frac{2}{e^t+e^{-t}}=f(t)\,.$$

Since $f(t)$ is even and $f(0)=1$, we have the expansion

$$f(t)=1+\sum_{n=1}^{\infty}\frac{E_{2n}}{(2n)!}\,t^{2n}.$$

Therefore, using formula (2),

$$g(t) = \sum_{n=1}^{\infty} c_n t^{2n} = 1 - \frac{1}{f(t)} = 1 - \frac{e^t + e^{-t}}{2} = \sum_{n=1}^{\infty} \frac{-1}{(2n)!}\, t^{2n},$$

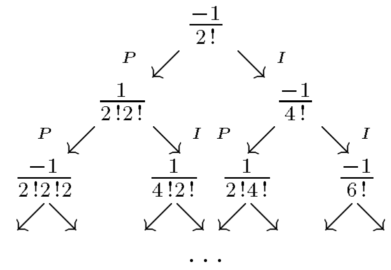
and $c_n = \frac{-1}{(2n)!}$. The PI-tree for generating $E_{2n}/(2n)!$ has the nodes

$$c_{k_1} \dots c_{k_m} = \frac{(-1)^m}{(2k_1)! \dots (2k_m)!}\,,$$

and each node has two children,

$$P\left(\frac{(-1)^m}{(2k_1)!\,(2k_2)! \dots (2k_m)!}\right) = \frac{(-1)^{m+1}}{2!\,(2k_1)!\,(2k_2)! \dots (2k_m)!}\,,$$
$$I\left(\frac{(-1)^m}{(2k_1)!\,(2k_2)! \dots (2k_m)!}\right) = \frac{(-1)^m}{(2k_1+2)!\,(2k_2)! \dots (2k_m)!}\,,$$

and the initial node is $c_1 = \frac{-1}{2!}$. The sum of the nodes at the nth level is $E_{2n}/(2n)!$.



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Received December 20, 1999

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