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APPROXIMATION OF MEASURABLE MULTIFUNCTIONS

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ABSTRACT. We investigate the problem of approximation of measurable multifunctions by sequences of continuous or measurable simple ones. We apply some known approximation theorems for single-valued maps to obtain results for multifunctions.

1. Preliminaries

We start with the definitions of topologies on hyperspaces, which will be considered later.

For a topological space Y we use, as in [1], the following notation:

$$\begin{aligned}
\mathrm{CL}(Y) &= \{A \in \mathscr{P}(Y) \colon A \text{ is a nonempty closed subset of } Y\},\\
K(Y) &= \{A \in \mathrm{CL}(Y) \colon A \text{ is compact}\},\\
V^- &= \{A \in \mathrm{CL}(Y) \colon A \cap V \neq \emptyset\},\\
V^+ &= \{A \in \mathrm{CL}(Y) \colon A \subset V\},
\end{aligned}$$

where V is a subset of Y. If (Y,ρ) is a metric space then for $y \in Y$ and $\varepsilon > 0$ we denote

$$\begin{aligned}
U(y,\varepsilon) &= \{A \in \mathrm{CL}(Y) \colon \rho(y,A) < \varepsilon\},\\
V(y,\varepsilon) &= \{A \in \mathrm{CL}(Y) \colon \rho(y,A) > \varepsilon\}.
\end{aligned}$$

Let Y be a topological space. *The lower topology* on $\mathrm{CL}(Y)$ is the topology defined by a subbase consisting of all sets V^-, where V is open in Y. Similarly, *the upper topology* is determined by a base consisting of all sets V^+ for open $V \subset Y$. The join of both lower and upper topologies is called *the Vietoris topology.* If (Y,ρ) is a metric space then *the Hausdorff metric* on the

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space of nonempty closed bounded subsets of Y is defined by the formula $\rho_H(A,B) = \max\{\sup_{a\in A}\rho(a,B),\ \sup_{b\in B}\rho(b,A)\}$. Note that the Hausdorff metric induces the Vietoris topology on the space $K(Y)$ (see [2, Theorem II-6]).

By *a multifunction* we mean any mapping $\varphi\colon X \to \mathscr{P}(Y)\setminus\{\emptyset\}$, where X and Y are arbitrary sets. If X and Y are topological spaces, then a multifunction $\varphi\colon X \to \mathrm{CL}(Y)$ is called *lower* (*upper*) *semicontinuous* if φ is continuous with respect to the lower (upper) topology on $\mathrm{CL}(Y)$.

Let (Y,ρ) be a metric space. *The Wijsman topology* $\mathscr{T}_{W_\rho}$ on $\mathrm{CL}(Y)$ is the smallest topology containing the sets $U(y,\varepsilon)$ and $V(y,\varepsilon)$ for all $y\in Y$, $\varepsilon>0$ (cf. [1]). A net of sets $\{A_\sigma\}_{\sigma\in\Sigma}\subset \mathrm{CL}(Y)$ converges to $A\in\mathrm{CL}(Y)$ with respect to the Wijsman topology iff for every $y\in Y$ the net $\{\rho(y,A_\sigma)\}_{\sigma\in\Sigma}$ converges to $\rho(y,A)$. One can observe that $\mathscr{T}_{W_\rho}$ is coarser than the Vietoris topology on $\mathrm{CL}(Y)$ (see also [1, Theorem 2.2.5]).

Let $(T,\mathfrak{M})$ be a measurable space. A multifunction $\varphi\colon T\to\mathrm{CL}(Y)$ is called *measurable* provided $\varphi^{-1}(V^-)\in\mathfrak{M}$ whenever V is open in Y. Observe that this definition differs from the definition of the measurability of a function. The idea is to consider a suitable topology on the space of subsets so that the measurability with respect to the Borel σ-algebra is equivalent to the measurability of a multifunction. For the theory of multifunctions we refer to [2] and [1].

A measure μ on a topological space X is *regular* provided all Borel subsets are measurable and for every measurable set $A\subset X$ and for every $\varepsilon>0$ there exist a closed set $F\subset A$ and an open set $G\supset A$ with $\mu(G\setminus F)<\varepsilon$.

A map $f\colon X\to Y$ is called *simple* (*generalized simple*) provided $f(X)$ is finite (countable).

Let $(T,\mathfrak{M},\mu)$ be a measure space and let $f_n, f\colon T\to Y$, where Y is any set. We say that a sequence $\{f_n\}_{n\in\mathbb{N}}$ is *almost everywhere* (abbreviated to a.e.) *discretely convergent* to f if there exists a measurable set $M\subset T$ with $\mu(M)=0$, and for each $t\in T\setminus M$ there exists $n_0=n_0(t)\in\mathbb{N}$ such that $f_n(t)=f(t)$ for $n\geqslant n_0$ (cf. discrete convergence in [3]).

2. Continuous approximation

In this section we give some generalizations of the following classical result.

FRÉCHET THEOREM. (see, e.g., [6]) *For every measurable function $f\colon D\to\mathbb{R}$ defined on a Lebesgue measurable set $D\subset\mathbb{R}$ there exists a sequence of continuous functions $f_n\colon D\to\mathbb{R}$, almost everywhere pointwise converging to f.*

Recent results on continuous approximation of measurable functions one can find in Nowak [7], Wiśniewski [9, 10] and Kawabe [5].

We now state a general Fréchet type result on approximation of measurable functions. The proof is quite similar to the proof of the Fréchet Theorem in Łojasiewicz [6].

We shall need Lusin's Theorem in the following general form.

Lusin's Theorem. (see, e.g., [11]) *Let X be a topological space with a regular measure μ and let Y be a second countable topological space. If $f\colon X \to Y$ is a measurable mapping then for every $\varepsilon > 0$ there exists a closed subset $F \subset X$ such that $\mu(X \setminus F) < \varepsilon$ and $f|F$ is continuous.*

Theorem 2.1. *Let X be a topological space with a regular measure μ and let Y be a second countable topological space. Assume that every continuous map $f\colon A \to Y$ from any closed subset A of X has a continuous extension $g\colon X \to Y$. Then for every measurable function $\varphi\colon X \to Y$ there exists a sequence of continuous functions $\varphi_n\colon X \to Y$ a.e., discretely convergent to φ.*

Proof. By Lusin's Theorem we obtain a sequence $\{F_n\}_{n\in\mathbb{N}}$ of closed subsets of X such that $\mu(X \setminus F_n) < \frac{1}{n}$ and $\varphi|F_n$ is continuous. Write $A_n = F_1 \cup \cdots \cup F_n$. Clearly, A_n is closed in X, $\varphi|A_n$ is continuous and $\mu(X \setminus A_n) < \frac{1}{n}$. By the assumption, for each $n \in \mathbb{N}$ there exists a continuous map $\varphi_n\colon X \to Y$ such that $\varphi_n|A_n = \varphi|A_n$. Consider $A = \bigcup_{n\in\mathbb{N}} A_n$. We have $\mu(X \setminus A) = 0$ and $\varphi_n(x) = \varphi(x)$ for $x \in A_k$, $n \geqslant k$. This completes the proof. $\square$

We now apply Theorem 2.1 to multifunctions. Our task is to verify the assumptions of Theorem 2.1 for suitable topologies on the space of closed or compact sets. We shall start with the case of the lower topology.

Proposition 2.2. *Let Y be a topological space. Then*

(i) *If $\mathscr{B}$ is an open base of Y then the family $\{B^- : B \in \mathscr{B}\}$ is a subbase of the lower topology in $\mathrm{CL}(Y)$.*

(ii) *If Y is second countable then $\mathrm{CL}(Y)$ with the lower topology is also second countable.*

Lemma 2.3. *Let A be a closed subset of a topological space X and Y be a topological space. Then every lower semicontinuous multifunction $\varphi\colon A \to \mathrm{CL}(Y)$ has a lower semicontinuous extension $\Phi\colon X \to \mathrm{CL}(Y)$.*

Proof. Define

$$\Phi(x) = \begin{cases} \varphi(x), & x \in A, \\ Y, & x \in X \setminus A. \end{cases}$$

It is easy to check that Φ is lower semicontinuous. $\square$

Lemma 2.4. *Let Y be a second countable topological space. Then every measurable multifunction with values in* $\mathrm{CL}(Y)$ *is a measurable function with respect to the lower topology (and conversely).*

Proof. It suffices to observe that, in view of Proposition 2.2, the collection $\{V^-: V \text{ open in } Y\}$ generates the σ-algebra of Borel sets in $\mathrm{CL}(Y)$ with respect to the lower topology. □

As a consequence of Proposition 2.2, Lemmas 2.3, 2.4, and Theorem 2.1 we obtain the following:

Theorem 2.5. *Let X be a topological space with a regular measure and let Y be a second countable topological space. Then for every measurable multifunction $\varphi: X \to \mathrm{CL}(Y)$ there exists a sequence of lower semicontinuous multifunctions $\varphi_n: X \to \mathrm{CL}(Y)$ a.e., discretely convergent to φ.*

Let us turn our consideration to the space of compact sets with the Vietoris topology.

Lemma 2.6. (cf. [2, Theorem II-8]) *If Y is a separable metric space then the space of nonempty compact subsets of Y with the Vietoris topology is second countable.*

Lemma 2.7. (cf. [2, Theorem II-10]) *Let T be a measurable space and let Y be a separable metric space. Then every measurable multifunction $\varphi: T \to K(Y)$ is a measurable function with respect to the Vietoris topology on $K(Y)$ (and conversely).*

The next result is a corollary to a general extension theorem of Curtis [4].

Theorem 2.8. *Let Y be a separable normed space and let X be a metric space. Then for every closed set $A \subset X$ every continuous (with respect to the Vietoris topology) multifunction $\varphi: A \to K(Y)$ can be continuously extended to a multifunction $\Phi: X \to K(Y)$.*

Applying Lemmas 2.6, 2.7, theorem of Curtis and Theorem 2.1 we get the following:

Theorem 2.9. *Let X be a metric space with a regular measure and let Y be a separable normed linear space. Then for every measurable multifunction $\varphi: X \to K(Y)$ there exists a sequence of multifunctions $\varphi_n: X \to K(Y)$, which are continuous with respect to the Vietoris topology, a.e., discretely convergent to φ.*

It would be interesting to obtain analogous results for other topologies on the space of subsets.

3. Approximation by simple multifunctions

We start with a general result for single-valued maps.

THEOREM 3.1. (see, e.g., [8]) *Let $(T, \mathfrak{M})$ be a measurable space and let (Y, ρ) be a separable metric space. Then for each measurable map $f\colon T \to Y$*

(i) *There exists a sequence $f_n\colon T \to Y$ of generalized simple measurable maps, uniformly convergent to f.*

(ii) *If (Y, ρ) is totally bounded then there exists a sequence $f_n\colon T \to Y$ of simple measurable maps, uniformly convergent to f.*

(iii) *There exists a sequence $f_n\colon T \to Y$ of simple measurable maps, pointwise convergent to f.*

An application of this theorem gives us the following result.

THEOREM 3.2. *Let $(T, \mathfrak{M})$ be a measurable space, let (Y, ρ) be a metric space and let $\varphi\colon T \to K(Y)$ be a measurable multifunction, where $K(Y)$ is considered with the Hausdorff metric. Suppose that Y is separable or, more generally, $\varphi(T)$ is a separable subspace of $K(Y)$. Then*

(i) *There exists a sequence of generalized simple measurable multifunctions uniformly convergent to φ.*

(ii) *There exists a sequence of simple measurable multifunctions pointwise convergent to φ.*

(iii) *If, moreover, Y is totally bounded then there exists a sequence of simple measurable multifunctions uniformly convergent to φ.*

Proof. Using Theorem 3.1 and Lemmas 2.6, 2.7 we obtain the first and the second statement. If Y is totally bounded then $K(Y)$ is also totally bounded (see the remark after Theorem II-4 in [2]). This proves (iii). □

Let us now consider the Wijsman topology on the space of closed subsets of a metric space.

PROPOSITION 3.3. ([1, Theorem 2.1.5]) *If (Y, ρ) is a separable metric space then $(\mathrm{CL}(Y), \mathscr{T}_{W_\rho})$ is metrizable and separable.*

LEMMA 3.4. (cf. [1, Hess' Theorem 6.5.14]) *Let $(T, \mathfrak{M})$ be a measurable space and let (Y, ρ) be a separable metric space. Then every measurable multifunction $\varphi\colon T \to \mathrm{CL}(Y)$ is a measurable function with respect to the Wijsman topology on $\mathrm{CL}(Y)$ (and conversely).*

Proof. As $(\mathrm{CL}(Y), \mathscr{T}_{W_\rho})$ has a countable base it is enough to show that the pre-image of each subbasic set is measurable. Notice that $U(y, \varepsilon) =$

$\left[\mathrm{B}(y,\varepsilon)\right]^{-}$. On the other hand

$$V(y,\varepsilon) = \bigcup_{n\in\mathbb{N}} \mathrm{CL}(Y)\setminus U\left(y,\varepsilon+\tfrac{1}{n}\right).$$

It follows that $\varphi^{-1}\left(U(y,\varepsilon)\right), \varphi^{-1}\left(V(y,\varepsilon)\right) \in \mathfrak{M}$. The converse of the above statement also holds, since each set W^{-}, where W is open in Y, is a countable union of sets of the form $U(y,\varepsilon)$. □

THEOREM 3.5. *Let $(T,\mathfrak{M})$ be a measurable space and let (Y,ρ) be a separable metric space. Then every measurable multifunction $\varphi\colon T \to \mathrm{CL}(Y)$ is the pointwise limit, with respect to the Wijsman topology, of a sequence of simple measurable multifunctions.*

Proof. It suffices to apply Lemma 3.4, Proposition 3.3 and Theorem 3.1 (iii). □

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