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ON INCLUSION BETWEEN WATERMAN CLASSES AND CHANTURYIA CLASSES

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ABSTRACT. A new convergence criterion for series is established and then applied to study inclusions between Waterman and Chanturyia classes of functions of bounded generalized variation.

1. A convergence criterion for series

This section in devoted solely to proving the following convergence criterion for series that somewhat resembles the well known Dini-Abel Theorem on the series $\sum a_n/S_n$.

Theorem 1. *Let (a_n) be a sequence of positive numbers that converges to 0 monotonically and let S_n denote the nth partial sum of the series $\sum a_n$. Then the following statements are equivalent:*

(a) *the series* $\sum\limits_{n=1}^{\infty} a_n$ *converges;*

(b) *the series* $\sum\limits_{n=1}^{\infty} \frac{n(a_n - a_{n+1})}{S_n}$ *converges.*

We precede the proof of the theorem with an auxiliary lemma.

Lemma. *Let (a_n) be a nonincreasing sequence of positive numbers that converges to 0. Then the following statements are equivalent:*

(i) $\lim\limits_{n\to\infty} \dfrac{a_{2n}}{a_n} = 1$;

(ii) $\lim\limits_{n\to\infty} \dfrac{\sum\limits_{k=n+1}^{2n} a_k}{\sum\limits_{k=1}^{n} a_k} = 1$;

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(iii) $\lim\limits_{n\to\infty} \dfrac{n a_{n+1}}{\sum\limits_{i=1}^{n} a_i} = 1$.

P r o o f. (i) $\Rightarrow$ (ii). We will show that

$$\limsup_{m\to\infty} \frac{\sum\limits_{i=1}^{m} a_i}{\sum\limits_{i=m+1}^{2m} a_i} = 1$$

which easily implies (ii). Given $\varepsilon \in \left(\frac{1}{2}, 1\right)$, let N_ε be a positive integer such that $a_{2n}/a_n \geq \varepsilon$ for $n \geq N_\varepsilon$. Set

$$S_k = \sum_{i=1+2^{k-1}N_\varepsilon}^{2^k N_\varepsilon} a_i \qquad \text{for} \quad k = 1, 2, \ldots \qquad \text{and} \qquad S_0 = \sum_{i=1}^{N_\varepsilon} a_i\,.$$

Then $S_{k+1} \geq 2\varepsilon S_k$ for $k \geq 1$ which implies that $S_n \geq (2\varepsilon)^k S_{n-k}$. Thus $S_{n+1} \geq (2\varepsilon)^n S_1 \to +\infty$ with $n \to +\infty$. Since

$$\frac{\sum\limits_{k=0}^{n} S_k}{S_{n+1}} \leq \frac{S_0}{S_{n+1}} + \sum_{k=1}^{n} \frac{1}{(2\varepsilon)^k}\,,$$

we get

$$\limsup_{k\to\infty} \frac{\sum\limits_{i=1}^{2^k N_\varepsilon} a_i}{\sum\limits_{i=1+2^k N_\varepsilon}^{2^{k+1}N_\varepsilon} a_i} \leq \frac{1}{2\varepsilon - 1}\,.$$

Given $m \geq N_\varepsilon$, there is a unique representation $m = 2^{p-1}N_\varepsilon + r$, where p is a positive integer and $0 \leq r \leq 2^{p-1}N_\varepsilon$. Setting

$$x = \sum_{i=1}^{r} a_{i+2^{p-1}N_\varepsilon}\,,$$

we get

$$\sum_{i=1}^{m} a_i = \sum_{k=0}^{p-1} S_k + x$$

and

$$\sum_{m+1}^{2m} a_i = S_p - x + \sum_{1+2^p N_\varepsilon}^{2m} a_i \geq S_p - x + 2\varepsilon \sum_{1+2^{p-1}N_\varepsilon}^{m} a_i\,.$$

Hence

$$\frac{\sum_{1}^{m} a_i}{\sum_{m+1}^{2m} a_i} \le \frac{\sum_{k=0}^{p-1} S_k + x}{S_p + (2\varepsilon - 1)x}.$$

Considering the expression on the right side of the above inequality as a function in x for $0 \le x \le S_p$, we estimate its maximum which leads to the inequality

$$\frac{\sum_{1}^{m} a_i}{\sum_{m+1}^{2m} a_i} \le \max \left\{ \frac{\sum_{k=0}^{p-1} S_k}{S_p}, \ \frac{\sum_{k=0}^{p-1} S_k + S_p}{2\varepsilon S_p} \right\},$$

and therefore

$$\limsup_{m\to\infty} \frac{\sum_{1}^{m} a_i}{\sum_{m+1}^{2m} a_i} \le \frac{1}{2\varepsilon - 1}.$$

Since ε can be arbitrarily close to 1, the proof of (i) $\Rightarrow$ (ii) is complete.

(ii) $\Rightarrow$ (iii). This implication is elementary since (a_n) is nonincreasing.

(iii) $\Rightarrow$ (i). Set

$$\varepsilon_p = 1 - \frac{p a_{p+1}}{\sum_{1}^{p} a_i} \qquad \text{for} \quad p = 1, 2, \dots$$

Since (iii) holds, we get $\varepsilon_p \to 0$ as $p \to +\infty$. Further, since

$$\sum_{n+1}^{2n} a_i \ge \frac{1 - \varepsilon_{2n}}{2} \sum_{1}^{2n} a_i \qquad \text{and} \qquad \sum_{1}^{n} a_i \le \frac{1 + \varepsilon_{2n}}{2} \sum_{1}^{2n} a_i,$$

we get

$$\frac{\sum_{n+1}^{2n} a_i}{\sum_{1}^{n} a_i} \ge \frac{1 - \varepsilon_{2n}}{1 + \varepsilon_{2n}} \to 1$$

as $n \to \infty$ which means that (ii) holds. Finally, we have

$$\frac{a_{2n}}{a_n} = \frac{2n a_{2n}}{\sum_{1}^{2n} a_k} \cdot \frac{\sum_{1}^{n} a_k}{n a_n} \cdot \frac{1}{2} \left(1 + \frac{\sum_{n+1}^{2n} a_k}{\sum_{1}^{n} a_k} \right) \to 1$$

which completes the proof of the lemma. $\square$

Proof of Theorem 1. (a) $\Rightarrow$ (b). This part of the proof is straightforward. Since the series $\sum a_n$ converges, convergence of the series in (b) is equivalent to convergence of the series $\sum n(a_n - a_{n+1})$ by the limit comparison test. However,

$$\sum_{n=1}^{m} n(a_n - a_{n+1}) = \sum_{n=1}^{m} a_n - m a_{m+1}$$

and it is well known that convergence of the series $\sum a_n$ implies convergence to 0 of the sequence (na_n). Thus, again by (a), the series $\sum n(a_n - a_{n+1})$ converges which completes the first part of the proof of Theorem 1.

(b) $\Rightarrow$ (a). Suppose that the series $\sum a_n$ diverges, that is $S_p \to +\infty$ as $p \to \infty$. Consider the case $\limsup a_n/a_{2n} > 1$ first. To prove that under these circumstances the series in (b) diverges, it will suffice to exhibit a constant $d > 0$ such that for every positive integer k there is an integer $p > k$ such that

$$\sum_{n=k}^{p} \frac{n(a_n - a_{n+1})}{S_n} \geq d\,.$$

Observe that

$$\begin{aligned}\sum_{n=k}^{p} \frac{n(a_n - a_{n+1})}{S_n} &\geq \frac{1}{S_p} \sum_{n=k}^{p} n(a_n - a_{n+1}) \\ &= \frac{(k-1)a_k + \sum\limits_{n=k}^{p} a_n - p a_{p+1}}{S_p}\,.\end{aligned}$$

Hence

$$\limsup_{p\to+\infty} \sum_{n=k}^{p} \frac{n(a_n - a_{n+1})}{S_n} \geq 1 - \liminf_{p\to+\infty} \frac{p a_{p+1}}{S_p}\,.$$

Thus we can take for d half of the quantity on the right side of the above inequality for it is positive by our Lemma.

It remains to prove divergence of the series in (b) in case $\lim a_n/a_{2n} = 1$. Then by our lemma $\lim\limits_{n\to\infty} na_n/S_n = 1$. Therefore, by the limit comparison test, the series (b) diverges if and only if the series $\sum(a_n - a_{n+1})/a_n$ diverges. Setting

$$d_n = \frac{1}{a_{n+1}} - \frac{1}{a_n} \qquad \text{and} \qquad D_n = \sum_{k=1}^{n} d_k = \frac{1}{a_{n+1}} - \frac{1}{a_1}\,,$$

we get

$$\sum_{n=1}^{\infty} \frac{a_n - a_{n+1}}{a_n} = \sum_{n=1}^{\infty} \frac{d_n}{D_n + \frac{1}{a_1}}\,.$$

Clearly the above series behaves identically with the series $\sum d_n/D_n$ and it is well known that the last series behaves identically with $\sum d_n$ which diverges since (a_n) tends to 0. $\square$

2. An application of the convergence criterion

We are now going to apply the theorem from the first section to study inclusions between Waterman classes and Chanturiya classes of functions of bounded generalized variation. Actually, this study gave rise to the new convergence criterion stated and proved in the first part of this paper. We start with recalling some basic definitions. The reader interested in more information on the role of these classes it the theory of Fourier series is referred to the remarkable survey paper [A2].

Let f be a real function of period 2π. Let $\Lambda = (\lambda_i)$ be a nondecreasing sequence of positive numbers such that the series $\sum 1/\lambda_i$ diverges. Let $\{I_n\}$ be a sequence of nonoverlapping intervals $I_n = [a_n, b_n] \subset [0, 2\pi]$ and let $f(I_n) = f(b_n) - f(a_n)$. In [W] D. Waterman has introduced the following concept of generalized bounded variation.

Definition 1. A function f is said to be of *bounded Λ-variation* (in symbols $f \in \Lambda\mathrm{BV}$) if

$$\sup \sum_{n=1}^{\infty} \frac{|f(I_n)|}{\lambda_n} < +\infty ,$$

where the supremum is taken over all possible choices of $\{I_n\}$ and is usually denoted by $V_\Lambda(f)$.

On the other hand, for everywhere bounded 2π-periodic functions, Z. A. Chanturiya has introduced the concept of modulus of variation in [Ch].

Definition 2. The modulus of variation of a function f is the sequence $\big(\nu(n, f)\big)_{n\in\mathbb{N}}$ defined by

$$\nu(n, f) = \sup \sum_{k=1}^{n} |f(I_k)| ,$$

where the supremum is taken over all families of nonoverlapping subintervals of $[0, 2\pi]$.

The modulus of variation of any function is nondecreasing and concave. Sequences with such properties are said to be moduli of variation. If a modulus of variation $\nu = (\nu_n)$ is given, then $V[\nu]$ denotes the Chanturiya class of functions satisfying $\nu(n, f) = O(\nu_n)$ as $n \to \infty$.

Inclusions between Waterman classes and Chanturyia classes were investigated by M. Avdispahić. In [A1] he proved that $\Lambda\mathrm{BV}$ is a subset of $V\Big[n/\sum\limits_1^n \frac{1}{\lambda_i}\Big]$. However, it is easy to see that the sequence $\Big(n/\sum\limits_1^n \frac{1}{\lambda_i}\Big)_{n\in\mathbb{N}}$ does not have to be concave. Thus, before analysing the inclusion, we need to justify the use of the sequence $\left(n/\sum\limits_1^n \frac{1}{\lambda_i}\right)_{n\in\mathbb{N}}$ in the symbol for a Chanturiya class. Of course, by $V\left[n/\sum\limits_1^n \frac{1}{\lambda_i}\right]$ we understand the class of all functions f such that $\nu(n,f) = O\left(n/\sum\limits_1^n \frac{1}{\lambda_i}\right)$, but this class could be not any of Chanturyia classes. We will show that it is not the case.

Z. A. Chanturiya proved that an equality $V[\nu] = V[w]$ holds if and only if there exist two positive constants A and B such that $A\nu_n \le w_n \le B\nu_n$ for all n. If the latter is the case, we will say that sequences (ν_n) and (w_n) are equivalent. The definition of equivalent sequences and the Chanturiya Theorem on them extend easily to non-concave sequences, and the following fact holds.

Proposition 1. *A necessary and sufficient condition for a nondecreasing sequence (a_n) of positive numbers to be equivalent to a concave sequence is that*

$$\exists c \ge 1 \quad \forall m\, \forall n \ge m \qquad \frac{a_n}{n} \le c\,\frac{a_m}{m}.$$

Proof. Necessity: Since every concave sequence of positive numbers is equivalent to a concave sequence satisfying the above condition with $c = 1$, the proof of necessity of the condition is straightforward and we leave it out.

Sufficiency: Careful analysis of the following considerations shows that a sequence $(\bar{a}_n)$ constructed in the proof is the smallest of all concave sequences that lie above the sequence (a_n). But we will not elaborate on the observation.

Set $b_1 = a_1$ and, for $k \ge 2$,

$$b_k = \sup_{n\ge k} \frac{a_n - \sum\limits_{i=1}^{k-1} b_i}{n-(k-1)}.$$

Then (b_k) is a nonincreasing sequence, since

$$\begin{aligned} b_{k+1} &= \sup_{n\ge k+1}\left(\frac{a_n - \sum\limits_{i=1}^{k-1} b_i}{n-k} - \frac{b_k}{n-k}\right) \\ &\le \sup_{n\ge k+1}\left(\frac{a_n - \sum\limits_{i=1}^{k-1} b_i}{n-k} - \frac{a_n - \sum\limits_{i=1}^{k-1} b_i}{(n-k)(n-k+1)}\right) \le b_k. \end{aligned}$$

Now suppose $b_{k+1} < 0$. Then, for $n \ge k+1$,

$$a_n - \sum_{i=1}^{k} b_i \le b_{k+1}(n-k)$$

which implies that

$$a_n - \sum_{i=1}^{k-1} b_i \le b_{k+1}(n-k) + b_k < b_k .$$

Thus, by the definition of b_k,

$$b_k = a_k - \sum_{i=1}^{k-1} b_i ,$$

and hence, $a_n < a_k$ for $n \ge k+1$, a contradiction. We have proven that (b_k) is a nonincreasing sequence of nonnegative numbers, and therefore

$$\bar{a}_k \overset{\text{def}}{=} \sum_{i=1}^{k} b_i$$

is a concave sequence.

In order to complete the proof it suffices to show that $a_k \le \bar{a}_k \le (c+1)a_k$. Clearly, $a_1 = \bar{a}_1 < (c+1)a_1$ and

$$\bar{a}_{k+1} = \bar{a}_k + \sup_{n \ge k+1} \frac{a_n - \bar{a}_k}{n-k+1} \ge \bar{a}_k + a_{k+1} - \bar{a}_k = a_{k+1}$$

for $k = 1, 2, \dots$

Given an integer $k \ge 2$, let $m = \sup\{p \in \{1, \dots, k-1\} \colon a_p = \bar{a}_p\}$. Then

$$\begin{aligned} \bar{a}_k &= a_m + (k-m) \sup_{n \ge k} \frac{a_n - a_m}{n-m} \\ &< a_k + (k-m) \sup_{n \ge k} \frac{c\frac{n}{k}\, a_k}{n-m} = (1+c)a_k \end{aligned}$$

which completes the proof of sufficiency and whence the whole proof. □

A necessary and sufficient condition for a sequence (a_n) of positive numbers to be equivalent to a nondecreasing sequence is that there is a constant $d \ge 1$ such that

$$\forall m \ \forall n \le m \quad a_n \le d a_m .$$

The suitable nondecreasing sequence can be then defined by $\bar{a}_n = \sup_{m \le n} a_m$, but the proof is so easy that it will be omitted.

Corollary 1. *A necessary and sufficient condition for a sequence (a_n) of positive numbers to be equivalent to a concave sequence is that there are constants c, $d \ge 1$ such that*

$$\forall m \ \forall n \ge m \qquad \frac{a_n}{n} \le c\, \frac{a_m}{m} \qquad \text{and} \qquad a_m \le d a_n .$$

P r o o f . Necessity is straightforward. By the virtue of our earlier observations, in order to prove sufficiency of the above conditions it suffices to show that a nondecreasing equivalent of (a_n) satisfies the condition of Proposition 1. Indeed,

$$\frac{\sup_{k\le n} a_k}{n} \le \frac{\sup_{k\le n} da_n}{n} \le dc\,\frac{a_m}{m} \le dc\,\frac{\sup_{k\le m} a_k}{m}\,.$$

□

The next corollary lets us not bother with possible non-concavity of the sequence $\left(n/\sum\limits_1^n \frac{1}{\lambda_i}\right)$.

COROLLARY 2. *For every λ-sequence Λ, there is a concave sequence (ν_n) (a modulus of variation) such that* $V\left[n/\sum\limits_{i=1}^n \frac{1}{\lambda_i}\right] = V[\nu_n]$.

It follows easily form the fact that $h_n = n/\sum\limits_{i=1}^n \frac{1}{\lambda_i}$ is a nondecreasing sequence with h_n/n monotonically convergent to 0 and from Chanturiya's Thm. 2.

We now return to the inclusions between Waterman and Chanturiya classes and we will use the convergence criterion from the first part of the paper.

PROPOSITION 2. *The Waterman class $\Lambda\mathrm{BV}$ is always a proper subset of its corresponding Chanturiya class* $V\left[n/\sum\limits_{i=1}^n \frac{1}{\lambda_i}\right]$.

P r o o f . Let (ν_n) be a concave sequence such that $V\left[n/\sum\limits_{i=1}^n \frac{1}{\lambda_i}\right] = V[\nu_n]$. Then there is a constant $c > 0$ such that $\nu_n \le cn/\sum\limits_{i=1}^n \frac{1}{\lambda_i}$ for all n. Let f be the sawtooth function such that $\nu(f,n) = \nu_n$ and

$$V(f) = \sum_{i=1}^{\infty} \frac{\nu_i - \nu_{i-1}}{\lambda_i} \qquad (\text{where } \nu_0 = 0)\,.$$

Since

$$\sum_{i=1}^{n} \frac{\nu_i - \nu_{i-1}}{\lambda_i} = \sum_{i=1}^{n-1} \nu_i\left(\frac{1}{\lambda_i} - \frac{1}{\lambda_{i+1}}\right) + \frac{\nu_n}{\lambda_n}$$

and since

$$0 < \frac{\nu_n}{\lambda_n} \le c\,\frac{n\frac{1}{\lambda_n}}{\sum\limits_{i=1}^{n} \frac{1}{\lambda_i}} \le c\,,$$

f is of bounded Λ-variation if and only if the series

$$\sum_{i=1}^{\infty} \nu_i\left(\frac{1}{\lambda_i} - \frac{1}{\lambda_{i+1}}\right)$$

converges. This, in turn, is equivalent to convergence of the series

$$\sum_{i=1}^{\infty} \frac{n}{\sum\limits_{k=1}^{n} \frac{1}{\lambda_k}} \left(\frac{1}{\lambda_i} - \frac{1}{\lambda_{i+1}} \right)$$

since the sequences (ν_n) and $n/\sum\limits_{k=1}^{n} \frac{1}{\lambda_k}$ are equivalent. However, since $\sum 1/\lambda_i$ diverges by the definition of a λ-sequence, our Theorem 1 leads to a conclusion of divergence of the above series. This means that f is not of bounded Λ-variation despite being of class $V\Big[n/\sum\limits_{i=1}^{n} \frac{1}{\lambda_i}\Big]$. □

As a concluding remark let us mention an open problem of similar kind. M. Avdispahić has shown in [A1] that

$$\bigcup V[\nu] \subset \Lambda\mathrm{BV}$$

where the union on the left side is taken over all moduli of variation $\nu = (\nu_n)$ that satisfy the condition $\sum\limits_{i=1}^{\infty} \big(\frac{1}{\lambda_i} - \frac{1}{\lambda_{i+1}}\big)\nu_i < +\infty$. It is still unknown whether the inclusion is always proper or not.

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