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THE CONVERGENCE KHINTCHINE THEOREM FOR POLYNOMIALS AND PLANAR p-ADIC CURVES

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ABSTRACT. A p-adic analogue of the convergence part of Khintchine's theorem for polynomials of degree n $(n \geq 2)$ with integer coefficients and for planar curves with non-zero curvature is proved.

1. Introduction

In this paper we will consider Diophantine approximation of p-adic numbers and generalize the convergence part of the metric theorem of Khintchine [1]. Similar problems were first investigated for $\mathbb{Z}_p$ by K. Mahler [2].

Let $p \geq 2$ be a prime number, $\mathbb{Q}_p$ be the field of p-adic numbers with the Haar measure, $\mathbb{Z}_p$ be the ring of p-adic integers, $|\cdot|_p$ be the p-adic valuation. Let $n \in \mathbb{N}$, $n \geq 2$, $\Psi(h)\colon \mathbb{N} \to \mathbb{R}^+$ be monotonic and $S_n(\Psi)$ be the set of $x \in \mathbb{Q}_p$ such that the inequality

$$\big|F(x)\big|_p < h^{-n}\Psi(h) \tag{1}$$

holds for infinitely many $F \in \mathbb{Z}[x]$, $\deg F = n$, where $h = h(F)$ is the height of F. Let $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$, $f(x) = \sum\limits_{n=1}^{\infty} c_n(x-c)^n$ be a normal function such that $f''(x) \neq 0$ almost everywhere on $\mathbb{Z}_p$. Let $G(x) = b_0 + b_1 x + b_2 f(x)$, where $b_i \in \mathbb{Z}$ $(i = 0,1,2)$, $H = \max |b_i| \neq 0$, and let $S_f(\Psi)$ be the set of $x \in \mathbb{Z}_p$ such that the inequality

$$\big|G(x)\big|_p < H^{-2}\Psi(H) \tag{2}$$

holds for infinitely many G.

We will prove two metric theorems. The first one establishes the right approximation to zero in $\mathbb{Q}_p$ by the values of polynomials F. The second theorem

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is about Diophantine approximation of general planar curves in $\mathbb{Z}_p \times \mathbb{Z}_p$. The condition on $f''(x)$ is equivalent to the condition that the curvature of the curve $(x, f(x))$ is non-zero almost everywhere.

Theorem 1. *If* $\sum_{h=1}^{\infty} \Psi(h) < \infty$ *then the set* $S_n(\Psi)$ *has zero Haar measure.*

Theorem 2. *If* $\sum_{h=1}^{\infty} \Psi(h) < \infty$ *then the set* $S_f(\Psi)$ *has zero Haar measure.*

In order to prove the theorems we will develop Sprindžuk's method of essential and inessential domains [3] and modify Melnichuk's method [4]. Theorem 1 is an analogue of Bernik's result for the field $\mathbb{R}$ [5]. A similar result for the field $\mathbb{C}$ was obtained by D. Vasilyev [7]. Theorem 2 is an analogue of [8] for the field $\mathbb{R}$ and it is sharper than [6].

Now we recall the definition of a normal function (by Mahler [2], see also [9]).

Definition. The function $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \ \left(|a|_p \le 1\right)$ is called a normal function if and only if 1) $|a_n|_p \le 1$ for all n, 2) $\lim_{n\to\infty} |a_n|_p = 0$.

The class of normal functions is quite wide: given any analytic function $g(z)$ we can find integers r, s such that $p^r g(p^s z)$ is a normal function.

Now suppose that $b \in T_p$, where T_p is the completion of the algebraic closure of $\mathbb{Q}_p$ with its valuation denote also by $|\cdot|_p$, and $|b|_p \le 1$. Then $f(z) = \sum_{n=0}^{\infty} f^{(n)}(b)(z-b)^n/n!$ and we can check that the series above is also normal. Hence, if $f(z)$ is normal, so is $f'(z)$.

Now let us discuss the history of the problems. Mahler's fundamental studies in the theory of transcendental numbers led him in 1932 to the conjecture that for any $\varepsilon > 0$ the inequality $\left|F(x)\right| < h(F)^{-n-\varepsilon}$ has at most a finite number of solutions in polynomials F for almost all $x \in \mathbb{R}$ (here $|\cdot|$ is the ordinary absolute value in $\mathbb{R}$, and F is defined as above).

This conjecture was proved by V. Sprindžuk [3] about 30 years later. A. Baker [10] proved the following more general result: if the function Ψ satisfies the conditions of Theorem 1 then the set of x such that

$$\left|F(x)\right| < \Psi^n\big(h(F)\big) \tag{3}$$

holds for infinitely many F has Lebesgue measure zero.

V. Bernik [5] extended Baker's result by replacing the right-hand side of (1) by $h^{-n+1}\Psi(h)$. By Groshev's theorem [11], the set of points $\overline{x} \in \mathbb{R}^n$ satisfying the inequality $|\overline{b}\cdot\overline{x} - p| < \Psi(|\overline{b}|)/|\overline{b}|^{n-1}$, where $|\overline{b}| = \max\{|b_1|, \cdot, |b_n|\}$, for infinitely many $\overline{b} \in \mathbb{Z}^n$ has full or zero Lebesgue measure according to divergence or convergence of $\sum_r \Psi(r)$, respectively.

Bernik's result [5] is evidently an analogue of the convergence part of Khintchine–Groshev's theorem and it is indeed the best possible result, as the complementary divergence case has recently been proved by V. Beresnevich [12].

Hence Theorem 1 is an analogue of the convergence part of Khintchine–Groshev's theorem in the field $\mathbb{Q}_p$. The detailed proof of it was published in [13].

Let $\Gamma = \{g_1(x), g_2(x) \colon x \in \mathbb{R}\}$ be a planar curve such that $g_1(x)$, $g_2(x)$ are the smooth functions with Wronskian $W(g_1', g_2')$ given by

$$W(g_1', g_2') = \det \begin{pmatrix} g_1' & g_2' \\ g_1'' & g_2'' \end{pmatrix}.$$

W. Schmidt [14] proved a remarkable result that if the Wronskian $W(g_1', g_2')$ is non-zero almost everywhere (equivalent to the curvature being non-zero almost everywhere), then for any positive ε the inequality $\left|G_\Gamma(x)\right| = \left|a_0 + a_1 g_1(x) + a_2 g_2(x)\right| < h^{-2-\varepsilon}$, where $h = \max(|a_0|, |a_1|, |a_2|)$, has only finitely many solutions $(a_0, a_1, a_2) \in \mathbb{Z}^3$ for almost all $x \in \mathbb{R}$. This inequality can also be expressed in the following equivalent form:

$$\left|G_\Gamma(x)\right| < h_\Gamma^{-2-\varepsilon}, \tag{4}$$

where $h_\Gamma = h$, holds for infinitely many Γ for almost no $x \in \mathbb{R}$. In other words, the planar curve Γ is extremal (see the definition in [15, p. 68]). For $\varepsilon = 0$ Dirichlet's theorem implies that every point of Γ satisfies the inequality (4) infinitely often. R. C. Baker [16] generalized this result by replacing inequality (4) by

$$\left|G_\Gamma(x)\right| < \Psi(h_\Gamma)^{-2}, \tag{5}$$

where the function Ψ is defined as above. A sharper result than that in [16] was obtained in [8]: instead of (5) they investigated the following inequality

$$\left|G_\Gamma(x)\right| < h_\Gamma^{-1}\Psi(h_\Gamma). \tag{6}$$

Let us return to the field of p-adic numbers $\mathbb{Q}_p$. Yu. Melnichuk [4] proved an analogue of W. Schmidt's result in $\mathbb{Q}_p$. Now Theorem 2 is shaper than [4] and it is an analogue of the result concerning the inequality (6) in $\mathbb{Q}_p$. The detailed proof of Theorem 2 will be published in [17]. Here we give the schemes of the proofs of both theorems and the lemmas.

2. Lemmas

Let $\mathcal{F}_n(H)$ be the set of primitive and irreducible polynomials F of degree n and height H with integer coefficients satisfying the following conditions: $|a_n| = H$, $|H|_p > p^{-n}$. Let be $\mathcal{F}_n = \bigcup\limits_{H=1}^{\infty} \mathcal{F}_n(H)$.

Lemma 1. *Without loss of generality we can restrict the solubility of* (1) *to polynomials* $F \in \mathcal{F}_n$.

The restriction to irreducible and primitive polynomials is based on a p-adic variant of Lemma 7 in [5] and on Lemma 10 in [3, p. 93].

Let $\{\alpha_n\}$ be a sequence of elements which are algebraic over $\mathbb{Q}$ and let $\mathbb{Q}^\star$ be the smallest extension field of $\mathbb{Q}$ containing all elements of this sequences. The field $\mathbb{Q}^\star$ can be defined by the following procedure: we consider the sequence of the fields $\mathbb{Q}_0 = \mathbb{Q}$, $\mathbb{Q}_1 = \mathbb{Q}_0(\alpha_1)\dots$, $\mathbb{Q}_n = \mathbb{Q}_{n-1}(\alpha_n), \dots$. Then $\mathbb{Q}^\star = \bigcup\limits_{i=1}^{\infty} \mathbb{Q}_i = \lim\limits_{n\to\infty} \mathbb{Q}_n$.

The valuation on $\mathbb{Q} = \mathbb{Q}_0$ can be extended to a valuation on the field $\mathbb{Q}_1$, a finite extension field of $\mathbb{Q}_0$. By induction we can extend the valuation to $\mathbb{Q}_2$, etc. This results in a valuation on the field $\mathbb{Q}^\star$, denoted by $|\cdot|_p$ as well.

We assume that all the roots $\kappa_1, \dots, \kappa_n$ of the polynomials $F \in \mathcal{F}_n(H)$ are different and ordered so that

$$|\kappa_1 - \kappa_2|_p \le |\kappa_1 - \kappa_3|_p \le \dots \le |\kappa_1 - \kappa_n|_p .$$

It is not difficult to prove that $\max\limits_{1\le i\le n} |\kappa_i|_p < p^n$ (see Lemma 4 [3, p. 86]). Let

$$S(\kappa_i) = \left\{\omega \in Q_p\colon |\omega - \kappa_i|_p = \min_{1\le j\le n} |\omega - \kappa_j|_p\right\} \qquad (1 \le i \le n),$$

$$K(p^n) = \left\{\omega \in Q_p\colon |\omega|_p < p^n\right\}.$$

Then it suffices to prove Theorem 1 for the points $\omega \in S(\kappa_1) \cap K(p^n)$ satisfying (1).

Lemma 2. *Let $\delta > 0$, $\sigma > 0$ be real numbers. Let F and Q be two coprime polynomials with integer coefficients of the degree at least n, $n \ge 2$, and $\max(h_F, h_Q) \le H$, where $H = H(\delta, n)$ is a sufficiently large number. If there is a number $\tau > 0$ such that*

$$\max\Bigl(\bigl|F(\omega)\bigr|_p,\ \bigl|Q(\omega)\bigr|_p\Bigr) < H^{-\tau},$$

for all ω from some disc with the radius p^{-t}, $p^{-t} \le H^{-\sigma} < p^{-t+1}$, then $\tau + 2\max(\tau-\sigma, 0) < 2n + \delta$.

For the proof see [18], [19, p. 132].

Let $0 < \varepsilon < 1$. Suppose that $\varepsilon_1 = \varepsilon/d$ where $d > 0$ is a sufficiently large number depending on n only. Let $T = [\varepsilon_1^{-1}]$. We fix the numbers $\rho_j \in \mathbb{R}$, $l_j \in \mathbb{Z}$, $r_j \in \mathbb{Q}$ such that

$$\begin{cases} |\kappa_1 - \kappa_j|_p = H^{-\rho_j}, \quad (l_j - 1)/T \le \rho_j < l_j/T \quad (2 \le j \le n), \\ r_j = (l_{j+1} + \cdots + l_n)/T, \quad (1 \le j \le n-1). \end{cases} \tag{7}$$

Now we associate each polynomial $F \in \mathcal{F}_n(H)$ with a vector $\boldsymbol{l} = (l_2, \ldots, l_n)$ having the nonnegative components. In [3, pp. 99–100] it is shown that the number of these vectors is finite and depends on n, p and T only.

Lemma 3. *Let be $\omega \in S(\kappa_1)$ and $F \in \mathcal{F}_n(H)$. Then*

$$|\omega - \kappa_1|_p \le \left|F(\omega)\right|_p \left|F'(\kappa_1)\right|_p^{-1}. \tag{8}$$

For the proof see [3, p. 77].

Lemma 4. *Let M_1 be the set of such points in $\mathbb{Z}_p$ for which the inequality (2) has infinitely many solutions in coprime integers (b_0, b_1, b_2) with $|b_2|_p < 1$. Then $\mu M_1 = 0$.*

The proof is similar to the proof of Lemma 1 in [4].

Lemma 5. *Let $f(x)$ be a normal function and $0 < \delta \le 1$, $0 < \lambda < 1$. Suppose that $\left|f''(x)\right|_p \ge \delta$ for all x in a disc $K \subset \mathbb{Z}_p$. Then the set $\sigma(f) = \left\{x \in K \colon \left|f(x)\right|_p < \lambda\right\}$ is a union of a finite number of disjoint discs, and if $\mu K \le \delta/2$ then $\sigma(f)$ is covered by at most two of discs with radius $r < (\delta^{-1}\lambda)^{-1/2}$.*

See Lemma 2 and Lemma 3 in [4].

Lemma 6. *Let $f(x)$ be a normal function and $\left|f''(x)\right|_p \ne 0$ for almost all $x \in \mathbb{Z}_p$. Let $G(x) = b_0 + b_1 x + b_2 f(x)$ be a function defined in the Introduction. Then the inequality $\left|G(x)\right|_p < H^{-3-\varepsilon}$, where ε is any positive number, has at most a finite number of integer solutions (b_0, b_1, b_2) for almost all $x \in \mathbb{Z}_p$.*

This lemma was proved in [4] as the main result.

3. Proof of Theorem 1

First of all we make a classification of polynomials $F \in \mathcal{F}_n(H)$ by collecting all the polynomials having the same vector $\boldsymbol{l}$ defined in (7) in one class $\mathcal{F}_n(H, \boldsymbol{l})$. The following investigation essentially depends on the value of $r_1 + l_2/T$.

1. Let $r_1 + l_2/T > n$. According to the condition on $\Psi(h)$ we have $\Psi(H) < cH^{-1}$ provided that H is sufficiently large, where $c > 0$ is a constant independent of H. Then instead of the set of $\omega \in S(\kappa_1) \cap K(p^n)$ satisfying the inequality (1) we can consider the set of $\omega \in S(\kappa_1) \cap K(p^n)$ satisfying the inequality

$$\left|F(\omega)\right|_p < ch^{-n-1}. \tag{9}$$

The second set was analysed in [3, pp. 108 – 111].

2. Let $n - 1 + 2n\varepsilon_1 < r_1 + l_2/T \leq n$. We consider the inequality (9) as in Proposition 1 from [5].

3. Let

$$2 - \varepsilon/2 < r_1 + l_2/T \leq n - 1 + 2n\varepsilon_1\,. \tag{10}$$

We introduce the values

$$\begin{aligned}\theta &= n + 1 - r_1 - l_2/T - \varepsilon_1\,,\\ \beta &= [\theta] - 1 + 0,2\{\theta\} - 0,1\varepsilon\,,\\ \sigma_1 &= l_2/T + 0,8\{\theta\} + (n+1)\varepsilon_1\,,\end{aligned}$$

where $[\theta]$ and $\{\theta\}$ are the integral and the fractional parts of θ respectively. We fix any sufficiently large integer H and divide the disc $S(\kappa_1) \cap K(p^n)$ into the elementary discs with the radius $H^{-\sigma_1}$. The number of these discs is estimated by $c_1 H^{\sigma_1}$, where c_1 is a constant independent of H.

We shall say that the polynomial $F \in \mathcal{F}_n(H, \boldsymbol{l})$ belongs to a disc I if there exists a point $\omega_0 \in I$ such that $\left|F(\omega_0)\right|_p < H^{-n-1}$. We shall consider discs which contain more than H^β polynomials from $\mathcal{F}_n(H, \boldsymbol{l})$. Let us fix such a disc I. All polynomials belonging to I will be divided into the subclasses in the following way. Two polynomials

$$\begin{aligned}F_1(x) &= Hx^n + a_{n-1}^{(1)}x^{n-1} + \cdots + a_1^{(1)}x + a_0^{(1)},\\ F_2(x) &= Hx^n + a_{n-1}^{(2)}x^{n-1} + \cdots + a_1^{(2)}x + a_0^{(2)}\end{aligned}$$

are in one class if $a_{n-1}^{(1)} = a_{n-1}^{(2)}, \ldots, a_{n-d}^{(1)} = a_{n-d}^{(2)}$ where $d = [\theta] - 1$. It is clear that the number of different classes is less than $(2H+1)^d$ and the number of polynomials under consideration is greater than H^β. We apply the so-called “box” or “pigeon-hole” principle. Then there exists a class which contain at least $c_2 H^{\beta - d}$ polynomials where $c_2 > 0$ is a constant independent of H. We denote polynomials of this class by $F_1(x), \ldots, F_{s+1}(x)$ and consider s new polynomials: $R_j(x) = F_{j+1}(x) - F_1(x)$ $(1 \leq j \leq s)$. Thus we can consider polynomials of degree at least $[r_1 + l_2/T] - 1$ with the p-adic valuation less than $H^{-r_1 - l_2/T + 2\varepsilon_1}$. We notice that the p-adic valuation of the polynomial F_j is estimated by using Taylor series for F_j and the condition (7).

The assumption that the number of polynomials $F \in \mathcal{F}_n(H, \boldsymbol{l})$ is large leads to a contradiction with Lemma 2. If the number of these polynomials $F \in \mathcal{F}_n(H, \boldsymbol{l})$ is not too large then the result is proved by the inequality (8) and the Borel – Cantelly lemma. This method is described in Proposition 2 from [5].

4. Let $\varepsilon \le r_1 + l_2/T \le 2 - \varepsilon/2$. We investigate the inequality (2) by the method of essential and inessential domains [3, Part 2, Ch. 2].

5. Let $0 \le r_1 + l_2/T < \varepsilon$. Then we construct a countable cover for the set of points $\omega \in S(\kappa_1) \cap K(p^n)$ satisfying the inequality (1) by the discs with the radius equal $\left((2n+1)H\left|F'(\kappa_1)\right|_p\right)^{-1}$. It is easy to obtain that the measure of this cover is less than the sum $\sum\limits_{H=1}^{\infty} \Psi(H)$. The Borel – Cantelly lemma finishes the proof.

4. Proof of Theorem 2

It is easy to prove that we can consider only the coprime integer solutions (b_0, b_1, b_2) of the inequality (2) and can use $\max\left(|b_1|_p, |b_2|_p\right) = 1$. According to Lemma 5 we investigate the set M_2 of $x \in \mathbb{Z}_p$ for which there are infinitely many coprime integer solutions (b_0, b_1, b_2) of the inequality (2) with $|b_2|_p = 1$ where $H = h_G$ is sufficiently large. This requires the condition on $f''(x)$. We note that the equality $|b_2|_p = 1$ implies the equality $\left|G''(x)\right|_p = \left|f''(x)\right|_p$.

According to the conditions of Theorem 2 we have $f''(x) \neq 0$ almost everywhere in $\mathbb{Z}_p$. Now we remove a set of arbitrarily small measure θ from $\mathbb{Z}_p$ in such a way that the inequality $\left|G''(x)\right|_p \ge \delta$ takes place in the complement part $\mathbb{Z}_p(\theta)$ of $\mathbb{Z}_p$ where $\delta = \delta(\theta)$ and $0 < \delta < 1$. We can represent the set $\mathbb{Z}_p(\theta)$ as a finite sum of disjoint discs K_s having the Haar measure $\mu K_s \le \delta/2$. The following investigation can be applied to any K_s, therefore we omit the index s.

The main idea of the proof is to cover the set $M_2 \cap K$ by a collection of countable sets with a small total measure. We note that the inequality $\Psi(H) < cH^{-1}$, where $c > 0$ is a constant independent of H, holds for a sufficiently large number H. Let $G(x)$ be a fixed function satisfying the conditions of Theorem 2 and $|b_2|_p = 1$. Let (b_0, b_1, b_2) be coprime integer and $0 < \varepsilon_1 < 1$. Given function $G(x)$, we introduce the sets

$$\sigma(G) = \left\{x \in K\colon \left|G(x)\right|_p < H^{-2}\Psi(H)\right\},$$
$$\tau(G) = \left\{x \in K\colon \left|G(x)\right|_p < H^{-2-\varepsilon_1}\right\},$$
$$\tau_1(G) = \left\{x \in K\colon \left|G(x)\right|_p < cH^{-3}\right\},$$

where $H = h_G$. We have $\sigma(F) \subseteq \tau_1(F) \subseteq \tau(F)$. As the set $\sigma(G)$ is compact, there is a point $\alpha_G \in \sigma(G)$ such that

$$\max_{x\in\sigma(G)} \left|G'(x)\right|_p = \left|G'(\alpha_G)\right|_p .$$

By Lemma 5 we can assume that $\tau(G) \subseteq K(G) = K\big(\alpha_G, H^{-1-2\varepsilon}\big)$ where $K\big(\alpha_G, H^{-1-2\varepsilon}\big)$ is the disc with the centre at α_G and radius $H^{-1-2\varepsilon}$, $0 < \varepsilon < \varepsilon_1/4$. We have $\tau_1(G) \subseteq K\big(\alpha_G, cH^{-3/2}\big)$ as well. Using Taylor series for $G(x)$ in the disc $K(G)$ and the properties of the non-archimedian valuation we obtain the inequality

$$|x-\alpha_G|_p = \left|G(x)-G(\alpha_G)\right|_p \left|G'(\alpha_G)\right|_p^{-1} \tag{11}$$

if $x \in K(G)$ and

$$\left|G'(\alpha_G)\right|_p > H^{-1-\varepsilon} . \tag{12}$$

Further we consider four cases according to the magnitude of $\left|G'(x)\right|_p$.

1. Let G be a function satisfying the condition

$$H^{-\varepsilon} \le \left|G'(x)\right|_p \le 1 , \qquad \text{if} \quad x \in K(G). \tag{13}$$

We divide functions G into the subclasses $\mathcal{G}_t(a)$ defined by the conditions $2^t \le h_G = H < 2^{t+1}$ $(t = 0,1,2,\dots)$ and $b_2 = b$. Now we construct a countable cover of $M_2 \cap K$ by the discs $K(\alpha_G, r_1)$ with $r_1 = (4H)^{-1}\left|G'(\alpha_G)\right|_p^{-1}$ using the sets $\tau(G)$ and the conditions (11)–(13). Analysis of the distribution of these discs is similar to Proposition 4 from [17].

2. Let G be a function satisfying the condition

$$H^{-1+2\varepsilon} \le \left|G'(x)\right|_p < H^{-\varepsilon} , \qquad \text{if} \quad x \in K(G). \tag{14}$$

Again we construct a covering of $M_2 \cap K$ by the discs $K(\alpha_G, r_2)$ with $r_2 = H^{-2}\left|G'(\alpha_G)\right|_p^{-1}$. Then we use the Sprindžuk's method of essential and inessential domains [3, Part 2, Ch. 2] as in Proposition 3 in [17] (see also the Section 3 of Paragraph 3).

3. Let G be a function satisfying the condition

$$H^{-1-\varepsilon} < \left|G'(x)\right|_p < H^{-1+2\varepsilon} , \qquad \text{if} \quad x \in K(G). \tag{15}$$

We divide functions G into the subclasses $\mathcal{G}_k(d)$ defined by the conditions $k^2 \le h_G = H < (k+1)^2$ $(k = 0,1,2,\dots)$ with $b_2 \equiv d \pmod{p^m}$, where $1 \le d \le p^{m-1}$, $(d,p) = 1$ and $m \in \mathbb{N}$, m is chosen so that $p^{m-1} \le H^{-1/2} < p^m$. Now we construct a countable cover of $M_2 \cap K$ by the discs $K(\alpha_G, r_3)$ with

$r_3 = c_1 k^{-5} \left| G'(\alpha_G) \right|_p^{-1}$ using the sets $\tau_1(G)$ and the conditions (11)–(13) where $c_1 = (c\delta^{-1})^{1/2}$. We apply Lemma 6 when the discs are inessential. The measure of the covering estimates above as the sum of the convergent series $\sum k^2 \Psi(k^2)$. The Borel – Cantelly lemma finishes the proof.

4. Let G be a function satisfying the condition

$$\left| G'(x) \right|_p < H^{-1-\varepsilon}, \qquad \text{if} \quad x \in K(G). \tag{16}$$

We construct a countable covering for $M_2 \cap K$ using the set $\tau_1(G)$ and according to (2), (16). The p-adic form of Proposition 4 in [12] implies that the measure of the covering equals zero.

All previous investigations can be considered for any δ, $0 < \delta < 1$, and for any θ, $0 < \theta < 1$.

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