Tatra Mt. Math. Publ. **19** (2000), 273–280

TATRA MOUNTAINS
Mathematical Publications

SLOWNESS — A MEASURE OF CONVERGENCE OF SEQUENCES

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ABSTRACT. The aim of this paper is to introduce a notion of slowness of convergence for sequences of real numbers. A method of calculation of slowness is derived and relation of this concept to metric dimension is investigated.

Although, there is a number of definitions of concepts of dimension, there is only one among them which can distinguish among countable sets, namely the metric dimension.

The upper and lower metric dimension (sometimes called box counting dimension, entropy dimension, limit capacity, Kolmogorov dimension, Minkowski dimension, Bouligand dimension) of totally bounded metric space X are defined by

$$\overline{\dim} X = \limsup_{r \to 0^+} \frac{\log N(r,X)}{-\log r},$$

and

$$\underline{\dim} X = \liminf_{r \to 0^+} \frac{\log N(r,X)}{-\log r}$$

respectively, where $N(r,X)$ denotes the least number of open balls covering the space X whose diameters are less than or equal to r.

The importance of this concept for countable sets, especially for converging sequences, can be demonstrated by a result in [MZ1; Theorem 4], where it is shown that every infinite compact metric space possesses a converging sequence with upper metric dimension equal to that of the whole space.

Calculation of metric dimension of a number of frequently occurring sequences might suggest that metric dimension should be used to measure the speed of

2000 Mathematics Subject Classification: 54F45, 54F50, 40A05.
Keywords: slowness, metric dimension, box-counting dimension, limit capacity, Hausdorff dimension, Kolmogorov dimension, Minkowski dimension, Bouligand dimension, summable sequences.

Partially supported by Grant Agency VEGA 1/4064/97 and 95/5305/471.

convergence of sequences. However, in [MZ2; Theorem 2] it is shown that it is not the case. To every sequence which has a positive metric dimension and converges to zero, there is a greater sequence whose metric dimension is zero. Note that sequences are compared in a usual sense, i.e., a sequence $\{a_n\}$ is greater than a sequence $\{b_n\}$ if $a_n > b_n$ holds for all but finite number of indices.

To get a means for measuring the speed of convergence of sequences, the following concept of slowness will be introduced, defined in a similar way as as metric dimension.

For any sequence $\{a_n\}$ of real numbers converging to zero we define two measures of "slowness" of its convergence. The numbers

$$\overline{\mathrm{s}}\{a_n\} = \limsup_{r \to 0+} \frac{\log\big(\mathrm{card}\{n \in \mathbb{N} \colon |a_n| \geq r\}\big)}{-\log r},$$

and

$$\underline{\mathrm{s}}\{a_n\} = \liminf_{r \to 0+} \frac{\log\big(\mathrm{card}\{n \in \mathbb{N} \colon |a_n| \geq r\}\big)}{-\log r},$$

where card A denotes the number of terms of finite set A, will be called *upper slowness* and *lower slowness* of the sequence $\{a_n\}$, respectively. If both these numbers equal then the resulting joint value $\mathrm{s}\{a_n\}$ is called *slowness* of the sequence $\{a_n\}$.

Probably, one could expect that the quickness of convergence should be introduced rather than slowness. The decision for concept of slowness arises from formal similarity with the definition of metric dimension and from a nature of results, see, e.g., Theorem 1 below.

Remarks.

i) From the above definitions, it is clear that the greater sequence cannot have smaller (upper, lower) slowness than the smaller one. Roughly spoken, the slower is the convergence of the sequence, the larger is value of slowness. Slowness then preserves monotonicity of partially ordered set of sequences converging to zero.

ii) The value of slowness of some sequence can be any nonnegative real number or infinity, as it is illustrated by examples below. A sequence with zero slowness converges very quickly, a sequence with infinite slowness converges very slowly.

iii) Reindexing of a sequence effects neither the value of slowness nor the value of metric dimension of this sequence.

iv) Since for a sequence $\{a_n\}$ the number of covering balls with diameter r is smaller than or equal to a number of terms of the sequence outside the ball with diameter $2r$ centered in zero, then hold

$$\overline{\dim}\{a_n\} \leq \overline{\mathrm{s}}\{a_n\}, \qquad \text{and} \qquad \underline{\dim}\{a_n\} \leq \underline{\mathrm{s}}\{a_n\}.$$

v) The definition of slowness can be extended to any Cauchy sequence in a metric space.

The above definition of slowness is illustrated by the following example.

EXAMPLE 1. Consider a sequence $\{a_n\} = \{1/n^p\}_{n=1}^{\infty}$, parametrized by $p > 0$. Then for an arbitrary r, $0 < r < 1$, there is a unique positive integer n_r such that

$$\frac{1}{(n_r + 1)^p} < r \le \frac{1}{n_r^p}\,.$$

The number of terms of the sequence $\{a_n\}$ lying outside of the interval $[0, r)$ is exactly n_r. Hence,

$$\frac{\log n_r}{\log(n_r + 1)^p} < \frac{\log\bigl(\operatorname{card}\{n \in \mathbb{N}\colon |1/n^p| \ge r\}\bigr)}{-\log r} \le \frac{\log n_r}{\log n_r^p} = \frac{1}{p}\,.$$

Since the limit of the first fraction exists and equals $1/p$, the slowness of $\{a_n\}$ also exists and $\mathrm{s}\{a_n\} = \frac{1}{p}$.

In the above example the slowness was calculated for a simple type of sequences. For more complicated ones, the sequence $\{\frac{1}{n \log n}\}$ for instance, it is not suitable to calculate directly by definition. A more convenient method to calculate the slowness for converging nonincreasing sequences in the real line is given by the following proposition.

PROPOSITION. *Let $\{a_n\}$ be a nonincreasing sequence of positive real numbers converging to zero. Then*

$$\overline{\mathrm{s}}\{a_n\} = \limsup_{n\to\infty} \frac{\log n}{-\log a_n}\,,$$

and

$$\underline{\mathrm{s}}\{a_n\} = \liminf_{n\to\infty} \frac{\log n}{-\log a_n}\,.$$

Proof. Denote by $\{n_k\}$ the increasing sequence of all indices such that $a_{n_k} < a_{n_k - 1}$, i.e., $a_{n_k} = a_{n_k + 1} = \cdots = a_{n_{k+1} - 1}$. Consider a function $\psi(r) = \frac{\log(\operatorname{card}\{n \in \mathbb{N}\colon |a_n| \ge r\})}{-\log r}$ defined for $r > 0$. This function is increasing and continuous on each interval $(a_{n_{k+1}}, a_{n_k}]$ and it is discontinuous at each point a_{n_k} where a negative jump occurs. We have

$$\overline{\mathrm{s}}\{a_n\} = \limsup_{r\to 0+} \psi(r) = \limsup_{k\to\infty} \frac{\log(n_{k+1} - 1)}{-\log a_{n_{k+1} - 1}}\,, \tag{1}$$

and

$$\underline{s}\{a_n\} = \liminf_{r\to 0^+} \psi(r) = \liminf_{k\to\infty} \frac{\log(n_k - 1)}{-\log a_{n_k}} = \liminf_{k\to\infty} \frac{\log n_k}{-\log a_{n_k}}\,. \tag{2}$$

Let an integer i be such that $n_k \le i \le n_{k+1} - 1$. Then

$$\frac{\log n_k}{\log a_{n_k}} \le \frac{\log i}{\log a_i} \le \frac{\log(n_{k+1} - 1)}{\log a_{n_{k+1}-1}}\,.$$

Hence

$$\limsup_{i\to\infty} \frac{\log i}{-\log a_i} = \limsup_{k\to\infty} \frac{\log(n_{k+1} - 1)}{-\log a_{n_{k+1}-1}}\,,$$

and

$$\liminf_{i\to\infty} \frac{\log i}{-\log a_i} = \liminf_{k\to\infty} \frac{\log n_k}{-\log a_{n_k}}\,,$$

which, together with (1) and (2), gives the desired proof of the Proposition. □

Example 2. The above Proposition can be now used to calculate the slowness of the sequence from Example 1:

$$s\left\{\frac{1}{n^p}\right\} = \lim_{n\to\infty} \frac{\log n}{\log n^p} = \frac{1}{p}\,.$$

Example 3. For a sequence $\{q^n\}$, where $0 < q < 1$, we obtain

$$s\{q^n\} = \lim_{n\to\infty} \frac{\log n}{-\log q^n} = -\lim_{n\to\infty} \frac{\log n}{n \log q} = 0\,.$$

Example 4. Consider a sequence $\{\frac{1}{n\log n}\}$. Then

$$s\left\{\frac{1}{n\log n}\right\} = \lim_{n\to\infty} \frac{\log n}{\log(n\log n)} = \lim_{n\to\infty} \frac{\log n}{\log n + \log(\log n)} = 1\,.$$

Example 5. To give an example of sequence which converges very slowly, consider a sequence $\{\frac{1}{\log n}\}$. Then

$$s\left\{\frac{1}{\log n}\right\} = \lim_{n\to\infty} \frac{\log n}{\log(\log n)} = \infty\,.$$

The statement of the following lemma is mentioned without proof in [H, Theorem 3.1].

Lemma 1. *Let $\{a_n\}$ be a nonincreasing sequence of positive numbers converging to zero. Then*

$$\overline{\mathrm{s}}\{a_n\} = \inf\Big\{p\colon \sum_{n=1}^{\infty} a_n^p < \infty\Big\}.$$

P r o o f . Denote by s the upper slowness of the sequence $\{a_n\}$, $s = \overline{\mathrm{s}}\{a_n\}$. It is sufficient to show that for all real numbers p greater than s the series $\sum\limits_{n=1}^{\infty} a_n^p$ converges and for all p smaller than s this series diverges.

From Proposition we have

$$\limsup_{n\to\infty} \frac{\log n^{\frac{1}{s}}}{-\log a_n} = 1\,. \tag{3}$$

Let $p > s$. (3) implies that for every $\varepsilon > 0$ there exists a positive integer n_0 such that for each $n > n_0$

$$1 + \varepsilon > \frac{\log n^{\frac{1}{s}}}{-\log a_n}$$

and, after rearrangement,

$$a_n < n^{-\frac{1}{s(1+\varepsilon)}}\,.$$

Then for a sufficiently small positive ε it is $\frac{p}{s(1+\varepsilon)} > 1$ which implies the convergence of the series $\sum\limits_{n=1}^{\infty} a_n^p$.

Now, let $p < s$. For every $\varepsilon > 0$, (3) guarantees the existence of the sequence $\{n_k\}$ such that

$$\frac{\log n_k^{\frac{1}{s}}}{-\log a_{n_k}} > 1 - \varepsilon\,.$$

Moreover, this sequence can be chosen in such a way that $n_{k+1} > 2n_k$ holds for all k. Take a sufficiently small positive ε fulfilling $\frac{p}{s(1-\varepsilon)} < 1$. Then

$$a_{n_k}^p > n_k^{-\frac{p}{s(1-\varepsilon)}} > n_k^{-1}. \tag{4}$$

Let $m = n_k$ be such that

$$a_m < 2^{-\frac{1}{s(1-\varepsilon)-p}}. \tag{5}$$

For any integer i, $\frac{m}{2} \le i \le m$,

$$\frac{\log i}{-\log a_i} \ge \frac{\log \frac{m}{2}}{-\log a_m} = \frac{\log m - \log 2}{-\log a_m} > s(1-\varepsilon) - \frac{\log 2}{-\log a_m} \tag{6}$$

from (4). Denote $w = s(1-\varepsilon) - \frac{\log 2}{-\log a_m}$. We can equivalently rewrite (6) as

$$a_i > i^{-\frac{1}{w}}$$

or

$$a_i^p > i^{-\frac{p}{w}} > \frac{1}{i},$$

since (5) is equivalent with

$$\frac{p}{w} < 1.$$

Hence

$$\sum_{n=1}^{\infty}(a_n)^p \ge \sum_{l=1}^{\infty}\sum_{j=\frac{n_l}{2}}^{n_l}(a_j)^p \ge \sum_{l=1}^{\infty}\sum_{j=\frac{n_l}{2}}^{n_l}\frac{1}{j} \ge \sum_{l=1}^{\infty}\frac{1}{2} = \infty.$$

□

The following corollary is a straightforward consequence of the previous lemma.

Corollary. *Let $\{a_n\}$ be a nonincreasing positive sequence converging to zero.*

(i) *If $\overline{\mathrm{s}}(\{a_n\}) < 1$ then $\sum\limits_{n=1}^{\infty} a_n < \infty$.*

(ii) *If $\overline{\mathrm{s}}(\{a_n\}) > 1$ then $\sum\limits_{n=1}^{\infty} a_n = \infty$.*

The first assertion of the corollary is known as the logarithmic test for convergence of series. On the other hand, it is evident that the second one, concerning divergence, is stronger than the logarithmic test for divergence of series in the class of series with nonincreasing terms.

The following example shows the existence of a sequence whose divergence can be decided by Corollary, but, however, cannot be decided by the Raabe's test for divergence, which states that the series $\sum\limits_{n=1}^{\infty}\{a_n\}$ diverges if $\frac{a_{n+1}}{a_n} \ge 1-\frac{1}{n}$ for almost all n.

Example 6. Let $n_k = 2^{2^k}$, $k = 1,2,\dots$. For every k and for all $i > 3$, $n_k < i \le n_{k+1}$, define $a_i = 2^{-2^k}$. Then

$$\overline{\mathrm{s}}\{a_n\} \ge \limsup_{k\to\infty}\frac{\log n_k}{-\log a_{n_k}} = \limsup_{k\to\infty}\frac{\log 2^{2^k}}{-\log 2^{2^{k-1}}} = 2.$$

So, $\sum a_n = \infty$ by Corollary. For the Raabe's test calculate the term $\frac{a_{n+1}}{a_n}$ for $n = 2^{2^k}$ and $k = 1,2,3,\dots$:

$$\frac{a_{n+1}}{a_n} = \frac{a_{2^{2^k}+1}}{a_{2^{2^k}}} = \frac{2^{-2^k}}{2^{-2^{k-1}}} = \frac{1}{2^{2^{k-1}}} \to 0, \qquad k\to\infty.$$

Hence, Raabe's test does not say anything about divergence of $\{a_n\}$.

In connection with the previous example there arises a problem of existence of a divergent series $\sum_{n=1}^{\infty}\{a_n\}$ for which $\lim_{n\to\infty} n\bigl(\frac{a_n}{a_{n+1}}-1\bigr)=1$ and divergence of which can be decided by Corollary ii).

The relation between metric dimension and summability is expressed in Theorem 4 in [MZ2]. This theorem asserts that upper metric dimension of a summable sequence is smaller than or equal to $1/2$. In the following theorem a relation among metric dimension, slowness, and summability is described.

THEOREM 1. *Let $\{a_n\}$ be a sequence such that the infinite series $\sum_{n=1}^{\infty} a_n$ is converging. Denote by $\{s_n\}$ the sequence of partial sums of the series $\sum_{n=1}^{\infty} a_n$. Then*

$$\overline{\mathrm{s}}\{a_n\} = \overline{\dim}\{s_n\}.$$

P r o o f . Reindex, if necessary, the sequence $\{a_n\}$ to get a new, nonincreasing sequence $\{b_n\}$. Since all terms of $\{a_n\}$ are positive, the sums of both sequences equal. Remark 1 iii) and application of Lemma 1 to the sequence $\{b_n\}$ gives

$$\overline{\mathrm{s}}\{a_n\} = \overline{\mathrm{s}}\{b_n\} = \inf\Bigl\{p\colon \sum_{n=1}^{\infty} b_n^p < \infty\Bigr\}.$$

The number on the right side of the last equation is called Besicovitch-Taylor index (see [BT]) of the compact set $\bigcup_{n=1}^{\infty}\{s_n\}\cup\{S\}$, where $S=\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$. In [H] it is proved that this index is equal to $\overline{\dim}\left\{\bigcup_{n=1}^{\infty}\{s_n\}\cup\{S\}\right\}$. Since the closure operation does not change the metric dimension, this completes the proof. $\square$

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Received November 17, 1998

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