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MOMENT PROBLEM FOR MAJORATED OPERATORS

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ABSTRACT. We present a moment problem in the context of majorated operators on the space of continuous functions. A generalization of the Hausdorff moment theorem is formulated and proved.

Let X be a normed space and (x_n) a sequence of linearly independent elements of X. If we have an operator U on X into some space Y we can form a sequence of elements of Y

$$U(x_n) = y_n\,, \qquad n = 0, 1, \dots \tag{1}$$

A moment problem in a wide sense may be stated as follows: Given a sequence (y_n) of elements of Y find an operator U satisfying (1). The solution of the problem stated in such a wide sense cannot be done in a satisfactory manner. Nevertheless even in a general case one can formulate some conditions of solvability and uniqueness of solution of the moment problem.

If the set (x_n) is fundamental, i.e., generating X, an operator U (if it exists) is uniquely determined by the sequence (y_n). It is easy to see that fundamentality of (x_n) is also a necessary condition of the unique solution of the moment problem. Of course we need some kind of linearity in the space Y. We shall suppose that Y is a normed vector space. If we denote by U an operator (not necessarily additive) defined on a set (x_n) by

$$U(x_n) = y_n\,, \qquad n = 0, 1, \dots \tag{2}$$

then, as the existence of an operator U is equivalent to the possibility of a linear extension of U to a linear hull $\mathrm{co}\big((x_n)\big)$, a necessary and sufficient condition for solving moment problem is the existence of such a constant M that for any complex numbers $c_0, c_1, \dots$, the inequality

$$\Big\|\sum c_k y_k\Big\| \le M \| c_k x_k \|_X \tag{3}$$

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is satisfied ($\|\cdot\|$ denotes a norm on Y and $\|\cdot\|_X$ a norm on X) or, equivalently,

$$\sup \frac{\left\| \sum c_k y_k \right\|}{\|c_k x_k\|_X} < M\,, \tag{4}$$

the supremum being taken over all possible

$$c_0, c_1, \dots, c_n\,, \qquad n = 0, 1, \dots$$

We shall consider only a power moment problem when $X = C[a,b]$ and

$$x_n(t) = t^n, \qquad n = 0, 1, \dots \tag{5}$$

Recall that the function $g(t)\colon [a,b] \to Y$ is said to have bounded variation $|g|$ if $\sup \sum\limits_j \left\| g(s_j) - g(s_{j-1}) \right\| < \infty$ where all possible partitions od $[a,b]$ are considered. We can formulate a task in the considered case as follows: Decide under which conditions there exists such a function of bounded variation $g(t)\colon [a,b] \to Y$ that

$$\int_a^b t^n \, dg(t) = y_n\,, \qquad n = 0, 1, \dots \tag{6}$$

The conditions (3) and (4) of solvability of the moment problem (6) can be written with t^k instead of x_k.

We shall derive a concrete result relating to a power moment problem in the interval $[0,1]$.

Theorem 1. *In order that there exists a function of bounded variation $g(t)\colon [0,1] \to Y$ such that*

$$\int_0^1 t^n \, dg(t) = y_n\,, \qquad y_n \in Y,\ n = 0, 1, \dots, \tag{7}$$

it is necessary and sufficient that there exists a constant M such that

$$\sum_{k=0}^{n} \binom{n}{k} \left\| \Delta^{n-k} y_k \right\| \le M\,, \qquad n = 0, 1, \dots\,, \tag{8}$$

where $\Delta^m y_k$ denotes the mth differences for the sequence (y_k) defined inductively by the equalities

$$\Delta^{m+1} y_k = \Delta^m y_k - \Delta^m y_{k+1}\,, \quad \Delta^0 y_k = y_k\,,$$
$$m = 0, 1, \dots, \quad k = 0, 1, \dots \tag{9}$$

P r o o f . *The necessity.* Let the moment problem (8) be solved. Denote by L a bounded linear operator on the space $C[0,1]$ generated by a function of bounded variation, $g(t)\colon [0,1] \to Y$,

$$L(f) = \int_0^1 f(t)\, dg(t)\,, \qquad f \in C[0,1]\,.$$

Put

$$x_k^{(m)}(t) = t^k (1-t)^m, \qquad m,k = 0,1,\dots \tag{10}$$

Since

$$\begin{aligned} x_k^{(m+1)}(t) &= t^k(1-t)^{m+1} = t^k(1-t)^m - t^{k+1}(1-t)^m \\ &= x_k^{(m)}(t) - x_{k+1}^{(m)}(t)\,, \end{aligned}$$

we have

$$L\big(x_k^{(m+1)}\big) = L\big(x_k^{(m)}\big) - L\big(x_{k+1}^{(m)}\big)\,, \qquad m,k = 0,1,\dots$$

Further

$$L\big(x_k^{(0)}\big) = y_k\,.$$

If we take into the consideration (10), we can easily see (by induction) that

$$L\big(x_k^{(m)}\big) = \Delta^m y_k\,, \qquad m,k = 0,1,\dots$$

From this we deduce that (8) is satisfied. Indeed,

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} \| \Delta^{n-k} y_k \| &= \sum_{k=0}^n \binom{n}{k} \left\| \int_0^1 x_k^{(n-k)}(t)\, dg(t) \right\| \\ &\le \sum_{k=0}^n \binom{n}{k} \int_0^1 \big| x_k^{(n-k)}(t) \big|\, d|g|(t) \\ &= \int_0^1 \left(\sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} \right) d|g|(t) \\ &= \int_0^1 \big[t + (1-t) \big]^n\, d|g|(t) \le |g|\big([0,1]\big)\,, \end{aligned}$$

where $n = 1,2,\dots$ The necessity of the condition is proved, if we put $M = |g|\big([0,1]\big)$.

The sufficiency. Let L_0 denote the operator defined on the set of functions (x_n), $x_n(t) = t^n$, $n = 0,1,\dots$ into Y by the formula $L_0(x_n) = y_n$, $n = 0,1,\dots$

Extend L to the linear hull of the set (x_n), i.e., to the set of all polynomials. Namely if $x(t) = c_0 + c_1 t + \dots + c_n t^n$, we put

$$L(x) = c_0 y_0 + c_1 a_1 = \dots + c_n y_n .$$

Since the functions x_n are linearly independent, the definition of L is unique.

The operator L defined above is clearly additive and homogeneous. We shall see that condition (9) implies that L is bounded.

Let us note that (even without this condition) the operator L is bounded on the set P_m of polynomials the degree of which is $\le m$, because P_m is a finite-dimensional space (as coordinates we take coefficients of polynomial), hence the convergence in P_m is coordinatewise.

We have

$$L(x_k^{(s)}) = \Delta y_k^s , \qquad s,k = 0,1,\dots$$

Take any polynomial $p(t)$. Let the degree of $p(t)$ be m. Form the sequence of corresponding Bernstein polynomials (of $p(t)$)

$$p_n(t) = B_n(p\,;t) = \sum_{k=0}^{n} \binom{n}{k} p\left(\tfrac{k}{n}\right) t^k (1-t)^{n-k} .$$

It is well-known that the degree of the polynomial $p_n(t)$ for any $n = 1,2,\dots$ is not greater than m, and since $p_n(t)$ uniformly converges to $p(t)$ for $n \to \infty$, we have (according to remarks above) $L(p_n) \to L(p)$.

To obtain our required function, we use Lemma (see [1], §19, p. 380, 383). For each subset A of $[a,b]$, let $C\big([a,b],A\big)$ denote the space of continuous functions on $[a,b]$ vanishing outside A. If $F\colon C\big([a,b]\big) \to X$ is a linear mapping, define for each A,

$$|||F_A||| = \sup \sum \|F(\psi_i)\| ,$$

where supremum is over all finite families ψ_i in $C\big([a,b],A\big)$ with $\sum |\psi_i| \le \chi_A(t)$ for all t in $[a,b]$. □

Lemma 1. *If $F\colon C([a,b]) \to X$ is a linear mapping, then there exists a (regular) Borel measure $\mu\colon \mathscr{B}\big([a,b]\big) \to X$ with finite variation such that*

$$F(\psi) = \int_a^b \psi(t)\, d\mu(t) , \qquad \psi \in C\big([a,b]\big) ,$$

if and only if

$$|||F_A||| < \infty$$

for all A in $\mathscr{B}\big([a,b]\big)$. A mapping F with such a property is called majorated (also dominated, [1]).

An equivalent definition is as follows.

If $F\colon C([a,b]) \to X$ is a linear mapping, it is *majorated* (*dominated*) if and only if there exists a nonnegative Borel measure μ on $\mathscr{B}[0,1]$ such that

$$\|F(\psi)\| \le \int_a^b |\psi(t)|\,d\mu(t)\,, \qquad \psi \in C\big([a,b]\big)\,,$$

To prove that

$$|||L_A||| = \sup \sum \|L(\psi_i)\|$$

is finite, we take for every ψ_i the corresponding Bernstein polynomial

$$p_n^i(t) = B_n(\psi_i;t) = \sum_{k=0}^{n} \binom{n}{k} \psi_i\big(\tfrac{k}{n}\big)\, t^k (1-t)^{n-k}\,.$$

Then we have

$$\sum \big\|L(p_n^i)\big\| \le \Big(\sum \big|\psi_i\big(\tfrac{k}{n}\big)\big|\Big) \cdot \sum_{k=0}^{n} \binom{n}{k} \big\|\Delta^{n-k} y_k\big\| \le M, \qquad n = 0,1,\dots$$

We have an equivalent form of the moment theorem:

Theorem 2. *In order that there exists a vector measure of bounded variation $\mu\colon \mathscr{B}[0,1] \to Y$ such that*

$$\int_0^1 t^n\,d\mu(t) = y_n\,, \qquad y_n \in Y, \quad n = 0,1,\dots, \tag{11}$$

it is necessary and sufficient that there exists a constant M such that

$$\sum_{k=0}^{n} \binom{n}{k} \big\|\Delta^{n-k} y_k\big\| \le M, \qquad (n = 0,1,\dots). \tag{12}$$

Remark 1. For suitable auxiliary readings, we refer to [2] and [3].

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