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SEQUENCES OF LOW FACTOR INTEGERS

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ABSTRACT. At the 2nd Czech and Polish Conference in Number Theory, A. Schinzel stated the problem to find $4n$ consecutive integers each with a prime factor less or equal to n [A. Schinzel; *Problem 1*, Annales Mathematicae Silesianae **12** (1998), 173]. This paper describes a simple algorithm that allows us to find such a sequence. The problem is extended to sequences of $k \cdot n$ consecutive integers. Examples of the least n found for $k = 2, \dots, 8$ are given.

1. The problem

It was proved by Westzynhius [4] that there exists a p_0 with the property that for every $p > p_0$ there exists an integer m such that $m + i$ has a prime factor not exceeding p for all nonnegative integers i less than $\frac{p \log\log p}{\log\log\log p}$.

This result implies the existence of large gaps between consecutive primes and it has been improved many times, for the last time by Pintz [1] (for an intermediate result see the proof of Theorem 5.1 in Chapter 5 of [2]). It follows hence that for every k there exists a prime p and an integer m such that $m+i$ has a prime factor not exceeding p for $i = 0, 1, \dots, k \cdot p - 1$. It is, however, a non-trivial task to find p and m for a given k, and A. Schinzel [3] has asked for an example with $k = 4$ (he asked for more than $4n$ consecutive integers, but a suitable sequence of $4n$ consecutive integers immediately extends to a sequence of $4 \cdot n + 1$ consecutive integers with the same property by adjusting the even number at one or at the other end).

Let $\mathrm{lf}(x)$ denote least prime factor of x, $\mathcal{P}(k, n, a_0)$ mean that the numbers

$$\mathrm{lf}(a_0 + i), \qquad i \in \{0, 1, \cdots, n \cdot k - 1\},$$

do not exceed n, and $\mathcal{P}(k, n)$ denote the problem of finding a_0 such that $\mathcal{P}(k, n, a_0)$ holds. It is clear that $\mathcal{P}(k, n, a_0)$ is equivalent to $\mathcal{P}(k, n, a)$ for

$$a \equiv a_0 \mod \prod_{p \le n} p .$$

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2. Algorithm

Let k be a given integer, p_j the jth prime ($p_1 = 2$, $p_2 = 3$), and suppose we want to solve $\mathcal{P}(k, p_s)$. The basic idea is to assign to all the numbers $x_i = x_0 + i$ in the sequence of length $l = k \cdot p_s$ their $\mathrm{lf}(x_i)$ $(i = 0, 1, \dots, l-1)$. Let m_j be the remainder of $x_0 \mod p_j$. If we decide that $m_1 = 0$, then for all numbers

$$x_i\,, \qquad i \in \{0, 2, 4, \dots\}\,,$$

$\mathrm{lf}(x_i) = 2$ holds, and all remaining x_i, with i odd, have so far unassigned $\mathrm{lf}(x_i)$. Now we should choose the value of m_2. The possible way is to count for each admissible value of m_2 the number of assignments $\mathrm{lf}(x_i) = 3$ to unassigned numbers x_i and to choose a value of m_2 with the maximal number of assignments. Next we choose the value of m_3, m_4, ... till all x_i have their $\mathrm{lf}(x_i) \le p_s$ assigned or p_s is processed. Finally we define the result r of the algorithm as the final number of unassigned numbers x_i or negatively taken number of unused primes p_j.

Thus, we have algorithm that gives for p_s and $l = k \cdot p_s$ the number r. If this number is less or equal to zero, than we can construct from the numbers m_1, ..., m_{s+r} a number

$$a_0 \equiv m_j \mod p_j \tag{1}$$

which satisfies $\mathcal{P}(k, p_s)$. Otherwise we know the number of *holes* x_i in our sequence with $\mathrm{lf}(x_i) > p_s$

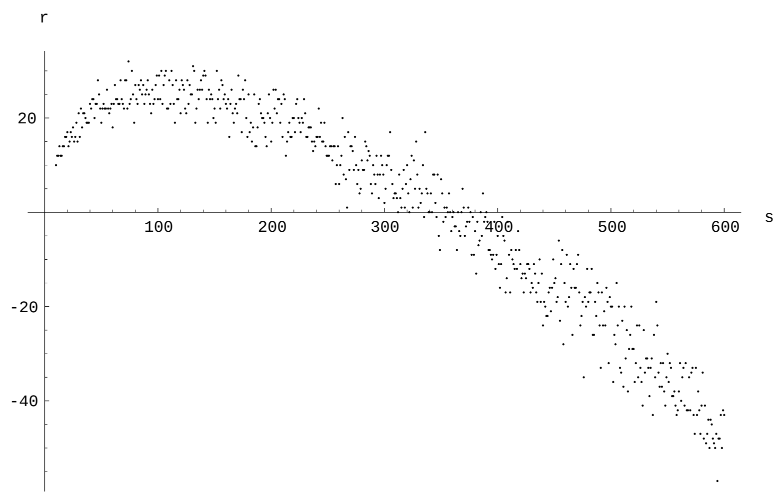

FIGURE 1. Dependence of result r of algorithm on number of used primes s for prime limit p_s and $k = 4$.

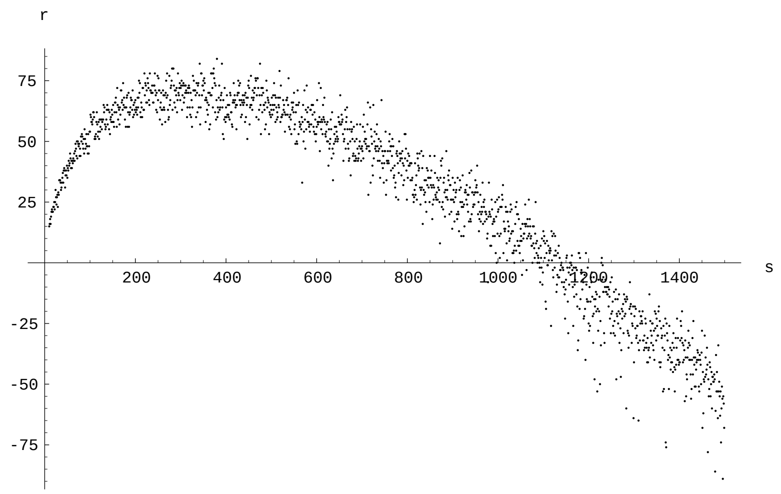

FIGURE 2. Dependence of result r of algorithm on number of used primes s for prime limit p_s and $k = 5$.

Now the solution of problem $\mathcal{P}(k, p_s, a_0)$ may be found using the described algorithm for a direct search over s. On the other hand, we can display the dependence of the algorithm result r on number s of used primes. It is for $k = 4$ shown in Figure 1, and for $k = 5$ in Figure 2.

Results

The search was concentrated on solving the problem $\mathcal{P}(k, p_s)$ for $k = 2, \ldots, 8$ with the least positive integer a_0 satisfying $\mathcal{P}(k, p_s, a_0)$.

Obtained results are summarized in Table 1.

TABLE 1. s—number of primes used, p_s—maximal prime used, $k \cdot p_s$—length of sequence, $\#d$—number of decimal digits of numbers in sequence.

k	s	p_s	$k \cdot p_s$	$\#d$
2	14	43	86	14
3	71	353	1059	143
4	269	1723	6892	731
5	926	7247	36235	3106
6	11861	126487	885409	54760
8	50002	611977	4895816	265473

Computational note

The main algorithm was written in C. Conversions from mods m_i to a_0 as in (1) were performed under PARI-GP. Unfortunately, internal procedure `chinese()` was very slow. Thus, mind that all equations (1) are taken modulo prime, we can write in PARI-GP the following routine

```
n=of[1]; l=length(of); pr=2;
for(j=2, l, p=prime(j);
 n=n+pr*lift((of[j]-n% p)*Mod(pr,p)^(p-2));
 pr=pr*p;
); n
```

where in array `of[]`, m_i are stored and resulting a_0 is in variable `n`. Even for 3000 primes this routine is hundred times faster than routine with `chinese()`, and for larger amount of primes the speed-up is much more significant.

All presented computations were established on my home computer[1] running Linux.

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[1]AMD K6/2–300MHz with FIC VA–503+ motherboard, 64MB of memory